AN ELEMENTARY TREATISE

ON THE

DYNAMICS OF A SYSTEM OF RIGID BODIES.

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DYNAMICS OF A SYSTEM OF RIGID BODIES.

With numerous Examples.

BY

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PREFACE.

In this edition I have made many additions to every part of the subject. I have been led to do this, because there are so many important applications which it did not seem proper to pass over without some notice. I have found how difficult it is not to render a book formidable to the student by its size and yet to supply some information at least on all the chief points of a great subject. I believe the reader will not find any portion treated at greater length than is necessary to render the argument intelligible.

As in the former editions, each chapter has been made as far as possible complete in itself, so that all that relates to any one part of the subject may be found in the same place. This arrangement will be found convenient for those who are already acquainted with the subject, as it will enable them to direct their attention to those parts in which they may feel most interested. It will also enable the student to select his own order of reading the subject. The student who is just beginning Dynamics may not wish to be delayed by a long chapter of preliminary analysis before he enters on the real subject of the book. He may therefore begin at D'Alembert's Principle and

only read those parts of Chapter I. to which reference is made. Other readers may also wish to pass on as soon as possible to the great principles of Angular Momentum and Vis Viva. Though a different order will be found advisable for different persons, I have ventured to indicate a list of Articles to which those who are just beginning the subject should first turn their attention.

It will be observed that a chapter has been devoted to the discussion of Motion in Two Dimensions. This course has been adopted because it seemed expedient to separate the difficulties of Dynamics from those of Solid Geometry.

I have attempted to give a slight historical notice whenever I felt it could be briefly done. This course, if not carried too far, will I believe be found to add greatly to the interest of the subject. But the success of this attempt is far from complete. In the earlier portions of the subject I had the guidance of Montuela, and further on there was Prof. Cayley's Report to the British Association. With the help of these the task became comparatively easy; but in some other portions the number of Memoirs which have been written is so vast, that anything but the slightest notice has been rendered impossible. A useful theorem is many times discovered, and probably each time with some variations. It is thus often difficult to ascertain who is the first author. I have therefore found it necessary to correct some of the references given in the second edition, and to add references where there were none before.

Throughout each chapter there will be found numerous examples, many very easy and others which are intended for the more advanced student. In order to obtain as great a variety of problems as possible, I have added a further collection at the end of each chapter, taken from the Examination Papers which have been set in the University and in the Colleges.

Some of these are such excellent illustrations of dynamical principles that they will certainly be of the greatest assistance to the student.

I cannot conclude without expressing how much I am indebted to Mr Webb, of St John's College, for the great assistance he has given me in correcting the proofs of the first eight chapters, and for the suggestions he has made to me. Most of the examples in these chapters have also been very kindly verified by him. Several others also of my friends have greatly assisted me by correcting some proof-sheets for me, particularly Mr Edwards, of Sidney Sussex College, who has read the proofs of the last three chapters.

Some portions of this edition have been written several years ago, and the printing has extended over two years. This course, though open to many objections, was rendered unavoidable by the pressure of other engagements. I have therefore found it necessary to add a few Notes, chiefly historical, at the end of the treatise.

EDWARD J. ROUTH.

PETERHOUSE,
April 24, 1877.

ERRATA.

Page 77.	Line 30.	*For* $h+h$ *read* $h+h'$.
,, 103.	,, 23.	*For* in that case *read* in these cases.
,, 113.	,, 18.	*For* $\cos 2\theta$ *read* $\cos\theta$.
,, 131.	,, 3.	*For* $\frac{-3\mu}{2-\mu^2}\tan^{-1}\frac{2}{\mu}$ *read* $\frac{-6\mu}{2-\mu^2}\tan^{-1}\frac{2}{\mu}$.
,, 141.		*For* v *read* V throughout the page.
,, 146.	,, 1.	*Omit the word* necessary.
,, 255.	,, 2.	*For* is *read* is parallel to.
,, 264.	,, 33.	*For* $\int^{\infty}$ *read* $\int_0^{\infty}$.
,, 287.	,, 14.	*For* single *read* simple.
,, 297.	,, 21.	*For* $\sqrt{h}$ *read* $\sqrt{2h}$.
,, 299.	,, 30.	*For* by Δ *read* by 2Δ.
,, 303.	,, 34.	*For* θ *read* $-\theta$.
,, ,,	,, 35.	*For* of three *read* of the axes referred to three.
,, ,,	,, 36.	*For* Art. 235 *read* Note to Art. 235.
,, 304.	,, 17.	*For* ψ'' *read* ϕ''.
,, 313.	,, 17.	*For* t *read* t, t_0.
,, 330.	,, 15.	*For* $-$ *read* $+$.
,, ,,	,, 17.	*For* x *read* $f(x)$.
,, 331.	,, 8.	*For* $-$ *read* $+$.

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NOTES.

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CHAPTER I.

ON FINDING MOMENTS OF INERTIA BY INTEGRATION.

1. In the subsequent pages of this work it will be found that certain integrals continually recur. It is therefore convenient to collect these into a preliminary chapter for reference. Though the bearing of these on Dynamics may not be obvious beforehand, yet the student may be assured that it is as useful to be able to write down moments of inertia with facility as it is to be able to quote the centres of gravity of the elementary bodies.

In addition however to these necessary propositions there are many others which are useful as giving a more complete view of the arrangement of the axes of inertia in a body. These also have been included in this chapter though they are not of the same importance as the former.

2. All the integrals used in Dynamics as well as those used in Statics and some other branches of Mixed Mathematics are included in the one form

$$\iiint x^\alpha y^\beta z^\gamma \, dx \, dy \, dz,$$

where (α, β, γ) have particular values. In Statics two of these three exponents are usually zero, and the third is either unity or zero, according as we wish to find the numerator or denominator of a coordinate of the centre of gravity. In Dynamics of the three exponents one is zero, and the sum of the other two is usually equal to 2. The integral in all its generality has not yet been fully discussed, probably because only certain cases have any real utility. In the case in which the body considered is a homogeneous ellipsoid the value of the general integral has been found in gamma functions by Lejeune Dirichlet in Vol. IV. of Liouville's Journal. His results were afterwards extended by Liouville in the same volume to the case of a heterogeneous ellipsoid in which the strata of uniform density are similar ellipsoids.

In this treatise, it is intended to restrict ourselves to the consideration of moments and products of inertia, as being the only cases of the integral which are useful in Dynamics.

3. If the mass of every particle of a material system be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the *moment of inertia* of the system about that line.

If M be the mass of a system and k be such a quantity that Mk^2 is its moment of inertia about a given straight line, then k is called the *radius of gyration* of the system about that line.

The term "moment of inertia" was introduced by Euler, and has now got into general use wherever Rigid Dynamics is studied. It will be convenient for us to use the following additional terms.

If the mass of every particle of a material system be multiplied by the square of its distance from a given plane or from a given point, the sum of the products so formed is called the moment of inertia of the system with reference to that plane or that point.

If two straight lines Ox, Oy be taken as axes, and if the mass of every particle of the system be multiplied by its *two* co-ordinates x, y, the sum of the products so formed is called the *product of inertia* of the system about those two axes.

This might, perhaps more conveniently, be called the product of inertia of the system with reference to the two co-ordinate planes xz, yz.

4. Let a body be referred to any rectangular axes Ox, Oy, Oz meeting in a point O, and let x, y, z be the co-ordinates of any particle m, then according to these definitions the moments of inertia about the axes of x, y, z respectively will be

$$A = \Sigma m\,(y^2 + z^2), \quad B = \Sigma m\,(z^2 + x^2), \quad C = \Sigma m\,(x^2 + y^2).$$

The moments of inertia with regard to the planes yz, zx, xy, respectively, will be

$$A' = \Sigma mx^2, \quad B' = \Sigma my^2, \quad C' = \Sigma mz^2.$$

The products of inertia with regard to the axes yz, zx, xy, will be

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad F = \Sigma mxy.$$

Lastly, the moment of inertia with regard to the origin will be

$$H = \Sigma m\,(x^2 + y^2 + z^2) = \Sigma mr^2,$$

where r is the distance of the particle m from the origin.

5. The following propositions may be established without difficulty, and will serve as illustrations of the preceding definitions.

(1) The three moments of inertia A, B, C about three rectangular axes are such that the sum of any two of them is greater than the third.

(2) The sum of the moments of inertia about any three rectangular axes meeting at a given point is always the same; and is equal to twice the moment of inertia with respect to that point.

For $A+B+C=2\Sigma m(x^2+y^2+z^2)=2\Sigma mr^2$, and is therefore independent of the directions of the axes.

(3) The sum of the moments of inertia of a system with reference to any plane through a given point and its normal at that point is constant and equal to the moment of inertia of the system with reference to that point.

Take the given point as origin and the plane as the plane of xy, then $C'+C=\Sigma mr^2$, which is independent of the direction of the axes.

Hence we infer that

$$A'=\tfrac{1}{2}(B+C-A),\quad B'=\tfrac{1}{2}(C+A-B),\ \text{and}\ C'=\tfrac{1}{2}(A+B-C).$$

(4) Any product of inertia as D cannot numerically be so great as $\frac{1}{2}A$.

(5) If A, B, F be the moments and product of inertia of a lamina about two rectangular axes in its plane, then AB is greater than F^2.

If t be any quantity we have $At^2+2Ft+B=\Sigma m(yt+x)^2=$ a positive quantity. Hence the roots of the quadratic $At^2+2Ft+B=0$ are imaginary, and therefore $AB>F^2$.

(6) Prove that for any body

$$(A+B-C)(B+C-A)>4E^2,$$
$$(A+B-C)(B+C-A)(C+A-B)=8DEF.$$

(7) Prove that the moment of inertia of the surface of a hemisphere of radius a and mass M about the diameter perpendicular to the base is $M\frac{2}{3}a^2$.

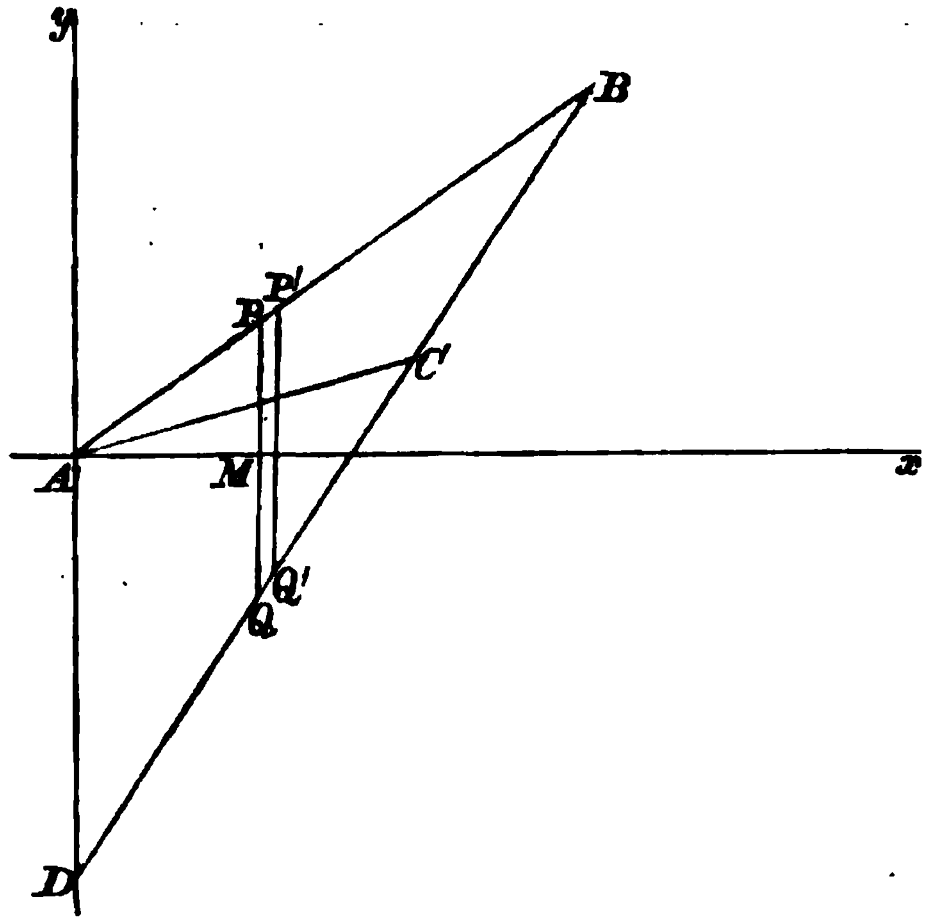
For, complete the sphere, then by (2) the moment of inertia about any diameter is two-thirds of the moment of inertia with respect to the point.

6. It is clear that the process of finding moments and products of inertia is merely that of integration. We may illustrate this by the following example.

To find the moment of inertia of a uniform triangular plate about an axis in its plane passing through one angular point.

Let ABC be the triangle, Ay the axis about which the moment is required. Draw Ax perpendicular to Ay and produce BC to meet Ay in D. The given triangle ABC may be regarded as the difference of the triangles ABD, ACD. Let us then first find the moment of inertia of ABD. Let $PQP'Q'$ be an elementary area whose sides PQ, $P'Q'$ are parallel to the base AD,

and let PQ cut Ax in M. Let β be the distance of the angular point B from the axis Ay, $AM = x$ and $AD = l$.



Then the elementary area $PQP'Q'$ is clearly $l\dfrac{\beta - x}{\beta}dx$, and its moment of inertia about Ay is $\mu l\dfrac{\beta - x}{\beta}dx \,.\, x^2$, μ being the mass per unit of area. Hence the moment of inertia of the triangle ABD

$$= \mu \int_0^\beta l\left(1 - \frac{x}{\beta}\right)x^2 dx = \mu l \frac{\beta^3}{12}.$$

Similarly if γ be the distance of the angular point C from the axis Ay, the moment of inertia of the triangle ACD is $\mu l\dfrac{\gamma^3}{12}$. Hence the moment of inertia of the given triangle ABC is $\mu\dfrac{l}{12}(\beta^3 - \gamma^3)$. Now $\dfrac{1}{2}l\beta$ and $\dfrac{1}{2}l\gamma$ are the areas of the triangles ABD, ACD. Hence if M be the mass of the triangle ABC, the moment of inertia of the triangle about the axis Ay is

$$\frac{M}{6}(\beta^2 + \beta\gamma + \gamma^2).$$

Ex. If each element of the mass of the triangle be multiplied by the nth power of its distance from the straight line through the angle A, then it may be proved in the same way that the sum of the products is

$$\frac{2M}{(n+1)(n+2)} \frac{\beta^{n+1} - \gamma^{n+1}}{\beta - \gamma}.$$

7. *When the body is a lamina the moment of inertia about an axis perpendicular to its plane is equal to the sum of the moments*

of inertia about any two rectangular axes in its plane drawn from the point where the former axis meets the plane.

For let the axis of z be taken as the normal to the plane, then, if A, B, C be the moments of inertia about the axes, we have

$$A = \Sigma m y^2, \quad B = \Sigma m x^2, \quad C = \Sigma m (x^2 + y^2),$$

and therefore $$C = A + B.$$

We may apply this theorem to the case of the triangle. Let β', γ' be the distances of the points B, C from the axis Ax. Then the moment of inertia of the triangle about a normal to the plane of the triangle through the point A is

$$= \frac{M}{6} (\beta^2 + \beta\gamma + \gamma^2 + \beta'^2 + \beta'\gamma' + \gamma'^2).$$

8. The following moments of inertia occur so frequently that they have been collected together for reference. The reader is advised to commit to memory the following table:

The moment of inertia of

(1) A rectangle whose sides are $2a$ and $2b$

about an axis through its centre in its plane perpendicular to the side $2a$ $\Big\} = \text{mass}\,\dfrac{a^2}{3}$,

about an axis through its centre perpendicular to its plane $\Big\} = \text{mass}\,\dfrac{a^2 + b^2}{3}$.

(2) An ellipse semi-axes a and b

about the major axis $a = \text{mass}\,\dfrac{b^2}{4}$,

.... minor axis $b = \text{mass}\,\dfrac{a^2}{4}$,

about an axis perpendicular to its plane through the centre $\Big\} = \text{mass}\,\dfrac{a^2 + b^2}{4}$.

In the particular case of a circle of radius a, the moment of inertia about a diameter is mass $\dfrac{a^2}{4}$, and about a perpendicular to its plane through the centre mass $\dfrac{a^2}{2}$.

(3) An ellipsoid semi-axes a, b, c

about the axis $a = \text{mass}\,\dfrac{b^2 + c^2}{5}$.

In the particular case of a sphere of radius a the moment of inertia about a diameter $= \text{mass}\,\dfrac{2}{5} a^2$.

(4) A right solid whose sides are $2a$, $2b$, $2c$

$$\left.\begin{array}{r}\text{about an axis through its centre perpendicular}\\ \text{to the plane containing the sides } b \text{ and } c\end{array}\right\} = \text{mass}\,\frac{b^2+c^2}{3}.$$

These results may be all included in one rule, which the author has long used as an assistance to the memory.

$$\left.\begin{array}{r}\text{Moment of inertia}\\ \text{about an axis}\\ \text{of symmetry}\end{array}\right\} = \text{mass}\,\frac{\text{(sum of squares of perpendicular semi-axes)}}{3,\ 4 \text{ or } 5}.$$

The denominator is to be 3, 4 or 5, according as the body is rectangular, elliptical or ellipsoidal.

Thus, if we wanted the moment of inertia of a circle of radius a about a diameter, we notice that the perpendicular semi-axis in its plane is the radius a, and the semi-axis perpendicular to its plane is zero, the moment of inertia required is therefore $M\frac{a^2}{4}$, if M be the mass. If we wanted the moment about a perpendicular to its plane through the centre, we notice that the perpendicular semi-axes are each equal to a and the moment required is therefore

$$M\frac{a^2+a^2}{4} = M\frac{a^2}{2}.$$

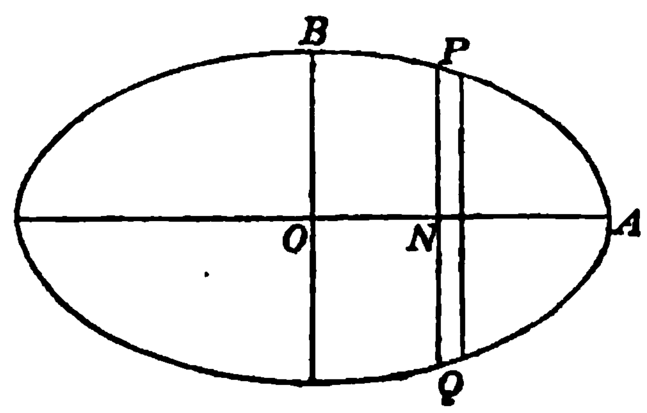
9. As the process for determining these moments of inertia is very nearly the same for all these cases, it will be sufficient to consider only two instances.

To determine the moment of inertia of an ellipse about the minor axis.

Let the equation to the ellipse be $y=\frac{b}{a}\sqrt{a^2-x^2}$. Take any elementary area PQ parallel to the axis of y, then clearly the moment of inertia is

$$4\mu\int_0^a x^2y\,dx = 4\mu\frac{b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx,$$

where μ is the mass of a unit of area.



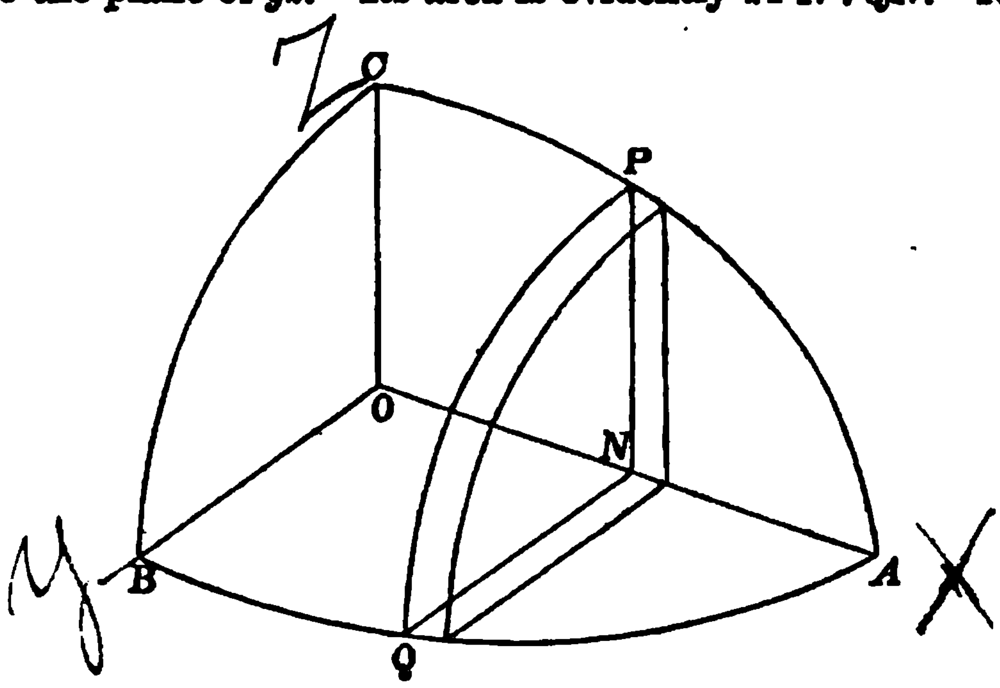
To integrate this, put $x=a\sin\phi$, then the integral becomes

$$a^4\int_0^{\frac{\pi}{2}}\cos^2\phi\sin^2\phi\,d\phi = a^4\int_0^{\frac{\pi}{2}}\frac{1-\cos 4\phi}{8}\,d\phi = \frac{\pi a^4}{16};$$

$$\therefore \text{ the moment of inertia} = \mu\pi ab\frac{a^2}{4} = \text{mass}\,\frac{a^2}{4}.$$

To determine the moment of inertia of an ellipsoid about a principal diameter.

Let the equation to the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Take any elementary area PNQ parallel to the plane of yz. Its area is evidently $\pi PN.QN$. Now PN is the



value of z when $y=0$, and QN the value of y when $z=0$, as obtained from the equation to the ellipsoid; $\therefore PN=\frac{c}{a}\sqrt{a^2-x^2}$, $QN=\frac{b}{a}\sqrt{a^2-x^2}$;

$$\therefore \text{ the area of the element}=\frac{\pi bc}{a^2}(a^2-x^2).$$

Let μ be the mass of a unit of volume, then the whole moment of inertia

$$=\mu\int_{-a}^{a}\frac{\pi bc}{a^2}(a^2-x^2)\frac{PN^2+QN^2}{4}dx$$

$$=\mu\frac{\pi}{4}\frac{bc}{a^2}\int_{-a}^{a}(a^2-x^2)\frac{b^2+c^2}{a^2}(a^2-x^2)\,dx$$

$$=\mu\frac{4}{3}\pi abc\frac{b^2+c^2}{5}$$

$$=\text{mass}\,\frac{b^2+c^2}{5}.$$

Ex. 1. The moment of inertia of an arc of a circle whose radius is a and which subtends an angle 2α at the centre

(a) about an axis through its centre perpendicular to its plane $=Ma^2$,

(b) about an axis through its middle point perpendicular to its plane

$$=2M\left(1-\frac{\sin\alpha}{\alpha}\right)a^2,$$

(c) about the diameter which bisects the arc $=M\left(1-\frac{\sin 2\alpha}{2\alpha}\right)\frac{a^2}{2}$.

Ex. 2. The moment of inertia of the part of the area of a parabola cut off by any ordinate at a distance x from the vertex is $M\frac{3}{7}x^2$ about the tangent at the vertex, and $M\frac{y^2}{5}$ about the principal diameter, where y is the ordinate corresponding to x.

Ex. 3. The moment of inertia of the area of the lemniscate $r^2=a^2\cos 2\theta$ about a line through the origin in its plane and perpendicular to its axis is $M\dfrac{3\pi+8}{48}a^2$.

Ex. 4. A lamina is bounded by four rectangular hyperbolas, two of them have the axes of co-ordinates for asymptotes, and the other two have the axes for principal diameters. Prove that the sum of the moments of inertia of the lamina about the co-ordinate axes is

$$\frac{1}{4}(\alpha^2-\alpha'^2)(\beta^2-\beta'^2),$$

where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the hyperbolas.

Take the equations $xy=u$, $x^2-y^2=v$, then the two moments of inertia are $A=\iint x^2 J\,du\,dv$ and $B=\iint y^2 J\,du\,dv$, where $\dfrac{1}{J}$ is the Jacobian of uv with regard to xy. This gives at once $A+B=\frac{1}{2}\iint du\,dv$, where the limits are clearly $u=\dfrac{\alpha^2}{2}$ to $\dfrac{\alpha'^2}{2}$, $v=\beta^2$ to $v=\beta'^2$.

Ex. 5. A lamina is bounded on two sides by two similar ellipses, the ratio of the axes in each being m, and on the other two sides by two similar hyperbolas, the ratio of the axes in each being n. These four curves have their principal diameters along the co-ordinate axes. Prove that the product of inertia about the co-ordinate axes is $\dfrac{(\alpha^2-\alpha'^2)(\beta^2-\beta'^2)}{4(m^2+n^2)}$, where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the curves.

10. Many moments of inertia may be deduced from those given in Art. 8 by the method of differentiation. Thus the moment of inertia of a solid ellipsoid of uniform density ρ about the axis of a is known to be $\frac{4}{3}\pi abc\rho\dfrac{b^2+c^2}{5}$. Let the ellipsoid increase indefinitely little in size, then the moment of inertia of the enclosed shell is

$$d\left\{\frac{4}{3}\pi abc\rho\frac{b^2+c^2}{5}\right\}.$$

This differentiation can be effected as soon as the law according to which the ellipsoid alters is given. Suppose the bounding ellipsoids to be similar, and let the ratio of the axes in each be $\dfrac{b}{a}=p$, $\dfrac{c}{a}=q$. Then

$$\text{moment of inertia of solid ellipsoid} = \frac{4}{3}\pi\rho pq\frac{p^2+q^2}{5}a^5;$$

$$\therefore \text{ moment of inertia of shell} = \frac{4}{3}\pi\rho pq(p^2+q^2)a^4da.$$

In the same way

$$\text{mass of solid ellipsoid} = \frac{4}{3}\pi\rho pqa^3;$$

$$\therefore \text{ mass of shell} = 4\pi\rho pqa^2da.$$

Hence the moment of inertia of an indefinitely thin ellipsoidal shell of mass M bounded by similar ellipsoids is $M\frac{b^2+c^2}{3}$.

By reference to Art. 8, it will be seen that this is the same as the moment of inertia of the circumscribing right solid of equal mass. These two bodies therefore have equal moments of inertia about their axes of symmetry at the centre of gravity.

11. The moments of inertia of a heterogeneous body whose boundary is a surface of uniform density may sometimes be found by the method of differentiation. Suppose the moment of inertia of a homogeneous body of density D, bounded by any surface of uniform density, to be known. Let this when expressed in terms of some parameter a be $\phi(a)D$. Then the moment of inertia of a stratum of density D will be $\phi'(a)\,Dda$. Replacing D by the variable density ρ, the moment of inertia required will be

$$\int \rho\phi'(a)\,da.$$

Ex. 1. Shew that the moment of inertia of a heterogeneous ellipsoid about the major axis, the strata of uniform density being similar concentric ellipsoids, and the density along the major axis varying as the distance from the centre, is $M\frac{2}{9}(b^2+c^2)$.

Ex. 2. The moment of inertia of a heterogeneous ellipse about the minor axis, the strata of uniform density being confocal ellipses and the density along the minor axis varying as the distance from the centre, is $\frac{3M}{20}\frac{4a^5+c^5-5a^2c^3}{2a^3+c^3-3ac^2}$.

Other methods of finding moments of inertia.

12. The moments of inertia given in the table in Art. 8 are only a few of those in continual use. The moments of inertia of an ellipse, for example, about its principal axes are there given, but we shall also frequently want its moments of inertia about other axes. It is of course possible to find these in each separate case by integration. But this is a tedious process, and it may be often avoided by the use of the two following propositions.

The moments of inertia of a body about certain axes through its centre of gravity, which we may take as axes of reference, are regarded as given in the table. In order to find the moment of inertia of *that body* about any other axis we shall investigate,

(1) A method of comparing the required moment of inertia with that about a parallel axis through the centre of gravity.

(2) A method of determining the moment of inertia about this parallel axis in terms of the given moments of inertia about the axes of reference.

13. Prop. I. *Given the moments and products of inertia about all axes through the centre of gravity of a body, to deduce the moments and products about all other parallel axes.*

The moment of inertia of a body or system of bodies about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity plus the moment of inertia of the whole mass collected at the centre of gravity about the original axis.

The product of inertia about any two axes is equal to the product of inertia about two parallel axes through the centre of gravity plus the product of inertia of the whole mass collected at the centre of gravity about the original axes.

Firstly, take the axis about which the moment of inertia is required as the axis of z. Let m be the mass of any particle of the body, which generally will be any small element. Let x, y, z be the co-ordinates of m, $\bar{x}$, $\bar{y}$, $\bar{z}$ those of the centre of gravity G of the whole system of bodies, x', y', z' those of m referred to a system of parallel axes through the centre of gravity.

Then since $\dfrac{\Sigma mx'}{\Sigma m}$, $\dfrac{\Sigma my'}{\Sigma m}$, $\dfrac{\Sigma mz'}{\Sigma m}$ are the co-ordinates of the centre of gravity of the system referred to the centre of gravity as the origin, it follows that $\Sigma mx' = 0$, $\Sigma my' = 0$, $\Sigma mz' = 0$.

The moment of inertia of the system about the axis of z is

$$
\begin{aligned}
&= \Sigma m\,(x^2 + y^2),\\
&= \Sigma m\,\{(\bar{x} + x')^2 + (\bar{y} + y')^2\},\\
&= \Sigma m\,(\bar{x}^2 + \bar{y}^2) + \Sigma m\,(x'^2 + y'^2), \; + 2\bar{x}\,.\,\Sigma mx' + 2\bar{y}\,.\,\Sigma my'.
\end{aligned}
$$

Now $\Sigma m\;(\bar{x}^2 + \bar{y}^2)$ is the moment of inertia of a mass Σm collected at the centre of gravity, and $\Sigma m\;(x'^2 + y'^2)$ is the moment of inertia of the system about an axis through G, also $\Sigma mx' = 0$, $\Sigma my' = 0$; whence the proposition is proved.

Secondly, take the axes of x, y as the axes about which the product of inertia is required. The product required is

$$
\begin{aligned}
&= \Sigma m\,xy \quad = \Sigma m\,(\bar{x} + x')\,(\bar{y} + y'),\\
&= \bar{x}\bar{y}\,.\,\Sigma m + \Sigma m\,x'y' + \bar{x}\Sigma my' + \bar{y}\Sigma mx'\\
&= \bar{x}\bar{y}\Sigma m + \Sigma mx'y'.
\end{aligned}
$$

Now $\bar{x}\bar{y}\,.\,\Sigma m$ is the product of inertia of a mass Σm collected at G and $\Sigma mx'y'$ is the product of the whole system about axes through G; whence the proposition is proved.

Let there be two parallel axes A and B at distances a and b from the centre of gravity of the body. Then, if M be the mass of the material system,

$$\left.\begin{matrix}\text{moment of inertia}\\ \text{about } A\end{matrix}\right\} - Ma^2 = \left\{\begin{matrix}\text{moment of inertia}\\ \text{about } B\end{matrix}\right. - Mb^2.$$

Hence when the moment of inertia of a body about one axis is known, that about any other parallel axis may be found. It is obvious that a similar proposition holds with regard to the products of inertia.

14. The preceding proposition may be generalised as follows. Let any system be in motion, and let x, y, z be the co-ordinates at time t of any particle of mass m, then $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are the velocities, and $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ the accelerations of the particle resolved parallel to the axes. Suppose

$$V = \Sigma m\phi\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, z, \frac{dz}{dt}, \frac{d^2z}{dt^2}\right)$$

to be a given function depending on the structure and motion of the system, the summation extending throughout the system. Also let ϕ be an algebraic function of the first or second order. Thus ϕ may consist of such terms as

$$Ax^2 + Bx\frac{dy}{dt} + C\left(\frac{dz}{dt}\right)^2 + Eyz + Fx + \dots\dots\dots$$

where A, B, C, &c. are some constants. Then the following general principle will hold.

"The value of V for any system of co-ordinates is equal to the value of V obtained for a parallel system of co-ordinates with the centre of gravity for origin plus the value of V for the whole mass collected at the centre of gravity with reference to the first system of co-ordinates."

For let $\bar{x}, \bar{y}, \bar{z}$, be the co-ordinates of the centre of gravity, and let $x = \bar{x} + x'$, &c. $\therefore \frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt}$, &c.

Now since ϕ is an algebraic function of the second order of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$; y, &c. it is evident that on making the above substitution and expanding, the process of squaring &c. will lead to three sets of terms, those containing only $\bar{x}, \frac{d\bar{x}}{dt}, \frac{d^2\bar{x}}{dt^2}$, &c., those containing the products of $\bar{x}, x'$ &c., and lastly those containing

only x', $\frac{dx'}{dt}$, &c. The first of these will on the whole make up $\phi\left(\bar{x}, \frac{d\bar{x}}{dt}, \text{ \&c.}\right)$, and the last $\phi\left(x', \frac{dx'}{dt}, \text{ \&c.}\right)$.
Hence we have

$$V = \Sigma m\phi\left(\bar{x}, \frac{d\bar{x}}{dt}\dots\right) + \Sigma m\phi\left(x', \frac{dx'}{dt} + \dots\right)$$
$$+ \Sigma m\left(A\bar{x}\frac{dx'}{dt} + B\frac{d\bar{x}}{dt}x' + C\bar{x}\frac{dy'}{dt} + \dots\right),$$

where A, B, C, &c. are some constants.

Now the term $\Sigma m\left(\bar{x}\frac{dx'}{dt}\right)$ is the same as $\bar{x}\Sigma m\frac{dx'}{dt}$, and this vanishes. For since $\Sigma mx' = 0$, it follows that $\Sigma m\frac{dx'}{dt} = 0$. Similarly all the other terms in the second line vanish.

Hence the value of V is reduced to two terms. But the first of these is the value of V at the origin for the whole mass collected at the centre of gravity, and the second of these the value of V for the whole system referred to the centre of gravity as origin. Hence the proposition is proved.

The proposition would obviously be true if $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, or any higher differential coefficients were also present in the function V.

15. Prop. II. *Given the moments and products of inertia about three straight lines at right angles meeting in a point, to deduce the moments and products of inertia about all other axes meeting in that point.*

Take these three straight lines as the axes of co-ordinates. Let A, B, C be the moments of inertia about the axes of x, y, z; D, E, F the products of inertia about the axes of yz, zx, xy. Let α, β, γ be the direction-cosines of any straight line through the origin, then the moment of inertia I of the body about that line will be given by the equation

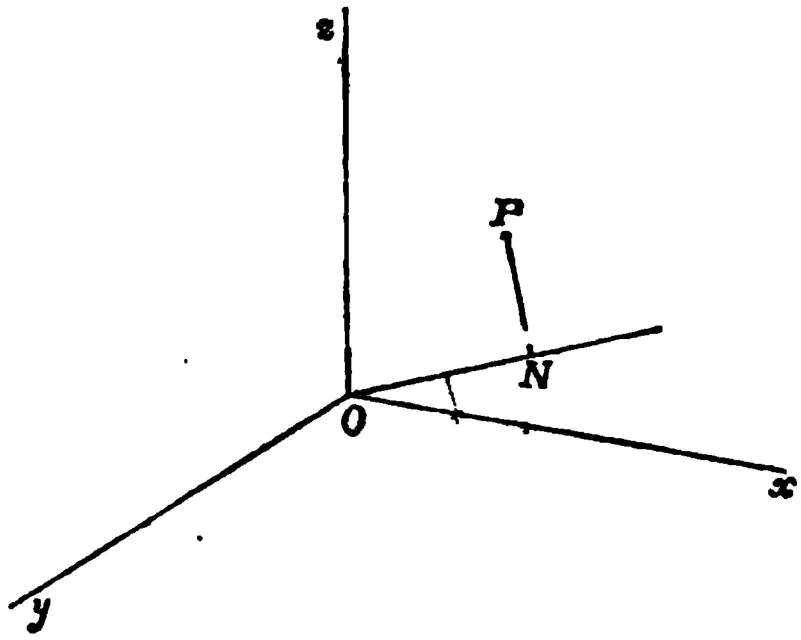
$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

Let P be any point of the body at which a mass m is situated, and let x, y, z be the co-ordinates of P. Let ON be the line whose direction-cosines are α, β, γ, draw PN perpendicular to ON.

Since ON is the projection of OP, it is clearly

$$= x\alpha + y\beta + z\gamma,$$

also $OP^2 = x^2 + y^2 + z^2$, and $1 = \alpha^2 + \beta^2 + \gamma^2$.



The moment of inertia I about $ON = \Sigma m\, PN^2$

$$= \Sigma m \{x^2 + y^2 + z^2 - (\alpha x + \beta y + \gamma z)^2\}$$

$$= \Sigma m \{(x^2 + y^2 + z^2)(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)^2\}$$

$$= \Sigma m (y^2 + z^2)\alpha^2 + \Sigma m (z^2 + x^2)\beta^2 + \Sigma m (x^2 + y^2)\gamma^2$$
$$- 2\Sigma myz \,.\, \beta\gamma - 2\Sigma mzx \,.\, \gamma\alpha - 2\Sigma mxy \,.\, \alpha\beta$$

$$= A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

It may be shewn in exactly the same manner that if $A'B'C'$ be the moments of inertia with regard to the *planes* yz, zx, xy, then the moment of inertia with regard to the *plane* whose direction-cosines are α, β, γ is

$$I' = A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta.$$

It should be remarked that this formula differs from the moment about a straight line in the signs of the three last terms.

16. When three straight lines at right angles and meeting in a given point are such that if they be taken as axes of co-ordinates the products Σmxy, Σmyz, Σmzx all vanish, these are said to be Principal Axes at the given point.

The three planes through any two principal axes are called the Principal Planes at the given point.

The moments of inertia about the principal axes at any point are called the Principal Moments of Inertia at that point.

17. The fundamental formula in Art. 15 may be much simplified if the axes of co-ordinates can be chosen so as to be principal axes at the origin. In this case the expression takes the simple form

$$I = A\alpha^2 + B\beta^2 + C\gamma^2.$$

A method will presently be given by which we can always find these axes, but in some simpler cases we may determine

their position by inspection. Let the body be symmetrical about the plane of xy. Then for every element m on one side of the plane whose co-ordinates are (x, y, z) there is another element of equal mass on the other side whose co-ordinates are $(x, y, -z)$. Hence for such a body $\Sigma mxz = 0$ and $\Sigma myz = 0$. If the body be a lamina in the plane of xy, then the z of every element is zero, and we have again $\Sigma mxz = 0$, $\Sigma myz = 0$.

Recurring to the table in Art. 8, we see that in every case the axes, about which the moments of inertia are given, are principal axes. Thus in the case of the ellipsoid, the three principal sections are all planes of symmetry, and therefore, by what has just been said, the principal diameters are principal axes of inertia. In applying the fundamental formula of Art. 15 to any body mentioned in the table, we may therefore always use the modified form given in this article.

18. Let us now consider how the two important propositions of Arts. 13 and 15 are to be applied in practice.

Ex. 1. Suppose we want the moment of inertia of an elliptic area of mass M and semiaxes a and b about a diameter making an angle θ with the major axis. The moments of inertia about the axes of a and b respectively are $M\frac{b^2}{4}$ and $M\frac{a^2}{4}$. Then by Art. 17 the moment of inertia about the diameter is $M\frac{b^2}{4}\cos^2\theta + M\frac{a^2}{4}\sin^2\theta$. If r be the length of the diameter this is known from the equation to the ellipse to be the same as $\frac{M}{4}\frac{a^2b^2}{r^2}$, which is a very convenient form in practice.

Ex. 2. Suppose we want the moment of inertia of the same ellipse about a tangent. Let p be the perpendicular from the centre on the tangent, then by Art. 13, the required moment is equal to the moment of inertia about a parallel axis through the centre together with $Mp^2 = \frac{M}{4}\frac{a^2b^2}{r^2} + Mp^2 = \frac{5M}{4}p^2$, since $pr = ab$.

Ex. 3. As an example of a different kind, let us find the moment of inertia of an ellipsoid of mass M and semiaxes (a, b, c) with regard to a diametral *plane* whose direction-cosines referred to the principal planes are (α, β, γ). By Art. 8, the moments of inertia with regard to the principal axes are $M\frac{b^2+c^2}{5}$, $M\frac{c^2+a^2}{5}$, $M\frac{a^2+b^2}{5}$. Hence by Art. 5, the moments of inertia with regard to the principal *planes* are $M\frac{a^2}{5}$, $M\frac{b^2}{5}$, $M\frac{c^2}{5}$. Hence the required moment of inertia is $\frac{M}{5}(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)$. If p be the perpendicular on the parallel tangent plane, we know by solid geometry that this is the same as $M\frac{p^2}{5}$.

Ex. 4. The moment of inertia of a rectangle whose sides are $2a$, $2b$ about a diagonal is

$$\frac{2M}{3}\frac{a^2b^2}{a^2+b^2}.$$

Ex. 5. If k_1, k_2 be the radii of gyration of an elliptic lamina about two conjugate diameters, then $\frac{1}{k_1^2}+\frac{1}{k_2^2}=4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$.

Ex. 6. The sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same.

Ex. 7. If M be the mass of a right cone, a its altitude and b the radius of the base, then the moment of inertia about the axis is $M\frac{3}{10}b^2$; that about a straight line through the vertex perpendicular to the axis is $M\frac{3}{5}\left(a^2+\frac{b^2}{4}\right)$, that about a slant side $M\frac{3b^2}{20}\frac{6a^2+b^2}{a^2+b^2}$; that about a perpendicular to the axis through the centre of gravity is $M\frac{3}{80}(a^2+4b^2)$.

Ex. 8. If a be the altitude of a right cylinder, b the radius of the base, then the moment of inertia about the axis is $M\frac{b^2}{2}$ and that about a straight line through the centre of gravity perpendicular to the axis is $\frac{M}{4}\left(\frac{a^2}{3}+b^2\right)$.

Ex. 9. The moment of inertia of a body of mass M about a straight line whose equation is $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ referred to any rectangular axes meeting at the centre of gravity is

$$Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm+M\{f^2+g^2+h^2-(fl+gm+hn)^2\},$$

where (l, m, n) are the direction-cosines of the straight line.

Ex. 10. The moment of inertia of an elliptic disc whose equation is

$$ax^2+2bxy+cy^2+2dx+2ey+1=0,$$

about a diameter parallel to the axis of x, is $\frac{M}{4}\cdot\frac{-Ha}{(ac-b^2)^2}$, where M is the mass and H is the determinant $ac-b^2+2bed-ae^2-cd^2$, usually called the discriminant.

Ex. 11. The moment of inertia of the elliptic disc whose equation in areal coordinates is $\phi\,(xyz)=0$ about a diameter parallel to the side a is

$$-M\left(\frac{\Delta}{a}\right)^2\frac{H}{2K^2}\left(\frac{d}{dy}-\frac{d}{dz}\right)^2\phi,$$

where Δ is the area, H the discriminant and K the bordered discriminant.

The Ellipsoids of Inertia.

19. The expression which has been found in Art. 15 for the moment of inertia I about a straight line whose direction-cosines are (α, β, γ),

$$I=A\alpha^2+B\beta^2+C\gamma^2-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta,$$

admits of a very useful geometrical interpretation.

Let a radius vector OQ move in any manner about the given point O, and be of such length that the moment of inertia about OQ may be *proportional* to the inverse square of the length. Then if R represent the length of the radius vector whose direction-cosines are (α, β, γ), we have $I = \frac{M\epsilon^4}{R^2}$, where ϵ is some constant introduced to keep the dimensions correct, and M is the mass. Hence the polar equation to the locus of Q is

$$\frac{M\epsilon^4}{R^2} = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta.$$

Transforming to Cartesian co-ordinates, we have

$$M\epsilon^4 = AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY,$$

which is the equation to a quadric. Thus to every point O of a material body there is a corresponding quadric which possesses the property that the moment of inertia about any radius vector is represented by the inverse square of that radius vector. The convenience of this construction is, that the relations which exist between the moments of inertia about straight lines meeting at any given point may be discovered by help of the known properties of a quadric.

Since a moment of inertia is essentially positive, being by definition the sum of a number of squares, it is clear that every radius vector R must be real. Hence the quadric is always an ellipsoid. It is called the *momental ellipsoid*, and was first used by Cauchy, *Exercices de Math.* Vol. II.

20. The momental ellipsoid is defined by a *geometrical* property, viz. that any radius vector is equal to some constant divided by the square root of the moment of inertia about that radius vector. Hence whatever co-ordinate axes are taken, we must always arrive at the same ellipsoid. If therefore the momental ellipsoid be referred to any set of rectangular axes, the coefficients of X^2, Y^2, Z^2, $-2YZ$, $-2ZX$, $-2XY$ in its equation will still represent the moments and products of inertia about the axes of co-ordinates.

Since the discriminating cubic determines the lengths of the axes of the ellipsoid, it also follows that its coefficients are unaltered by a transformation of axes. But these coefficients are

$$A + B + C,$$

$$AB + BC + CA - D^2 - E^2 - F^2,$$

$$ABC - 2DEF - AD^2 - BE^2 - CF^2.$$

Hence for all rectangular axes having the same origin, these are invariable and all greater than zero.

21. It should be noticed that the constant ϵ is arbitrary, though when once chosen it cannot be altered. Thus we have a series of similar and similarly situated ellipsoids, any one of which may be used as *a* momental ellipsoid.

When the body is a plane lamina, a section of the ellipsoid corresponding to any point in the lamina by the plane of the lamina, is called a *momental ellipse* of that point.

22. If principal axes at any point O of a body be taken as axes of co-ordinates, the equation to the momental ellipsoid takes the simple form $AX^2 + BY^2 + CZ^2 = M\epsilon^4$, where M is the mass and ϵ^4 any constant. Let us now apply this to some simple cases.

Ex. 1. To find the momental ellipsoid at the centre of a material elliptic disc. Taking the same notation as before, we have $A = M\frac{b^2}{4}$, $B = M\frac{a^2}{4}$, $C = M\frac{a^2+b^2}{4}$. Hence the ellipsoid is

$$M\frac{b^2}{4}X^2 + M\frac{a^2}{4}Y^2 + M\frac{a^2+b^2}{4}Z^2 = M\epsilon^4.$$

Since ϵ is any constant, this may be written

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \left(\frac{1}{a^2} + \frac{1}{b^2}\right)Z^2 = \epsilon'.$$

When $Z = 0$, this becomes an ellipse similar to the boundary of given disc. Hence we infer that the momental ellipse at the centre of an elliptic area is any similar and similarly situated ellipse. This also follows from Art. 18, Ex. 1.

Ex. 2. To find the momental ellipsoid at any point O of a material straight rod AB of mass M and length $2a$. Let the straight line OAB be the axis of x, O the origin, G the middle point of AB, $OG = c$. If the material line can be regarded as indefinitely thin, $A = 0$, $B = M\left(\frac{a^2}{3} + c^2\right) = C$, hence the momental ellipsoid is $Y^2 + Z^2 = \epsilon'^2$, where ϵ' is any constant. The momental ellipsoid is therefore an elongated spheroid, which becomes a right cylinder having the straight line for axis, when the rod becomes indefinitely thin.

Ex. 3. The momental ellipsoid at the centre of a material ellipsoid is

$$(b^2 + c^2)X^2 + (c^2 + a^2)Y^2 + (a^2 + b^2)Z^2 = \epsilon^4,$$

where ϵ is any constant. It should be noticed that the longest and shortest axes of the momental ellipsoid coincide in direction with the longest and shortest axes respectively of the material ellipsoid.

23. By a consideration of some simple properties of ellipsoids, the following propositions are evident:

I. Of the moments of inertia of a body about axes meeting at a given point, the moment of inertia about one of the principal axes is greatest and about another least.

For, in the momental ellipsoid, the moment of inertia about any radius vector from the centre is least when that radius vector

is greatest and *vice versâ*. And it is evident that the greatest and least radii vectores are two of the principal diameters.

It follows by Art. 5 that of the moments of inertia with regard to all *planes* passing through a given point, that with regard to one principal plane is greatest and with regard to another is least.

II. If the three principal moments at any point O be equal to each other, the ellipsoid becomes a sphere. Every diameter is then a principal diameter, and the radii vectores are all equal. Hence every straight line through O is a principal axis at O, and the moments of inertia about them are all equal.

For example, the perpendiculars from the centre of gravity of a cube on the three faces are principal axes; for, the body being referred to them as axes, we clearly have $\Sigma mxy = 0$, $\Sigma myz = 0$, $\Sigma mzx = 0$. Also the three moments of inertia about them are by symmetry equal. Hence every axis through the centre of gravity of a cube is a principal axis, and the moments of inertia about them are all equal.

Next suppose the body to be a regular solid. Consider two planes drawn through the centre of gravity each parallel to a face of the solid. The relations of these two planes to the solid are in all respects the same. Hence also the momental ellipsoid at the centre of gravity must be similarly situated with regard to each of these planes, and the same is true for planes parallel to all the faces. Hence the ellipsoid must be a sphere and the moment of inertia will be the same about every axis.

24. *At every point of a material system there are always three principal axes at right angles to each other.*

Construct the momental ellipsoid at the given point. Then it has been shown that the products of inertia about the axes are half the coefficients of $-XY$, $-YZ$, $-ZX$ in the equation to the momental ellipsoid referred to these straight lines as axes of co-ordinates. Now if an ellipsoid be referred to its principal diameters as axes, these coefficients vanish. Hence the principal diameters of the ellipsoid are the principal axes of the system. But every ellipsoid has at least three principal diameters, hence every material system has at least three principal axes.

25. Ex. 1. The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point (p, q, r) is

$$\left(\frac{A}{M}+q^2+r^2\right)X^2+\left(\frac{B}{M}+r^2+p^2\right)Y^2+\left(\frac{C}{M}+p^2+q^2\right)Z^2$$
$$-2qr\,YZ-2rp\,ZX-2pq\,XY=\epsilon^4,$$

when referred to its centre as origin.

Ex. 2. Show that the cubic equation to find the three principal moments of inertia at any point (p, q, r) may be written in the form of a determinant

$$\begin{vmatrix} \frac{I-A}{M}-q^2-r^2 & pq & rp \\ pq & \frac{I-B}{M}-r^2-p^2 & qr \\ rp & qr & \frac{I-C}{M}-p^2-q^2 \end{vmatrix}=0.$$

If (l, m, n) be proportional to the direction-cosines of the axis corresponding to any one of the values of I, their values may be found from the equations

$$\left.\begin{aligned} \{I-(A+Mq^2+Mr^2)\}\,l+Mpqm+Mrpn&=0,\\ Mpql+\{I-(B+Mr^2+Mp^2)\}\,m+Mqrn&=0,\\ Mrpl+Mqrm+\{I-(C+Mp^2+Mq^2)\}\,n&=0. \end{aligned}\right\}$$

Ex. 3. If $S=0$ be the equation to the momental ellipsoid at the centre of gravity O referred to any rectangular axes written in the form given in Art. 19, then the momental ellipsoid at the point P whose co-ordinates are (p, q, r) is

$$S+M(p^2+q^2+r^2)(X^2+Y^2+Z^2)-M(pX+qY+rZ)^2=0.$$

Hence show (1) that the conjugate planes of the straight line OP in the momental ellipsoids at O and P are parallel and (2) that the sections perpendicular to OP have their axes parallel.

26. The reciprocal surface of the momental ellipsoid is another ellipsoid, which has also been employed to represent, geometrically, the positions of the principal axes and the moment of inertia about any line.

We shall require the following elementary proposition. The reciprocal surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is the ellipsoid $a^2x^2+b^2y^2+c^2z^2=\epsilon^4$.

Let ON be the perpendicular from the origin O on the tangent plane at any point P of the first ellipsoid, and let l, m, n be the direction-cosines of ON, then $ON^2=a^2l^2+b^2m^2+c^2n^2$. Produce ON to Q so that $OQ=\frac{\epsilon^2}{ON}$, then Q is a point on the reciprocal surface. Let $OQ=R$; $\therefore$ $\frac{\epsilon^4}{R^2}=a^2l^2+b^2m^2+c^2n^2$. Changing this to rectangular co-ordinates, we get $\epsilon^4=a^2x^2+b^2y^2+c^2z^2$.

To each point of a material body there corresponds a series of similar momental ellipsoids. If we reciprocate these we get another series of similar ellipsoids coaxial with the first, and such that the moment of inertia of the body about the perpendiculars on the tangent planes to any one ellipsoid are proportional to the squares of those perpendiculars. It is, however, convenient to call that particular ellipsoid the *ellipsoid of gyration* which makes the moment of inertia about a perpendicular on a tangent plane *equal* to the product of the mass into the square

of that perpendicular. If M be the mass of the body and A, B, C the principal moments, the equation to the ellipsoid of gyration is

$$\frac{X^2}{A}+\frac{Y^2}{B}+\frac{Z^2}{C}=\frac{1}{M}.$$

It is clear that the constant on the right-hand side must be $\frac{1}{M}$, for when Y and Z are put equal to zero, X^2 must by definition be $\frac{A}{M}$.

27. Conversely, the series of momental ellipsoids at any point of a body may be regarded as the reciprocals, with different constants, of the ellipsoid of gyration at that point. They are all of an opposite shape to the ellipsoid of gyration, having their longest axes in the direction of the shortest axis and their shortest axes in the direction of the longest axis of the ellipsoid of gyration. The momental ellipsoids however resemble the general shape of the body more nearly than the ellipsoid of gyration. They are protuberant where the body is protuberant and compressed where the body is compressed. The exact reverse of this is the case in the ellipsoid of gyration. See Art. 22, Ex. 3.

28. Ex. 1. To find the ellipsoid of gyration at the centre of a material elliptic disc. Taking the values of A, B, C given in Art. 22, Ex. 1, we see that the ellipsoid of gyration is $\frac{X^2}{b^2}+\frac{Y^2}{a^2}+\frac{Z^2}{a^2+b^2}=\frac{1}{4}$.

Ex. 2. The ellipsoid of gyration at any point O of a material rod AB is $\frac{X^2}{0}+\frac{Y^2}{\frac{1}{3}a^2+c^2}+\frac{Z^2}{\frac{1}{3}a^2+c^2}=1$, taking the same notation as in Art. 22, Ex. 2. This is a very flat ellipsoid which when the rod is indefinitely thin becomes a circular area whose centre is at O, whose radius is $\sqrt{\frac{1}{3}a^2+c^2}$ and whose plane is perpendicular to the rod.

Ex. 3. It may be shown that the general equation to the ellipsoid of gyration referred to any set of rectangular axes meeting at the given point of the body is

$$\begin{vmatrix} A & -F & -E & MX \\ -F & B & -D & MY \\ -E & -D & C & MZ \\ MX & MY & MZ & M \end{vmatrix}=0,$$

or when expanded

$$\begin{aligned}(BC-D^2)X^2+(CA-E^2)Y^2+(AB-F^2)Z^2+2(AD+EF)YZ \\ +2(BE+FD)ZX+2(CF+DE)XY \\ =\frac{1}{M}(ABC-AD^2-BE^2-CF^2-2DEF).\end{aligned}$$

The right-hand side, when multiplied by M, is the discriminant obtained by leaving out the last row and the last column, and the coefficients of X^2, Y^2, Z^2, $2ZX$, $2XY$, $2YZ$ are the minors of this discriminant.

29. The use of the ellipsoid whose equation referred to the principal axes at the centre of gravity is

$$\frac{X^2}{\Sigma mx^2}+\frac{Y^2}{\Sigma my^2}+\frac{Z^2}{\Sigma mz^2}=\frac{5}{M},$$

has been suggested by Legendre in his *Fonctions Elliptiques.* This ellipsoid is to be regarded as a homogeneous solid of such density that its mass is equal to that of the body. By Art. 8, Ex. 3, it possesses the property that its moments of inertia with regard to its principal axes, and therefore by Art. 15 its moments of inertia with regard to all planes and axes, are the same as those of the body. We may call this ellipsoid the *equimomental* ellipsoid or *Legendre's* ellipsoid.

Ex. If a plane move so that the moment of inertia with regard to it is always proportional to the square of the perpendicular from the centre of gravity on the plane, then this plane envelopes an ellipsoid similar to Legendre's ellipsoid.

30. There is another ellipsoid which is sometimes used. By Art. 15 the moment of inertia with reference to a plane whose direction-cosines are (α, β, γ) is

$$I'=\Sigma mx^2.\alpha^2+\Sigma my^2.\beta^2+\Sigma mz^2.\gamma^2+2\Sigma myz.\beta\gamma+2\Sigma mzx.\gamma\alpha+2\Sigma mxy.\alpha\beta.$$

Hence, as in Art. 19, we may construct the ellipsoid

$$\Sigma mx^2.X^2+\Sigma my^2.Y^2+\Sigma mz^2.Z^2+2\Sigma myz.YZ+2\Sigma mzx.ZX+2\Sigma mxy.XY=M\epsilon^4.$$

Then the moment of inertia with regard to any plane through the centre of the ellipsoid is represented by the inverse square of the radius vector perpendicular to that plane.

If we compare the equation of the momental ellipsoid with that of this ellipsoid, we see that one may be obtained from the other by subtracting the same quantity from each of the coefficients of X^2, Y^2, Z^2. Hence the two ellipsoids have their circular sections coincident in direction.

This ellipsoid may also be used to find the moments of inertia about any straight line through the origin. For we may deduce from Art. 5 that the moment of inertia about any radius vector is represented by the difference between the inverse square of that radius vector and the sum of the inverse squares of the semi-axes. This ellipsoid is a reciprocal of Legendre's ellipsoid. All these ellipsoids have their principal diameters coincident in direction, and any one of them may be used to determine the directions of the principal axes at any point.

31. When the body considered is a lamina, the section of the ellipsoid of gyration at any point of the lamina by the plane of the lamina is called the *ellipse* of gyration. If the plane of the lamina be the plane of xy, we have $\Sigma mz^2=0$. The section of the fourth ellipsoid is then clearly the same as a momental ellipse at the point. If any momental ellipse be turned round its centre through a right angle it evidently becomes similar and similarly situated to the ellipse of gyration. So that, in the case of a lamina, any one of these ellipses may be easily changed into the others.

32. *A straight line passes through a fixed point* O *and moves about it in such a manner that the moment of inertia about the line is always the same and equal to a given quantity* I. *To find the equation to the cone generated by the straight line.*

Let the principal axes at O be taken as the axes of co-ordinates, and let (α, β, γ) be the direction-cosines of the straight line in any position. Then by Art. 17 we have $A\alpha^2 + B\beta^2 + C\gamma^2 = I$.

Hence the equation to the locus is

$$(A - I)\,\alpha^2 + (B - I)\,\beta^2 + (C - I)\,\gamma^2 = 0,$$

or, transforming to Cartesian co-ordinates,

$$(A - I)\,x^2 + (B - I)\,y^2 + (C - I)\,z^2 = 0.$$

It appears from this equation that the principal diameters of the cone are the principal axes of the body at the given point.

The given quantity I must be less than the greatest and greater than the least of the moments A, B, C. Let A, B, C be arranged in descending order of magnitude; then if I be less than B, the cone has its concavity turned towards the axis C, if I be greater than B the concavity is turned towards the axis A, if $I = B$ the cone becomes two planes which are coincident with the central circular sections of the momental ellipsoid at the point O.

The geometrical peculiarity of this cone is that its circular sections in all cases are coincident in direction with the circular sections of the momental ellipsoid at the vertex.

This cone is called an *equimomental cone* at the point at which its vertex is situated.

33. The properties of products of inertia of a body about different sets of axes are not so useful as to require a complete discussion. The following theorems will serve as exercises.

Ex. 1. If any point O be given and any plane drawn through it, then two straight lines at right angles Ox, Oy can always be found such that the product of inertia about these lines is zero.

These are the axes of the section of the momental ellipsoid at the point O formed by the given plane.

Ex. 2. If two other straight lines at right angles Ox', Oy' be taken in the same plane making an angle θ measured in the positive direction with Ox, Oy respectively, then the product of inertia F' about Ox', Oy' is given by the equation

$$F' = \frac{1}{2}\sin 2\theta\,(A - B),$$

where A, B are the moments of inertia about Ox, Oy.

Ex. 3. If I be the moment of inertia about any line in this plane making an angle θ with Ox, then

$$I = A\cos^2\theta + B\sin^2\theta.$$

For the section of the momental ellipsoid by the plane is the ellipse whose equation is $Ax^2 + By^2 = M\epsilon^4$, whence the property follows at once.

Ex. 4. Let $(\lambda\mu\nu)$ $(\lambda'\mu'\nu')$ be the direction-cosines of two straight lines Ox', Oy' at right angles passing through the origin O and referred to the principal axes at O as axes of co-ordinates. Then the product of inertia about these lines is

$$F' = \lambda\lambda'\Sigma mx^2 + \mu\mu'\Sigma my^2 + \nu\nu'\Sigma mz^2.$$

For let $(x'y'z')$ be the co-ordinates of any point (xyz) referred to Ox', Oy' and a third line Oz' as new axes of co-ordinates. Then

$$x' = \lambda x + \mu y + \nu z, \text{ and } y' = \lambda' x + \mu' y + \nu' z.$$

Hence, since $F' = \Sigma mx'y'$, the theorem follows by simple multiplication.

Since $\lambda\lambda' + \mu\mu' + \nu\nu' = 0$, we have

$$-F' = A\lambda\lambda' + B\mu\mu' + C\nu\nu'.$$

Ex. 5. If $(\lambda\mu\nu)$ be the direction-cosines of an axis Ox', then the direction-cosines $(\lambda'\mu'\nu')$ of another axis Oy' at right angles such that the product of inertia about Ox', Oy' is zero, are given by the equations

$$\frac{\lambda'}{(B-C)\mu\nu} = \frac{\mu'}{(C-A)\nu\lambda} = \frac{\nu'}{(A-B)\lambda\mu}.$$

For by (4) the equations to find $\lambda'\mu'\nu'$ are

$$\left.\begin{aligned} A\lambda\lambda' + B\mu\mu' + C\nu\nu' &= 0, \\ \lambda\lambda' + \mu\mu' + \nu\nu' &= 0, \end{aligned}\right\}$$

whence the theorem follows by cross multiplication.

By (1) this is equivalent to the geometrical theorem. Given a radius vector Ox' of an ellipsoid, find another radius vector Oy' such that Ox', Oy' are principal diameters of the section $x'Oy'$.

Ex. 6. Let (lmn) be the direction-cosines of any given straight line Oz', and let D', E' be the products of inertia about Oz', Oy'; Oz', Ox', where Ox', Oy' are any two straight lines at right angles. Then as Ox', Oy' turn round Oz', $D'^2 + E'^2$ is constant, and

$$D'^2 + E'^2 = (A-B)^2 (lm)^2 + (B-C)^2 (mn)^2 + (C-A)^2 (nl)^2.$$

For by (4), $\quad -D' = Al\lambda + Bm\mu + Cn\nu, \quad -E' = Al\lambda' + Bm\mu' + Cn\nu'$;

$$\therefore D'^2 + E'^2 = A^2 l^2 (\lambda^2 + \lambda'^2) + 2ABlm (\lambda\mu + \lambda'\mu') + \&c.$$

But

$$\left.\begin{aligned} \lambda^2 + \lambda'^2 &= 1 - l^2 = m^2 + n^2, \\ \lambda\mu + \lambda'\mu' &= -lm, \end{aligned}\right\}$$

whence by substitution the theorem follows at once.

Ex. 7. If A', B' be the moments of inertia about Ox', Oy', then as Ox', Oy' turn round Oz', the value of $A'B' - F'^2$ is constant, and

$$A'B' - F'^2 = BCl^2 + CAm^2 + ABn^2.$$

On Equimomental Bodies.

34. Two bodies or systems of bodies are said to be equimomental when their moments of inertia about all straight lines are equal each to each.

35. If two systems have the same centre of gravity, the same mass, the same principal axes and principal moments at the centre of gravity, it follows from the two fundamental propositions of Arts. 13 and 15 that their moments of inertia about all straight lines are equal, each to each.

That the converse theorem is also true may be shown thus. We know by Art. 13 that of all straight lines having a given direction in a body, that straight line has the least moment of inertia which passes through the centre of gravity. It is clear that these least moments of inertia could not be equal in two bodies for *all* directions unless they had a common centre of gravity. Of all straight lines through the centre of gravity those which have the greatest and least moments of inertia are two of the principal axes, hence these and therefore also the third principal axis must be coincident in direction if the two bodies are equimomental. The principal moments of inertia must then be equal, because all moments are equal. Lastly, by Art. 13, the two systems could not have equal moments about two parallel axes, each to each, unless their masses were equal.

It is easy to see that two equimomental systems must have the same momental ellipsoid, and therefore the same principal axes at every point.

36. *To find the moments and products of inertia of a triangle about any axes whatever.*

If β and γ be the distances of the angular points B, C, of a triangle ABC from any straight line AX through the angle A, in the plane of the triangle, it is known that the moment of inertia of the triangle about AX is $\frac{M}{6}(\beta^2+\beta\gamma+\gamma^2)$, where M is the mass of the triangle.

Let three equal particles, the mass of each being $\frac{M}{3}$, be placed at the middle points of the three sides. Then it is easily seen, that the moment of inertia of the three particles about AX is

$$\frac{M}{3}\left\{\left(\frac{\beta+\gamma}{2}\right)^2+\left(\frac{\gamma}{2}\right)^2+\left(\frac{\beta}{2}\right)^2\right\},$$

which is the same as that of the triangle. The three particles treated as one system, and the triangle, have the same centre of gravity. Let this point be called O. Draw any straight line OX' through the common centre of gravity O parallel to AX, then it is evident that the moments of inertia of the two systems about OX' are also equal.

Since this equality exists for all straight lines through O in the plane of the triangle, it will be true for two straight lines OX',

OY' at right angles, and therefore also for a straight line OZ' perpendicular to the plane of the triangle.

One of the principal axes at O of the triangle, and of the system of three particles, is normal to the plane, and therefore the same for the two systems. The principal axes at O in the plane, are those two straight lines about which the moments of inertia are greatest and least, and therefore by what precedes these axes are the same for the two systems. If at any point two systems have the same principal axes and principal moments, they have also the same moments of inertia about all axes through that point, and the same products of inertia about any two straight lines meeting in that point. And if this point be the centre of gravity of both systems, the same thing will also be true for any other point.

If then a particle whose mass is one-third that of the triangle be placed at the middle point of each side, the moment of inertia of the triangle about any straight line, is the same as that of the system of particles, and the product of inertia about any two straight lines meeting one another, is the same as that of the system of particles about the same straight lines.

37. Three points D, E, F can always be found such that the products and moments of inertia of three equal particles placed at D, E, F, may be the same as the products and moments of inertia of any plane area. For let O be the centre of gravity of the area, Ox, Oy the principal axes at O in the plane of the area, and $M\alpha^2$ and $M\beta^2$ be the moments of inertia about these axes.

Let (xy), $(x'y')$, $(x''y'')$ be the co-ordinates of D, E, F, m the mass of a particle, so that $M = 3m$.

Then we must have
$$m(x^2 + x'^2 + x''^2) = M\beta^2,$$
$$m(y^2 + y'^2 + y''^2) = M\alpha^2,$$
$$xy + x'y' + x''y'' = 0.$$

Also, since the two systems must have the same centre of gravity, $x + x' + x'' = 0$, $y + y' + y'' = 0$.

Eliminating $x'y'$, $x''y''$ from these equations, we get

$$\alpha^2x^2 + \beta^2y^2 = \tfrac{2}{3}\frac{M}{m}\cdot\alpha^2\beta^2, = 2\alpha^2\beta^2,$$

which is the equation to a momental ellipse. It easily follows, that D may be taken any where on this ellipse, and E and F are at the opposite extremities of that chord which is bisected in some point N by the produced radius DO, so that $ON = \frac{1}{2}OD$.

38. A momental ellipsoid at the centre of gravity of any triangle may be found as follows.

Let an ellipse be inscribed in the triangle touching two of the sides AB, BC in their middle points F, D. Then, by Carnot's Theorem, it touches the third side CA in its middle point E. Since DF is parallel to CA the tangent at E, the straight line joining E to the middle point N of DF passes through the centre, and therefore the centre of the conic is the centre of gravity of the triangle.

This conic may be shown to be a momental ellipse of the triangle at O. To prove this, let us find the moment of inertia of the triangle about OE. Let $OE = r$, and let the semi-conjugate diameter be r', and ω the angle between r and r'. Now $ON = \frac{1}{2}r$, and hence from the equation to the ellipse $FN^2 = \frac{3}{4}r'^2$,

$$\text{therefore moment of inertia about } OE = \tfrac{2}{3}M \cdot \tfrac{3}{4}r'^2 \sin^2\omega, = \frac{M}{2} \cdot \frac{\Delta'^2}{\pi^2 r^2};$$

where Δ' is the area of the ellipse, so that the moments of inertia of the system about OE, OF, OD are proportional inversely to OE^2, OF^2, OD^2. If we take a momental ellipse of the right dimensions, it will cut the inscribed conic in E, F, and D, and therefore also at the opposite ends of these diameters. But two conics cannot cut each other in six points unless they are identical. Hence this conic is a momental ellipse at O of the triangle.

A normal at O to the plane of the triangle is a principal axis of the triangle (Art. 17). Hence a momental ellipsoid of the triangle has the inscribed conic for one principal section. If a and b be the lengths of the axes of this conic, c that of the axis of the ellipsoid which is perpendicular to the plane of the lamina, we have by Arts. 7 and 19

$$\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

If the triangle be an equilateral triangle, the momental ellipsoid becomes a spheroid, and every axis through the centre of gravity in the plane of the triangle is a principal axis.

Since any similar and similarly situated ellipse is also a momental ellipse, we might take the ellipse circumscribing the triangle, and having its centre at the centre of gravity, as the momental ellipse of the triangle.

39. Ex. 1. A momental ellipse at an angular point of a triangular area touches the opposite side at its middle point and bisects the adjacent sides.

Ex. 2. The principal radii of gyration at the centre of gravity of a triangle are the roots of the equation

$$x^4 - \frac{a^2+b^2+c^2}{36}x^2 + \frac{\Delta^2}{108} = 0,$$

where Δ is the area of the triangle.

Ex. 3. The direction of the principal axes at the centre of gravity O of a triangle may be constructed thus. Draw at the middle point D of any side BC lengths $DH=\frac{6k^2}{p}$, $DH'=\frac{6k'^2}{p}$ along the perpendicular, where p is the perpendicular from A on BC and k^2, k'^2 are the principal radii of gyration found by the last example. Then OH, OH' are the directions of the principal axes at O, whose moments of inertia are respectively Mk^2 and Mk'^2.

Ex. 4. The directions of the principal axes and the principal moments at the centre of gravity may also be constructed thus. Draw at the middle point D of any side BC a perpendicular $DK=\frac{BC}{2\sqrt{3}}$. Describe a circle on OK as diameter and join D to the middle point of OK cutting the circle in R and S, then OR, OS are the directions of the principal axes, and the moments of inertia about them are respectively $M\frac{DS^2}{2}$, and $M\frac{DR^2}{2}$.

Ex. 5. Let four particles each one-sixth of the mass of the area of a parallelogram be placed at the middle points of the sides and a fifth particle one-third of the same mass be placed at the centre of gravity, then these five particles and the area of the parallelogram are equimomental systems.

Ex. 6. Let four particles each one-twelfth of the mass of the area of a quadrilateral be placed at each corner and let a negative mass also one-twelfth be placed at the intersection of the diagonals and a sixth particle three-quarters of the same mass be placed at the centre of gravity, then these six particles and the area of the quadrilateral are equimomental systems.

Ex. 7. Let three particles each one-sixth of the mass of an elliptic area be placed one at one extremity of the major axis and the other two at the extremities of the ordinate which bisects the semi-axis major, and let a fourth particle whose mass is one-half that of the area be placed at the centre of gravity. Then the moments and products of inertia of the system of four particles and of the elliptic area are the same for all axes whatever.

Ex. 8. Any sphere of radius a and mass M is equimomental to a system of four particles each of mass $\frac{3M}{20}\left(\frac{a}{r}\right)^2$ placed so that their distances from the centre make equal angles with each other and are each equal to r and a fifth particle equal to the remainder of the mass of the sphere placed at the centre.

40. *To find the moments and products of inertia of a tetrahedron about any axes whatever.*

Let $ABCD$ be the tetrahedron. Through one angular point D draw any plane and let it be taken as the plane of xy. Let D be the area of the base ABC; α, β, γ the distances of its angular points from the plane of xy, and p the length of the perpendicular from D on the base ABC.

Let PQR be any section parallel to the base ABC and of thickness du, where u is the perpendicular from D on PQR. The moment of inertia of the triangle PQR with respect to the *plane*

of xy is the same as that of three equal particles, each one-third its mass, placed at the middle points of its sides. The volume of the element $PQR = \frac{u^2}{p^2}\rho du$. The ordinates of the middle points of the sides AB, BC, CA are respectively $\frac{\alpha+\beta}{2}$, $\frac{\beta+\gamma}{2}$, $\frac{\gamma+\alpha}{2}$. Hence, by similar triangles, the ordinates of the middle points of PQ, QR, RP are also $\frac{\beta+\gamma}{2}\frac{u}{p}$, $\frac{\gamma+\alpha}{2}\frac{u}{p}$, $\frac{\alpha+\beta}{2}\frac{u}{p}$.

The moment of inertia of the triangle PQR with regard to the plane xy is therefore

$$= \frac{1}{3}\frac{u^2}{p^2}\rho du \left\{\left(\frac{\beta+\gamma}{2}\frac{u}{p}\right)^2 + \left(\frac{\gamma+\alpha}{2}\frac{u}{p}\right)^2 + \left(\frac{\alpha+\beta}{2}\frac{u}{p}\right)^2\right\}.$$

Integrating from $u=0$ to $u=p$, we have the moment of inertia of the tetrahedron with regard to the plane xy

$$= \frac{V}{10}\{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta\},$$

where V is the volume.

If particles each one-twentieth of the mass of the tetrahedron were placed at each of the angular points and the rest of the mass, viz. four-fifths, were collected at the centre of gravity, the moment of inertia of these five particles with regard to the plane of xy would be

$$= V\frac{4}{5}\left(\frac{\alpha+\beta+\gamma}{4}\right)^2 + \frac{V}{20}\alpha^2 + \frac{V}{20}\beta^2 + \frac{V}{20}\gamma^2,$$

which is the same as that of the tetrahedron.

The centre of gravity of these five particles is the centre of gravity of the tetrahedron, and they together make up the mass of the tetrahedron. Hence, by Art. 13, the moments of inertia of the two systems with regard to any plane through the centre of gravity are the same, and by the same article this equality will exist for all planes whatever. It follows by Art. 5, that the moments of inertia about any straight line are also equal. The two systems are therefore equimomental.*

41. If the distance of every point in a given figure in space from some fixed plane be increased in a fixed ratio, the figure thus altered is called the *projection* of the given figure. By pro-

* This result was proposed as a Problem in the Mathematical Tripos in an interval of the publication of the preceding and following results, thus anticipating the author by a few days.

jecting a figure from three planes as base planes at right angles in succession, the figure may be often much simplified. Thus an ellipsoid can always be projected into a sphere, and any tetrahedron into a regular tetrahedron.

It is clear that if the base plane from which the figure is projected be moved parallel to itself into a position distant D from its former position, no change of form is produced in the projected figure. If n be the fixed ratio of projection the projected figure has merely been moved through a space nD perpendicular to the base plane. We may therefore suppose the base plane to pass through any given point which may be convenient.

42. *If two bodies are equimomental, their projections are also equimomental.*

Let the origin be the common centre of gravity, then the two bodies are such that $\Sigma m = \Sigma m'$; $\Sigma mx = 0$, $\Sigma m'x' = 0$, &c., $\Sigma mx^2 = \Sigma m'x'^2$, $\Sigma myz = \Sigma m'y'z'$, &c., unaccented letters referring to one body and accented letters to the other. Let both the bodies be projected from the plane of xy in the fixed ratio $1 : n$. Then any point whose co-ordinates are (x, y, z) is transferred to (x, y, nz) and (x', y', z') to (x', y', nz'). Also the elements of mass m, m' become nm and nm'. It is evident that the above equalities are not affected by these changes, and that therefore the projected bodies are equimomental.

The projection of a momental ellipse of a plane area is a momental ellipse of the projection.

Let the figure be projected from the axis of x as base line, so that any point (x, y) is transferred to (x, y') where $y' = ny$, and any element of area m becomes m' where $m' = nm$. Then

$$\Sigma mx^2 = \frac{1}{n}\Sigma m'x^2, \quad \Sigma mxy = \frac{1}{n^2}\Sigma m'xy', \quad \Sigma my^2 = \frac{1}{n^3}\Sigma m'y'^2.$$

The momental ellipses of the primitive and the projection are

$$\Sigma my^2X^2 - 2\Sigma mxyXY + \Sigma mx^2Y^2 = M\epsilon^4,$$

$$\Sigma m'y'^2X'^2 - 2\Sigma m'xy'X'Y' + \Sigma m'x^2Y'^2 = M'\epsilon'^4.$$

To project the former we put $X' = X$, $Y' = nY$, its equation then becomes identical with the latter by virtue of the above equalities if we put $\epsilon'^4 = \epsilon^4 n^2$.

43. Ex. 1. A momental ellipse of the area of a square at its centre of gravity is easily seen to be the inscribed circle. By projecting these first with one side as base line, and secondly with a diagonal as base, the square becomes successively a rectangle and then a parallelogram. Hence a momental ellipse at the centre of gravity of a parallelogram is the inscribed conic touching at the middle points of the sides.

Ex. 2. By projecting an equilateral triangle into any triangle, we may infer the results of some of the previous articles, but the method will be best explained by its application to a tetrahedron.

Ex. 3. Since any ellipsoid may be obtained by projecting a sphere, we infer by Art. 39, Ex. 8, that any solid ellipsoid of mass M is equimomental to a system of four particles each of mass $\frac{3M}{20}\frac{1}{n^2}$ placed on a similar ellipsoid whose linear dimensions are n times as great as those of the material ellipsoid, so that the eccentric lines of the particles make equal angles with each other and a fifth particle equal to the remainder of the mass of the sphere placed at the centre of gravity.

If this material ellipsoid be the Legendre's ellipsoid of any given body, we see that any body whatever is equimomental to a system of five particles placed as above described on an ellipsoid similar to the Legendre's ellipsoid of the body.

Ex. 4. Show that a solid oblique cone on an elliptic base of mass M is equimomental to a system of three particles each $\frac{1}{10}M$ placed on the circumference of the base so that the differences of their eccentric angles are equal, a fourth particle $\frac{3}{10}M$ placed at the middle point of the straight line joining the vertex to the centre of gravity of the base, and a fifth particle to make up the mass of the cone placed at the centre of gravity of the volume.

44. *To find the equimomental ellipsoid of any tetrahedron.*

The moments of inertia of a regular tetrahedron with regard to all *planes* through the centre of gravity O are equal by Art. 23. If r be the radius of the inscribed sphere, the moment with regard to a plane parallel to one face is easily seen by Art. 40 to be $M\frac{3r^2}{5}$. If then we describe a sphere of radius $\rho=\sqrt{3}\,r$, with its centre at the centre of gravity, and its mass equal to that of the tetrahedron; this sphere and the tetrahedron will be equimomental. Since the centre of gravity of any face projects into the centre of gravity of the projected face, we infer that the ellipsoid to which any tetrahedron is equimomental, is similar and similarly situated to that inscribed in the tetrahedron and touching each face in its centre of gravity, but has its linear dimensions greater in the ratio $1:\sqrt{3}$. It may also be easily seen that the sphere whose radius is $\rho=\sqrt{3}r$, touches each edge of the regular tetrahedron at its middle point. Hence we infer that the equimomental ellipsoid of any tetrahedron touches each edge at its middle point and has its centre at the centre of gravity of the volume.

These results may also be deduced from Art. 25, Ex. 2, without the use of projections.

45. Ex. 1. If E^2 be the sum of the squares of the edges of a tetrahedron, F^2 the sum of the squares of the areas of the faces and V the volume, show that the semi-axes of the ellipsoid inscribed in the tetrahedron, touching each face in the centre of gravity and having its centre at the centre of gravity of the tetrahedron, are the roots of

$$\rho^6 - \frac{E^2}{2^4 . 3}\rho^4 + \frac{F^2}{2^4 . 3^2}\rho^2 - \frac{V^2}{2^6 . 3} = 0,$$

and if the roots be $\pm\rho_1 \pm \rho_2 \pm \rho_3$, then the moments of inertia with regard to the principal *planes* of the tetrahedron are $M\frac{3\rho_1^2}{5}$, $M\frac{3\rho_2^2}{5}$, $M\frac{3\rho_3^2}{5}$.

Ex. 2. If a perpendicular EP be drawn at the centre of gravity E of any face $= \frac{4\rho}{p}$ where p is the perpendicular from the opposite corner of the tetrahedron on that face, then P is a point on the principal plane corresponding to the root ρ of the cubic.

46. *To explain how the theory of inversion can be applied to find moments of inertia.*

Let a radius vector drawn from some fixed origin O to any point P of a figure be produced to P' where the rectangle $OP . OP' = \kappa^2$ where κ is some given quantity. Then as P travels all over the given figure, P' traces out another which is called the inverse of the given figure.

Let (x, y, z) be the coordinates of P, (x', y', z') those of P'; r, r' the radii vectores, dv, dv' corresponding polar elements of volume; ρ, ρ', dm, dm' their respective densities and masses. Let $d\omega$ be the solid angle subtended at O by either dv or dv'. Then

$$dv' = r'^2\, d\omega\, dr' = \left(\frac{\kappa}{r}\right)^6 r^2\, d\omega\, dr = \left(\frac{\kappa}{r}\right)^6 dv,$$

and since $\frac{x'}{r'} = \frac{x}{r}$ we have $x'^2\, dv' = \left(\frac{\kappa}{r}\right)^{10} x^2\, dv$. Now $dm = \rho dv$, $dm' = \rho' dv'$. If then we take $\rho' = \left(\frac{r}{\kappa}\right)^{10}\rho$ we have $\Sigma x'^2\, dm' = \Sigma x^2\, dm$, with similar equalities in the case of all the other moments and products of inertia.

Hence we infer, that if a homogeneous body be inverted with regard to a point O, and the density of the new body vary inversely as the tenth power of the distance from O, then these two bodies have the same moments of inertia about all straight lines through O.

Ex. The density of a solid sphere varies inversely as the tenth power of the distance from an external point O. Prove that its moments of inertia about any straight line through O is the same as if the sphere were homogeneous and equal in density to that of the heterogeneous sphere at a point where the tangent from O meets the sphere. Prove that if the density had varied inversely as the sixth power of the distance from O, the masses of the two spheres would have been equal. What is the condition they should have a common centre of gravity?

47. The theory of equimomental particles is of considerable use in finding the centre of pressure of any area vertically immersed in a homogeneous fluid under the action of gravity. It may be proved from hydrostatical principles that if the axis of

x be in the effective surface, and the axis of y vertically downwards, the co-ordinates of the centre of pressure are

$$X = \frac{\text{Product of inertia about the axes}}{\text{moment of area about } Ox},$$

$$Y = \frac{\text{Moment of inertia about } Ox}{\text{moment of area about } Ox}.$$

We see therefore that two equimomental areas have the same centre of pressure. The preceding proposition may be used with considerable effect.

Ex. Prove that the centre of pressure of any triangle wholly immersed is the centre of gravity of three weights placed at the middle points of the sides and each proportional to the depth of the point at which it is placed.

On the positions of the Principal Axes of a system.

48. PROP. *A straight line being given it is required to find at what point in its length it is a principal axis of the system, and if any such point exist to find the other two principal axes at that point.*

Take the straight line as axis of z, and any point O in it as origin. Let C be the point at which it is a principal axis, and let Cx', Cy' be the other two principal axes.

Let $CO = h$, $\theta =$ angle between Cx' and Ox. Then

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z - h \end{aligned}\right\}.$$

Hence

$$\left.\begin{aligned} \Sigma mx'z' = \cos\theta\Sigma mxz + \sin\theta\Sigma myz \\ - h(\cos\theta\Sigma mx + \sin\theta\Sigma my) \end{aligned}\right\} = 0 \text{..........} \quad (1)$$

$$\left.\begin{aligned} \Sigma my'z' = -\sin\theta\Sigma mxz + \cos\theta\Sigma myz \\ - h(-\sin\theta\Sigma mx + \cos\theta\Sigma my) \end{aligned}\right\} = 0 \text{..........} \quad (2)$$

$$\Sigma mx'y' = \Sigma m(y^2 - x^2)\frac{\sin 2\theta}{2} + \Sigma mxy\cos 2\theta = 0 \text{ ....} \quad (3)$$

The last equation shows that

$$\tan 2\theta = \frac{2\Sigma mxy}{\Sigma m(x^2 - y^2)} \text{..........} (4)$$

$$= \frac{2F}{B - A},$$

according to the previous notation.

The equations (1) and (2) must be satisfied by the same value of h. Eliminating h we get $\Sigma mxz\,\Sigma my = \Sigma myz\,\Sigma mx$ as the condition that the axis of z should be a principal axis at some point in its length. Substituting in (1) we have

$$h = \frac{\Sigma myz}{\Sigma my} = \frac{\Sigma mxz}{\Sigma mx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The equation (5) expresses the condition that the axis of z should be a principal axis at some point in its length; and the value of h gives the position of this point. The positions of the other two principal axes may then be found by equation (4).

If $\Sigma mxz = 0$ and $\Sigma myz = 0$, the equations (1) and (2) are both satisfied by $h = 0$. These are therefore the sufficient and necessary conditions that the axis of z should be a principal axis at the origin.

If the system be a plane lamina and the axis of z be a normal to the plane at any point, we have $z = 0$. Hence the conditions $\Sigma mxz = 0$ and $\Sigma myz = 0$ are satisfied. Therefore one of the principal axes at any point of a lamina is a normal to the plane at that point.

In the case of a surface of revolution bounded by planes perpendicular to the axis, the axis is a principal axis at any point of its length.

Again equation (4) enables us, when one principal axis is given, to find the other two. If $\theta = \alpha$ be the first value of θ, all the others are included in $\theta = \alpha + n\dfrac{\pi}{2}$; hence all these values give only the same axes over again.

49. Since (4) does not contain h, it appears that if the axis of z be a principal axis at more than one point, the principal axes at those points are parallel. Again, in that case (5) must be satisfied by more than one value of h. But since h enters only in the first power, this cannot be unless

$$\Sigma mx = 0, \quad \Sigma my = 0,$$
$$\Sigma mxz = 0, \quad \Sigma myz = 0;$$

so that the axis must pass through the centre of gravity and be a principal axis at the origin, and therefore (since the origin is arbitrary) a principal axis at every point in its length.

If the principal axes at the centre of gravity be taken as the axes of x, y, z, (1) and (2) are satisfied for all values of h. Hence, if a straight line be a principal axis at the centre of gravity, it is a principal axis at every point in its length.

50. Let the system be projected on a plane perpendicular to the given straight line, so that the ratios of the elements of mass to each other are unaltered. The given straight line, which has been taken as the axis of z, cuts this plane in O, and will be a principal axis of the projection at O, because the projected system being a plane lamina, the conditions $\Sigma mxz = 0$, $\Sigma myz = 0$ are both satisfied. Since z does not appear in equation (4), it follows that if the given straight line be a principal axis at some point C in its length, the other two principal axes at C will be parallel to the principal axes of the projected system at O. These last may often be conveniently found by the next proposition.

51. Ex. 1. The principal axes of a right-angled triangle at the right angle are, one perpendicular to the plane and two others inclined to its sides at the angles $\frac{1}{2}\tan^{-1}\frac{ab}{a^2-b^2}$, where a and b are the sides of the triangle adjacent to the right angle.

Take the formula $\tan 2\theta = \frac{2F}{B-A}$, Art. 48, then by Art. 8, $A = M\frac{a^2}{6}$, $B = M\frac{b^2}{6}$, $F = M\frac{ab}{12}$.

Ex. 2. The principal axes of a quadrant of an ellipse at the centre are, one perpendicular to the plane and two others inclined to the principal diameters at the angles $\frac{1}{2}\tan^{-1}\frac{4}{\pi}\frac{ab}{a^2-b^2}$, where a and b are the semi-axes of the ellipse.

Ex. 3. The principal axes of a cube at any point P are, the straight line joining P to O the centre of gravity of the cube, and any two straight lines at P perpendicular to PO, and perpendicular to each other.

Ex. 4. Prove that the locus of a point P at which one of the principal axes is parallel to a given straight line is a rectangular hyperbola in the plane of which the centre of gravity of the body lies, and one of the asymptotes is parallel to the given straight line. But if the given straight line be parallel to one of the principal axes at the centre of gravity, the locus of P is that principal axis and the perpendicular principal plane.

Take the origin at the centre of gravity, and one axis of co-ordinates parallel to the given straight line.

Ex. 5. An edge of a tetrahedron will be a principal axis at some point in its length, only when it is perpendicular to the opposite edge. [Jullien.]

Conversely if this condition be satisfied, the edge will be a principal axis at a point C such that $OC = \frac{2}{5}ON$ where N is the middle point of the edge and O is the foot of the perpendicular distance between it and the opposite edge.

52. Prop. *Given the positions of the principal axes* Ox, Oy, Oz *at the centre of gravity* O, *and the moments of inertia about them, to find the positions of the principal axes at any point* P *in the plane of* xy, *and the moments of inertia about those axes.*

Let the mass of the body be M, and let A, B be the moments of inertia about the axes Ox, Oy, of which we shall suppose A the greater.

Take two points S and H in the axis of greatest moment, one on each side of the origin so that

$$OS = OH = \sqrt{\frac{A-B}{M}}.$$

These points may be called the foci of inertia for that principal plane.

Because these points are in one of the principal axes at the centre of gravity, the principal axes at S and H are parallel to the axes of co-ordinates, and the moments of inertia about those in the plane of xy are respectively A and $B + M.\overline{OS}|^2 = A$, and these being equal, any straight line through S or H in the plane of xy is a principal axis at that point, and the moment of inertia about it is equal to A.

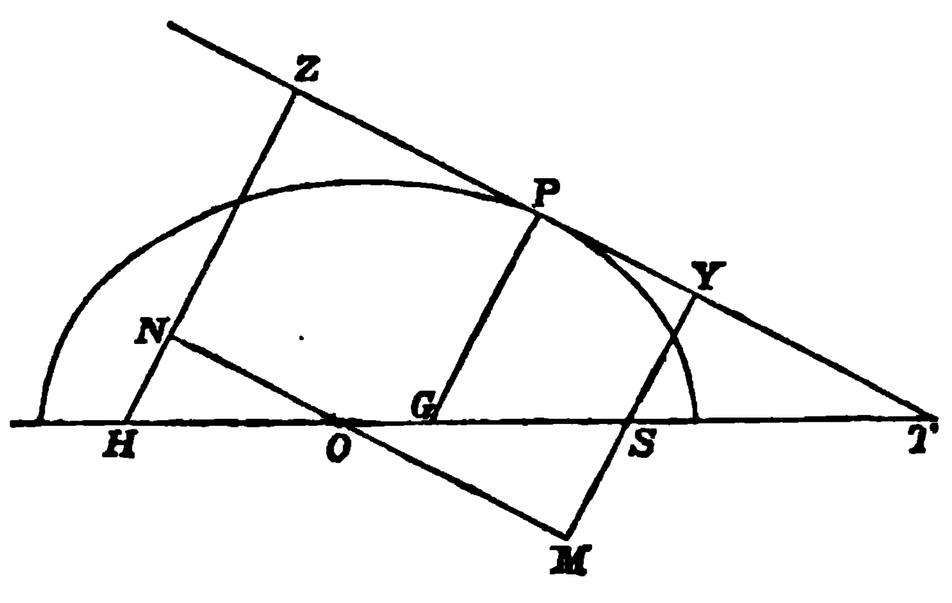
If P be any point in the plane of xy, then one of the principal axes at P will be perpendicular to the plane xy. For if p, q be the co-ordinates of P, the conditions that this line is a principal axis are

$$\left.\begin{aligned}\Sigma m(x-p)z = 0\\ \Sigma m(y-q)z = 0\end{aligned}\right\},$$

which are obviously satisfied because the centre of gravity is the origin, and the principal axes the axes of co-ordinates.

The other two principal axes may be found thus. If two straight lines meeting at a point P be such that the moments of inertia about them are equal, then provided they are in a principal plane the principal axes at P bisect the angles between these two straight lines. For if with centre P we describe the momental ellipse, then the axes of this ellipse bisect the angles between any two equal radii vectores.

Join SP and HP; the moments of inertia about SP, HP are each equal to A. Hence, if PG and PT are the internal and



external bisectors of the angle SPH; PG, PT are the principal axes at P. *If therefore with* S *and* H *as foci we describe any ellipse or hyperbola, the tangent and normal at any point are the principal axes at that point.*

53. Take any straight line MN through the origin, making an angle θ with the axis of x. Draw SM, HN perpendiculars on MN. The moment of inertia about it is

$$\begin{aligned} &= A \cos^2\theta + B \sin^2\theta \\ &= A - (A - B)\sin^2\theta \\ &= A - M.(OS\sin\theta)^2 \\ &= A - M.SM^2. \end{aligned}$$

Through P draw PT parallel to MN, and let SY and HZ be the perpendiculars from S and H on it. The moment of inertia about PT is then

$$\begin{aligned} &= \text{moment about } MN + M.MY^2 \\ &= A + M(MY - SM)(MY + SM) \\ &= A + M.SY.HZ. \end{aligned}$$

In the same way it may be proved that the moment of inertia about a line PG passing between H and S is *less* than A by the mass into the product of the perpendiculars from S and H on PG.

If therefore with S *and* H *as foci we describe any ellipse or hyperbola, the moments of inertia about any tangent to either of these curves is constant.*

It follows from this that the moments of inertia about the principal axes at P are equal to $B + M\left(\dfrac{SP \pm HP}{2}\right)^2$.

For if a and b be the axes of the ellipse we have $a^2 - b^2 = OS^2$ $= \dfrac{A-B}{M}$, and hence

$$A + M.SY.HZ = A + Mb^2 = B + Ma^2 = B + M\left(\frac{SP + HP}{2}\right)^2,$$

and the hyperbola may be treated in a similar manner.

54. This reasoning may be extended to points lying in any given plane passing through the centre of gravity O of the body. Let Ox, Oy be the axes in the given plane such that the product of inertia about them is zero (Art. 33). Construct the points S and H as before, so that OS^2 and OH^2 are each equal to the difference of the moments of inertia about Ox and Oy divided by the mass. Draw Sy' a parallel through S to the axis of y, the

product of inertia about Sx, Sy' is equal to that about Ox, Oy together with the product of inertia of the whole mass collected at O. Both these are zero, hence the section of the momental ellipsoid at S is a circle, and the moment of inertia about every straight line through S in the plane xOy is the same and equal to that about Ox. We can then show that the moments of inertia about PH and PS are equal; so that PG, PT, the internal and external bisectors of the angle SPH are the principal diameters of the section of the momental ellipsoid at P by the given plane. And it also follows that the moments of inertia about the tangents to a conic whose foci are S and H are the same.

55. Ex. 1. To find the foci of inertia of an elliptic area. The moments of inertia about the major and minor axes are $M\frac{b^2}{4}$ and $M\frac{a^2}{4}$. Hence the minor axis is the axis of greatest moment. The foci of inertia therefore lie in the *minor* axis at a distance from the centre $=\frac{1}{2}\sqrt{a^2-b^2}$, i.e. half the distance of the geometrical foci from the centre.

Ex. 2. Two particles each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that the principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse, provided $\frac{m}{M}=\frac{5}{8}\frac{e^2}{1-2e^2}$.

Ex. 3. At the points which have been called foci of inertia *two* of the principal moments are equal. Show that it is not in general true that a point exists such that the moments of inertia about all axes through it are the same, and find the conditions that there may be such a point.

Refer the body to the principal axes at the centre of gravity. Let P be the point required, (x, y, z) its co-ordinates. Since the momental ellipsoid at P is to be a sphere, the products of inertia about all rectangular axes meeting at P are zero. Hence, by Art. 13, $xy=0$, $yz=0$, $zx=0$. It follows that two of the three x, y, z must be zero, so that the point must be on one of the principal axes at the centre of gravity. Let this be called the axis of z. Since the moments of inertia about three axes at P parallel to the co-ordinate axes are $A+Mz^2$, $B+Mz^2$ and C, we see that these cannot be equal unless $A=B$ and each is less than C. There are then two points on the axis of unequal moment which are equimomental for all axes. [Poisson and Binet.]

56. *Given the positions of the principal axes at the centre of gravity O and the moments of inertia about them, to find the positions of the principal axes*, and the principal moments at any other point P.*

Let the body be referred to its principal axes at the centre of gravity O, let A, B, C be its principal moments, the mass of the

* Some of the following theorems were given by Sir William Thomson and Mr Townsend, in two articles which appeared at the same time in the *Mathematical Journal*, 1846. Their demonstrations are different from those given in this treatise.

body being taken as unity. Construct a quadric confocal with the ellipsoid of gyration, and let the squares of its semi-axes be $a^2 = A + \lambda$, $b^2 = B + \lambda$, $c^2 = C + \lambda$. Let us find the moment of inertia with regard to any tangent plane.

Let (α, β, γ) be the direction angles of the perpendicular to any tangent plane. The moment of inertia, with regard to a parallel plane through O, is

$$\frac{A+B+C}{2} - (A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma).$$

The moment of inertia, with regard to the tangent plane, is formed by adding the square of the perpendicular distance between the planes, viz.

$$(A+\lambda)\cos^2\alpha + (B+\lambda)\cos^2\beta + (C+\lambda)\cos^2\gamma,$$

we get

$$\left.\begin{array}{r}\text{moment of inertia with re-}\\ \text{gard to a tangent plane}\end{array}\right\} = \frac{A+B+C}{2} + \lambda$$

$$= \frac{B+C-A}{2} + a^2.$$

Thus the moments of inertia with regard to all tangent planes to any one quadric confocal with the ellipsoid of gyration are the same.

These planes are all principal planes at the point of contact. For draw any plane through the point of contact P, then in the case in which the confocal is an ellipsoid, the tangent plane parallel to this plane is more remote from the origin than this plane. Therefore, the moment of inertia with regard to any plane through P is less than the moment of inertia with regard to a tangent plane to the confocal ellipsoid through P. That is, the tangent plane to the ellipsoid is the principal plane of greatest moment. In the same way the tangent plane to the confocal hyperboloid of two sheets through P is the principal plane of least moment. It follows that the tangent plane to the confocal hyperboloid of one sheet is the principal plane of mean moment.

Through a given point P, three confocals can be drawn, the normals to these confocals are, by Art. 16, the principal axes at P. By Art. 5, Ex. 3, the principal axis of *least* moment is normal to the confocal *ellipsoid* and of greatest moment normal to the confocal hyperboloid of two sheets.

57. The moment of inertia with regard to the *point* P is, by Art. 14, $\dfrac{A+B+C}{2} + OP^2$. Hence, by Art. 5, Ex. 3, the moments

of inertia about the normals to the three confocals through P whose parameters are $\lambda_1, \lambda_2, \lambda_3$ are respectively

$$OP^2-\lambda_1,\quad OP^2-\lambda_2,\quad OP^2-\lambda_3.$$

58. If we describe any other confocal and draw a tangent cone to it whose vertex is P, the axes of this cone are known to be the normals to the three confocals through P. This gives another construction for the principal axes at P.

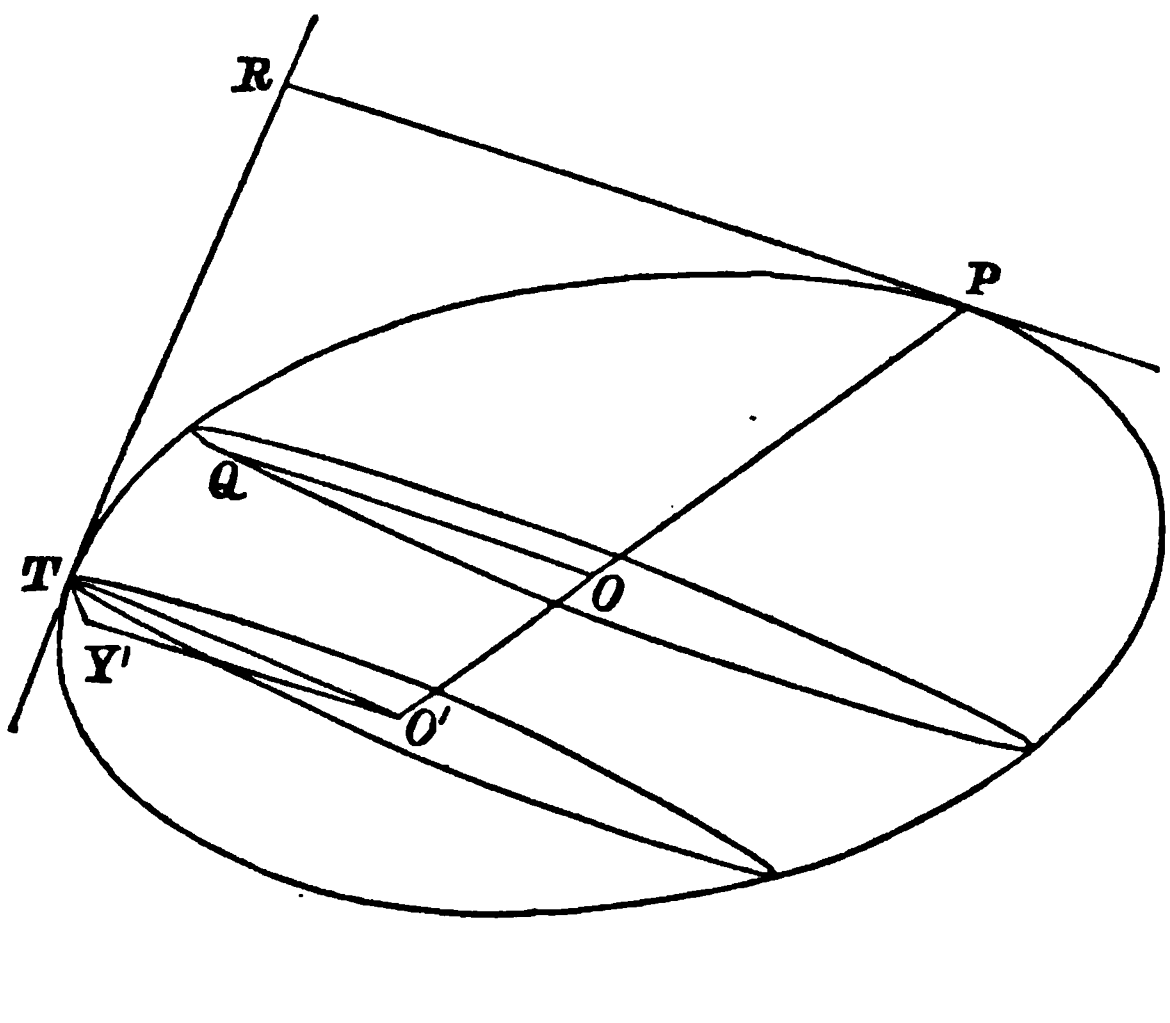
If this confocal diminish without limit, until it becomes a focal conic, then the principal diameters of the system at P are the principal diameters of a cone whose vertex is P and base a focal conic of the ellipsoid of gyration at the centre of gravity.

59. If we wish to use only one quadric, we may consider the confocal ellipsoid through P. We know* that the normals to the

* These propositions are to be found in books on Solid Geometry, they may also be proved as follows.

Let the confocal ellipsoid pass near P and approach it indefinitely. The base of the enveloping cone is ultimately the Indicatrix; and as the cone becomes ultimately a tangent plane, one of its axes is ultimately a perpendicular to the plane of the Indicatrix. Now in any cone two of its axes are parallel to the principal diameters of any section perpendicular to the third axis. Hence the axes of the enveloping cone are the normal to the surface and parallels to the principal diameters of the Indicatrix. But all parallel sections of an ellipsoid are similar and similarly situated, hence the principal diameters of the Indicatrix are parallel to the principal diameters of the diametral section parallel to the tangent plane at P.

To find the principal moments, we may reason as follows. Let a tangent plane to the ellipsoid be drawn perpendicular to any radius vector OQ of the diametral section of OP, then the point of contact T, OQ and OP will lie in one plane when



other two confocals are tangents to the lines of curvature on the ellipsoid, and are also parallel to the principal diameters of the diametral section made by a plane parallel to the tangent plane at P. And if D_1D_2 be these principal semi-diameters, we know that

$$\lambda_2 = \lambda_1 - D_1^2, \quad \lambda_3 = \lambda_1 - D_2^2.$$

Hence, if through any point P we describe the quadric

$$\frac{x^2}{A+\lambda} + \frac{y^2}{B+\lambda} + \frac{z^2}{C+\lambda} = 1,$$

the axes of co-ordinates being the principal axes at the centre of gravity, then the principal axes at P are the normal to this quadric, and parallels to the axes of the diametral section made by a plane parallel to the tangent plane at P. And if these axes be $2D_1$ and $2D_2$, the principal moments at P are

$$OP^2 - \lambda, \quad OP^2 - \lambda + D_1^2, \quad OP^2 - \lambda + D_2^2.$$

Ex. If two bodies have the same centre of gravity, the same principal axes at the centre of gravity and the differences of their principal moments equal, each to each, then these bodies have the same principal axes at all points.

60. *The axes of co-ordinates being the principal axes at the centre of gravity it is required to express the condition that any given straight line may be a principal axis at some point in its length and to find that point.*

Let the equations to the given straight line be

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n} \quad \text{.....................} \quad (1),$$

OQ is an axis of the section. For draw through T a section parallel to the diametral section, and let O' be its centre, and let $O'Y'$ be a perpendicular from O' on the tangent plane, which touches at T. Then OQ, $O'Y'$ and OP are in one plane. Now consider the section whose centre is O'; $O'Y'$ is the perpendicular on the tangent to an ellipse whose point of contact is T. Hence $O'Y'$, $O'T$ do not coincide unless $O'Y'$ be the direction of the axis of the ellipse. But this section is similar to the diametral section to which it was drawn parallel. Hence OQ is an axis of the diametral section.

Let PR be a straight line drawn through P parallel to OQ to meet in R the tangent plane which touches in T. Then RP, RT are two tangents at right angles to the ellipse PQT. Hence

$$\begin{aligned} OR^2 &= \text{sum of the squares of the semi-axes of the ellipse} \\ &= OP^2 + OQ^2 \end{aligned}$$

because OP, OQ are conjugate diameters.

The moment of inertia about PR, a perpendicular to a tangent plane, has been proved above to be $OR^2 - \lambda$, hence the moment of inertia about a parallel through P to the axis OQ is $OP^2 + OQ^2 - \lambda$.

then it must be a normal to the quadric

$$\frac{x^2}{A+\lambda}+\frac{y^2}{B+\lambda}+\frac{z^2}{C+\lambda}=1 \quad\dots\dots\dots\dots\dots\dots(2)$$

at the point at which the straight line is a principal axis.

Hence comparing the equation to the normal to (2) with (1), we have

$$\frac{x}{A+\lambda}=\mu l, \quad \frac{y}{B+\lambda}=\mu m, \quad \frac{z}{C+\lambda}=\mu n \quad\dots\dots\dots(3),$$

these six equations must be satisfied by the same values of x, y, z, λ and μ.

Substituting for x, y, z from (3) in (1), we get

$$A\mu-\frac{f}{l}=B\mu-\frac{g}{m}=C\mu-\frac{h}{n},$$

eliminating μ from these last equations we have

$$\frac{\frac{f}{l}-\frac{g}{m}}{A-B}=\frac{\frac{g}{m}-\frac{h}{n}}{B-C}=\frac{\frac{h}{n}-\frac{f}{l}}{C-A}=\mu \quad\dots\dots\dots\dots\dots(4).$$

This clearly amounts to only one equation, and is the required condition that the straight line should be a principal axis at some point in its length.

Substituting for x, y, z from (3) in (2), we have

$$\lambda(l^2+m^2+n^2)=\frac{1}{\mu^2}-(Al^2+Bm^2+Cn^2),$$

which gives one value only to λ. The values of λ and μ having been found, equations (3) will determine x, y, z, the co-ordinates of the point at which the straight line is a principal axis.

The geometrical meaning of this condition may be found by the following considerations, which were given by Mr Townsend in the *Mathematical Journal*. The normal and tangent plane at every point of a quadric will meet any principal plane in a point and a straight line, which are pole and polar with regard to the focal conic in that plane. Hence to find whether any assumed straight line is a principal axis or not, draw any plane perpendicular to the straight line and produce both the straight line and the plane to meet any principal plane at the centre of gravity. If the line of intersection of the plane be parallel to the polar line of the point of intersection of the straight line with respect to the focal conic, the axis will be a principal axis, if otherwise it will not be so. And the point at which the assumed straight line is a principal axis may be found by drawing a plane through the

polar line perpendicular to the straight line. The point of intersection is the required point.

The analytical condition (4) exactly expresses the fact that the polar line is parallel to the intersection of the plane.

61. Ex. 1. Given a plane $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}-1=0$, there is always some point in it at which it is a principal plane. Also this point is its intersection with the straight line $fx-A=gy-B=hz-C$.

Ex. 2. Let two points P, Q be so situated that a principal axis at P intersects a principal axis at Q. Then if two planes be drawn at P and Q perpendicular to these principal axes, their intersection will be a principal axis at the point where it is cut by the plane containing the principal axes at P and Q. [Mr Townsend.]

For let the principal axes at P, Q meet any principal plane at the centre of gravity in p, q, and let the perpendicular planes cut the same principal plane in LN, MN. Also let the perpendicular planes intersect each other in RN. Then RN is perpendicular to the plane containing the points P, Q, p, q. Also since the polars of p and q are LN, MN, it follows that pq is the polar of the point N. Hence the straight line RN satisfies the criterion of the last Article.

Ex. 3. If P be any point in a principal plane at the centre of gravity, then every axis which passes through P, and is a principal axis at some point, lies in one of two perpendicular planes. One of these planes is the principal plane at the centre of gravity, and the other is a plane perpendicular to the polar line of P with regard to the focal conic. Also the locus of all the points Q at which QP is a principal axis is a circle passing through P and having its centre in the principal plane. [Mr Townsend.]

Ex. 4. The edge of regression of the developable surface which is the envelope of the normal planes of any line of curvature drawn on a confocal quadric is a curve such that all its tangents are principal axes at some point in each.

62. *To find the locus of the points at which two principal moments of inertia are equal to each other.*

The principal moments at any point P are

$$I_1 = OP^2 - \lambda, \quad I_2 = OP^2 - \lambda + D_1^2, \quad I_3 = OP^2 - \lambda + D_2^2.$$

If we equate I_1 and I_2 we have $D_1 = 0$, and the point P must lie on the elliptic focal conic of the ellipsoid of gyration.

If we equate I_2 and I_3 we have $D_1 = D_2$, so that P is an umbilicus of any ellipsoid confocal with the ellipsoid of gyration. The locus of these umbilici is the hyperbolic focal conic.

In the first of these cases we have $\lambda = -C$, and D_2 is the semi-diameter of the focal conic conjugate to OP. Hence $D_2^2 + OP^2 =$ sum of squares of semi-axes $= A - C + B - C$. The three principal moments are therefore $I_1 = I_2 = OP^2 + C$, $I_3 = A + B - C$, and the axis of unequal moment is a tangent to the focal conic.

The second case may be treated in the same way by using a confocal hyperboloid, we therefore have $I_2 = I_3 = OP^2 + B$,

$I_1 = A + C - B$, and the axis of unequal moment is a tangent to the focal conic.

63. *To find the curves on any confocal quadric at which a principal moment of inertia is equal to a given quantity* I.

Firstly. The moment of inertia about a normal to a confocal quadric is $OP^2 - \lambda$. If this be constant, we have OP constant, and therefore the required curve is the intersection of that quadric with any concentric sphere. Such a curve is a sphero-conic.

Secondly. Let us consider those points at which the moment of inertia about a tangent is constant.

Construct any two confocals whose semi-major axes are a and a'. Draw any two tangent planes to these which cut each other at right angles. The moment of inertia about their intersection is the sum of the moments of inertia with regard to the two planes, and is therefore

$$= B + C - A + a^2 + a'^2.$$

Thus the moments of inertia about the intersections of perpendicular tangent planes to the same confocals are the same.

Let a, a', a'' be the semi-major axes of the three confocals which meet at any point P, then since confocals cut at right angles, the moment of inertia about the intersection of the confocals a', a'' is

$$I_1 = B + C - A + a'^2 + a''^2.$$

The intersection of these two confocals is a line of curvature on either. *Hence the moments of inertia about the tangents to any line of curvature are equal to one another; and these tangents are principal axes at the point of contact.*

On the quadric a draw a tangent PT making any angles ϕ and $\frac{\pi}{2} - \phi$ with the tangents to the lines of curvature at the point of contact P. If I_2, I_3 be the moments about the tangents to these lines of curvature, the moment of inertia about the tangent PT

$$\begin{aligned} &= I_2 \cos^2\phi + I_3 \sin^2\phi \\ &= B + C - A + (a''^2 + a^2)\cos^2\phi + (a^2 + a'^2)\sin^2\phi. \end{aligned}$$

But along a geodesic on the quadric a, $a'^2 \sin^2\phi + a''^2 \cos^2\phi$ is constant. Hence the moments of inertia about the tangents to any geodesic on the quadric are equal to each other.

64. **Ex. 1.** If a straight line touch any two confocals whose semi-major axes are a, a', the moment of inertia about it is $B + C - A + a^2 + a'^2$.

Ex. 2. When a body is referred to its principal axes at the centre of gravity, show how to find the coordinates of the point P at which the three principal moments are equal to three given quantities $I_1 I_2 I_3$. [Jullien's Problem.]

The elliptic co-ordinates of P are evidently $a^2 = \frac{1}{2}(I_2 + I_3 - I_1 - B - C + A)$ &c.; and the co-ordinates (x, y, z) may then be found by Dr Salmon's formulæ,

$$x^2 = \frac{a^2 a'^2 a''^2}{(A-B)(A-C)} \text{ \&c.}$$

Ex. 3. Let two planes at right angles touch two confocals whose semi-major axes are a, a'; and let a, a' be the values of a, a', when the confocals touch the intersection of the planes; then $a^2 + a'^2 = \mathrm{a}^2 + \mathrm{a}'^2$, and the product of inertia with regard to the two planes is $a^2 a'^2 - \mathrm{a}^2 \mathrm{a}'^2$.

65. The locus of all those points at which one of the principal moments of inertia of the body is constant is called an *equi-momental surface*.

To find the equation to such a surface we have only to put I_1 constant, this gives $\lambda = r^2 - I$. Substituting in the equation to the subsidiary quadric, the equation to the surface becomes

$$\frac{x^2}{x^2 + y^2 + z^2 + A - I} + \frac{y^2}{x^2 + y^2 + z^2 + B - I} + \frac{z^2}{x^2 + y^2 + z^2 + C - I} = 1.$$

Through any point P on an equi-momental surface describe the confocal quadric such that the principal axis is a tangent to a line of curvature on the quadric. By Art. 63 one of the intersections of the equi-momental surface and this quadric is the line of curvature. Hence the principal axis at P about which the moment of inertia is I is a tangent to the equi-momental surface.

Again, construct the confocal quadric through P such that the principal axis is a normal at P, then one of the intersections of the momental surface and this quadric is the sphero-conic through P. The normal to the quadric, being the principal axis, has just been shown to be a tangent to the surface. Hence the tangent plane to the equi-momental surface, is the plane which contains the normal to the quadric and the tangent to the sphero-conic.

To draw a perpendicular from the centre O on this tangent plane, we may follow Euclid's rule. Take PP' a tangent to the sphero-conic, drop a perpendicnlar from O on PP', this is the radius vector OP, because PP' is a tangent to the sphere. At P in the tangent plane draw a perpendicular to PP', this is the normal PQ to the quadric. From O drop a perpendicular OQ on this normal, then OQ is a normal to the tangent plane. Hence this construction,

If P *be any point on an equi-momental surface whose parameter is* I *and* OQ *a perpendicular from the centre on the tangent*

plane, then PQ *is the principal axis at* P *about which the moment of inertia is the constant quantity* I.

The equi-momental becomes Fresnel's wave surface when I is greater than the greatest principal moment of inertia at the centre of gravity. The general form of the surface is too well known to need a minute discussion here. It consists of two sheets, which become a concentric sphere and a spheroid when two of the principal moments at the centre of gravity are equal. When the principal moments are unequal, there are two singularities in the surface.

(1) The two sheets meet at a point P in the plane of the greatest and least moments. At P there is a tangent cone to the surface. Draw any tangent plane to this cone, and let OQ be a perpendicular from the centre of gravity O on this tangent plane. Then PQ is a principal axis at P. Thus there are an infinite number of principal axes at P because an infinite number of tangent planes can be drawn to the cone. But at any given point there cannot be more than three principal axes unless two of the principal axes be equal, and then the locus of the principal axes is a plane. Hence the point P is situated on a focal conic, and the locus of all the lines PQ is a normal plane to the conic. The point Q lies on a sphere whose diameter is OP, hence the locus of Q is a circle.

(2) The two sheets have a common tangent plane which touches the surface along the curve. This curve is a circle whose plane is perpendicular to the plane of greatest and least moments. Let OP' be a perpendicular from O on the plane of the circle, then P' is a point on the circle. If R be any other point on the circle the principal axis at R is RP'. Thus there is a circular ring of points at each of which the principal axis passes through the same point and the moments of inertia about these principal axes are all equal.

The equation to the equi-momental surface may also be used for the purpose of finding the three principal moments at any point whose co-ordinates (x, y, z) are given. If we clear the equation of fractions, we have a cubic to determine I whose roots are the three principal moments.

Thus let it be required to find the locus of all those points in a body at which any symmetrical function of the three principal moments is equal to a given quantity. We may express this symmetrical function in terms of the coefficients by the usual rules, and the equation to the locus is found.

Ex. 1. If an equi-momental surface cut a quadric confocal with the ellipsoid of gyration at the centre of gravity, then the intersections are a sphero-conic and a line of curvature. But if the quadric be an ellipsoid, both these cannot be real.

For if the surface cut the ellipsoid in both, let P be a point on the line of curvature, and P' a point on the sphero-conic, then by Art. 59, $OP^2 + D_1^2 = OP'^2$, which is less than $A + \lambda$. But $OP^2 + D_1^2 + D_2^2 = A + B + C + 3\lambda$, therefore $D_2^2 > B + C + 2\lambda$, which is $> A + 2\lambda$. Hence $D_2 >$ than the greatest radius vector of the ellipsoid, which is impossible.

Ex. 2. Find the locus of all those points in a body at which

(1) the sum of the principal moments is equal to a given quantity I.

(2) the sum of the products of the principal moments taken two and two together, is equal to I^2.

(3) the product of the principal moments is equal to I^3.

The results are

(1) a sphere whose radius is $\sqrt{\dfrac{I - (A + B + C)}{2M}}$, Art. 18.

(2) the surface

$$\left.\begin{matrix}(x^2+y^2+z^2)^2 + (A+B+C)(x^2+y^2+z^2) \\ + Ax^2 + By^2 + Cz^2 + AB + BC + CA\end{matrix}\right\} = I^2, \text{ Art. 65.}$$

(3) the surface $A'B'C' - A'y^2z^2 - B'z^2x^2 - C'x^2y^2 - 2x^2y^2z^2 = I^3$, where $A' = A + y^2 + z^2$, with similar expressions for B', C'.

CHAPTER II.

D'ALEMBERT'S PRINCIPLE, &c.

66. THE principles, by which the motion of a single particle under the action of given forces can be determined, will be found discussed in any treatise on Dynamics of a Particle. These principles are called the three laws of motion. It is shown that if (x, y, z) be the co-ordinates of the particle at any time t referred to three rectangular axes fixed in space, m its mass; X, Y, Z the forces resolved parallel to the axes, the motion may be found by solving the simultaneous equations,

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z.$$

If we regard a rigid body as a collection of material particles connected by invariable relations, we might write down the equations of the several particles in accordance with the principles just stated. The forces on each particle are however no longer known, some of them being due to the mutual actions of the particles.

We assume (1) that the action between two particles is along the line which joins them, (2) that the action and reaction between any two are equal and opposite. Suppose there are n particles, then there will be $3n$ equations, and, as shown in any treatise on Statics, $3n - 6$ unknown reactions. To find the motion it will be necessary to eliminate these unknown quantities. We may expect to find six resulting equations, and these will be shown, a little further on, to be sufficient to determine the motion of the body.

When there are several rigid bodies which mutually act and re-act on each other the problem becomes still more complicated. But it is unnecessary for us to consider in detail, either this or the preceding case, for D'Alembert has proposed a method by which all the necessary equations may be obtained without writing down the equations of motion of the several particles, and without making any assumption as to the nature of the mutual actions except the following, which may be regarded as a natural consequence of the laws of motion.

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

67. *To explain D'Alembert's Principle.*

In the application of this principle it will be convenient to use the term *effective force*, which may be defined as follows.

When a particle is moving as part of a rigid body, it is acted on by the external impressed forces and also by the molecular reactions of the other particles. If we considered this particle to be separated from the rest of the body, and all these forces removed, there is some one force which, under the same initial conditions, would make it move in the same way as before. This force is called the effective force on the particle. It is evidently the resultant of the impressed and molecular forces on the particle.

Let m be the mass of the particle, (x, y, z) its co-ordinates referred to any fixed rectangular axes at the time t. The accelerations of the particle are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$. Let f be the resultant of these, then, as explained in Dynamics of a Particle, the effective force is measured by mf.

Let F be the resultant of the impressed forces, R the resultant of the molecular forces on the particle. Then mf is the resultant of F and R. Hence if mf be reversed, the three F, R, and mf are in equilibrium.

We may apply the same reasoning to every particle of each body of the system. We thus have a group of forces similar to R, a group similar to F and a group similar to mf, these three groups will form a system of forces in equilibrium. Now by D'Alembert's principle the group R will itself form a system of forces in equilibrium. Whence it follows that the group F will be in equilibrium with the group mf. Hence

If forces equal to the effective forces but acting in exactly opposite directions were applied at each point of the system these would be in equilibrium with the impressed forces.

68. By this principle the solution of a dynamical problem is reduced to a problem in Statics. The process would be as follows. We first choose some quantities by means of which the position of the system in space may be fixed. We then express the effective forces on each element in terms of these quantities. These reversed will be in equilibrium with the given impressed forces. Lastly, the equations of motion for each body may be formed, as is usually done in Statics, by resolving in three directions and taking moments about three straight lines.

69. Before the publication of D'Alembert's principle a vast number of Dynamical problems had been solved. These may be found scattered through the early volumes of the Memoirs of St Petersburg, Berlin and Paris, in the works of John

Bernoulli and the *Opuscules* of Euler. They require for the most part the determination of the motions of several bodies with or without weight which push or pull each other by means of threads or levers to which they are fastened or along which they can glide, and which having a certain impulse given them at first are then left to themselves or are compelled to move in given lines or surfaces.

The postulate of Huyghens, "that if any weights are put in motion by the force of gravity they cannot move so that the centre of gravity of them all shall rise higher than the place from which it descended," was generally one of the principles of the solution: but other principles were always needed in addition to these, and it required the exercise of ingenuity and skill to detect the most suitable in each case. Such problems were for some time a sort of trial of strength among mathematicians. The *Traité de Dynamique* published by D'Alembert in 1743, put an end to this kind of challenge by supplying a direct and general method of resolving or at least throwing into equations any imaginable problem. The mechanical difficulties were in this way reduced to difficulties of Pure Mathematics. See Montucla, Vol. III. page 615, or Whewell's version of the same in his History of the Inductive Sciences.

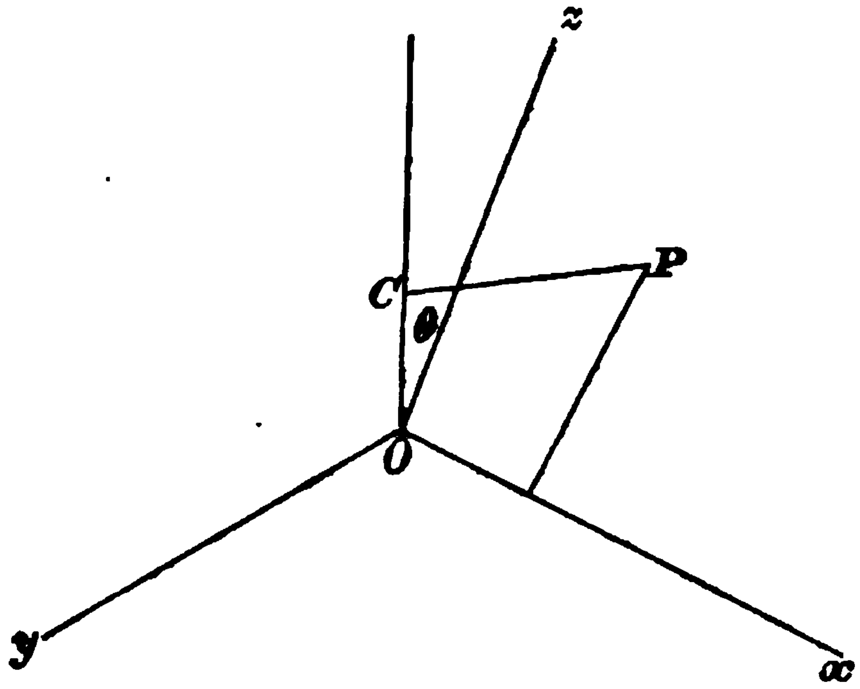
D'Alembert uses the following words:—"Soient A, B, C, &c. les corps qui composent le systême, et supposons qu'on leur ait imprimé les mouvemens a, b, c, &c. qu'ils soient forcés, à cause de leur action mutuelle, de changer dans les mouvemens a, b, c, &c. Il est clair qu'on peut regarder le mouvement a imprimé au corps A comme composé du mouvement a, qu'il a pris, et d'un autre mouvement α; qu'on peut de même regarder les mouvemens b, c, &c. comme composés des mouvemens b, β; c, γ; &c., d'ou il s'ensuit que le mouvement des corps A, B, C, &c. entr'eux auroit été le même, si au lieu de leur donner les impulsions a, b, c, on leur eût donné à-la-fois les doubles impulsions a, α; b, β; &c. Or par la supposition les corps A, B, C, &c. ont pris d'eux-mêmes les mouvemens a, b, c, &c. donc les mouvemens α, β, γ, &c. doivent être tels qu'ils ne dérangent rien dans les mouvemens a, b, c, &c. c'est-à-dire que si les corps n'avoient reçu que les mouvemens α, β, γ, &c. ces mouvemens auroient dû se détruire mutuellement, et le systême demeurer en repos. De là resulte le principe suivant pour trouver le mouvement de plusieurs corps qui agissent les uns sur les autres. Décomposez les mouvemens a, b, c &c. imprimés à chaque corps, chacun en deux autres a, α; b, β; c, γ; etc. qui soient tels que si l'on n'eût imprimé aux corps que les mouvemens a, b, c, &c. ils eussent pu conserver les mouvemens sans se nuire réciproquement; et que si on ne leur eût imprimé que les mouvemens α, β, γ, &c. le systême fût demeuré en repos; il est clair que a, b, c, &c. seront les mouvemens que ces corps prendront en vertu de leur action. Ce qu'il falloit trouver."

70. As an example of D'Alembert's principle let us consider the following problem.

A heavy body is capable of motion by two hinges about a horizontal axis, which axis is made to rotate with a uniform angular velocity ω about a vertical axis intersecting it in a point O. It is required to find the conditions that the body may be inclined at a constant angle to the vertical.

Let the horizontal axis which is fixed in the body be taken as axis of y, and let two other axes also fixed in the body be taken as a set of rectangular axes with origin O. Let θ be the angle

the plane of yz makes with a vertical plane through Oy. Our object is to find the relation between θ and ω.



By hypothesis each particle P describes a horizontal circle whose centre C is in the vertical through O. If r be the radius CP of this circle, and m the mass, the effective force on the particle is $m\omega^2 r$ and is directed along the radius. When reversed this will act in the direction CP.

The impressed forces on the body are its weight which may be supposed to act at the centre of gravity and the actions at the hinges. To avoid these last, we shall take moments about the axis Oy. Then the moment of the reversed effective forces together with the moment of the weight will be zero.

Let M be the mass of the body, $\bar{x}, \bar{y}, \bar{z}$ the co-ordinates of the centre of gravity, ξ its distance from the vertical plane through Oy. The moment of the weight is $Mg\xi$. The resolved part of the effective force parallel to Oy has no moment about Oy. The moment of the resolved part perpendicular to the vertical plane through Oy is $m\omega^2\rho$ if ρ be the distance of the particle from that plane. The equation of moments gives if $CO = u$

$$Mg\xi + \Sigma m\omega^2 \rho u = 0.$$

By projecting the co-ordinates on CO and CP we have

$$u = -x \sin\theta + z\cos\theta,$$

$$\rho = \quad x\cos\theta + z\sin\theta,$$

$$\xi = \quad \bar{x}\cos\theta + \bar{z}\sin\theta.$$

Substituting we get

$$Mg(\bar{x}\cos\theta + \bar{z}\sin\theta) = \omega^2 \{\tfrac{1}{2}\sin 2\theta \Sigma m(x^2 - z^2) - \cos 2\theta \Sigma mxz\}.$$

When the shape and structure of the body are known, the integrals $\Sigma m(x^2 - z^2)$ and $\Sigma m xz$ can be found by the methods of

the preceding chapter or by direct integration. This equation will then give the required relation between θ and ω.

It may be noticed that the only manner in which the form of the body enters into the equation is through its moments and products of inertia. If we change the body into any equi-momental one, the equation connecting θ and ω will be unaltered. So far as this problem is concerned, a body may be said to be given *Dynamically* when its mass, centre of gravity, principal axes, and principal moments at the centre of gravity are given. This remark will be found to be of general application.

Ex. 1. If the body be pushed along the axis of y and made to rotate about the vertical with the same angular velocity as before, show that no effect is produced on the inclination of the body to the vertical.

Ex. 2. If the body be a heavy disc capable of turning about a horizontal axis Oy in its own plane, show that the plane of the disc will be vertical unless $\omega^2 > \frac{gh}{k^2}$ where h is the distance of the centre of gravity of the disc from Oy and k the radius of gyration about Oy.

Ex. 3. If the body be a circular disc capable of turning about a horizontal axis perpendicular to its plane and intersecting the disc in its circumference, show that if the tangent to the disc at the hinge make an angle θ with the vertical, the angular velocity ω must be $\sqrt{\frac{g}{a \sin\theta}}$.

Ex. 4. Two equal balls A and B are attached to the extremities of two equal thin rods Aa, Bb. The ends a and b are attached by hinges to a fixed point O and the whole is set in rotation about a vertical through O as in the Governor of the Steam Engine. If the mass of the rods be neglected show that the time of rotation is equal to the time of oscillation of a pendulum whose length is the vertical distance of either sphere below the hinges at O.

Ex. 5. If in the last example m be the mass of either thin rod and M that of either sphere, l the length of a rod, r the radius of a sphere, h the depth of either centre below the hinge, then the length of the pendulum is $\frac{h}{l+r}\frac{M(l+r)^2+\frac{1}{3}ml^2}{M(l+r)+\frac{1}{2}ml}$.

71. *To apply D'Alembert's principle to obtain the equations of motion of a system of rigid bodies.*

Let (x, y, z) be the co-ordinates of the particle m at the time t referred to any set of rectangular axes fixed in space. Then $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$, will be the accelerations of the particle. Let X, Y, Z be the impressed accelerating forces on the same particle resolved parallel to the axes. By D'Alembert's principle the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right),$$

together with similar forces on every particle will be in equilibrium. Hence by the principles of Statics we have the equation

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma m X,$$

and two similar equations for y and z; these are obtained by resolving parallel to the axes. Also we have

$$\Sigma m \left(y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2}\right) = \Sigma m (yZ - zY),$$

and two similar equations for zx and xy; these are obtained by taking moments about the axes.

These equations may be written in the more convenient forms

$$\left.\begin{aligned} \frac{d}{dt} \Sigma m \frac{dx}{dt} &= \Sigma m X \\ \frac{d}{dt} \Sigma m \frac{dy}{dt} &= \Sigma m Y \\ \frac{d}{dt} \Sigma m \frac{dz}{dt} &= \Sigma m Z \end{aligned}\right\} \text{.................. (A)},$$

$$\left.\begin{aligned} \frac{d}{dt} \Sigma m \left(y \frac{dz}{dt} - z \frac{dy}{dt}\right) &= \Sigma m (yZ - zY) \\ \frac{d}{dt} \Sigma m \left(z \frac{dx}{dt} - x \frac{dz}{dt}\right) &= \Sigma m (zX - xZ) \\ \frac{d}{dt} \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) &= \Sigma m (xY - yX) \end{aligned}\right\} \text{............(B)}.$$

In a precisely similar manner by taking the expressions for the accelerations in polar co-ordinates we should have obtained another but equivalent set of equations of motion.

72. Let us consider the meaning of these equations without reference to axes of co-ordinates. The effective forces are to be equivalent to the impressed forces. But as shown in Statics any system of forces and therefore each of these is equivalent to a single force and a single couple at some point taken as origin. These resultant forces and couples must therefore be equivalent, each to each.

If we multiply the mass m of any particle P by its velocity v we have the momentum mv of the particle. Let us represent this in direction and magnitude by a straight line PP'. Then, just as in Statics, this momentum is equivalent to an equal and parallel *linear* momentum at O which we may represent by OM, and a *couple* whose moment is mvp, where p is the perpendicular distance between OM and PP'. The plane of this couple is the

plane containing OM and PP', and it may as usual be represented in direction and magnitude by an axis ON perpendicular to its plane. This couple is sometimes called an angular momentum.

Let OM', ON' be the positions of these two lines after an interval of time dt. Then MM', NN' will represent in direction and magnitude the linear momentum and the angular or couple momentum added on in the time dt. Hence the *effective force* on any particle m is equivalent to a single linear effective force acting at O represented by $\frac{MM'}{dt}$, and a single effective couple represented by $\frac{NN'}{dt}$.

Let OV, OH be two straight lines drawn through the origin O to represent in direction and magnitude the resultant linear momentum and resultant couple momentum of the whole system at any time t. Let OV', OH' be the positions of these lines at the time $t+dt$. Then OV is the resultant of the group OM corresponding to all the particles of the system, and OV' the resultant of the group OM'. Hence $\frac{VV'}{dt}$ represents the whole linear *effective force* of the system at the time t. By similar reasoning $\frac{HH'}{dt}$ represents the resultant *effective couple* of the system. Thus it appears that the points V and H trace out two curves in space whose properties are analogous to those of the hodograph in Dynamics of a particle. From this reasoning it follows, that if V_x be the resolved part of the momentum of a system in the direction of any straight line Ox, and H_x the moment of the momentum about that straight line, then $\frac{dV_x}{dt}$ and $\frac{dH_x}{dt}$ are respectively the resolved part along, and the moment about that straight line, of the effective force of the whole system.

Let us now refer the whole system to Cartesian co-ordinates as in Art. 71. We see that $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, $m\frac{dz}{dt}$ are the resolved parts of the momentum of the particle m. Hence OV is the resultant of $\Sigma m\frac{dx}{dt}$, $\Sigma m\frac{dy}{dt}$, and $\Sigma m\frac{dz}{dt}$. Also $m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$ is the moment of the momentum of the particle m about the axis of z. Hence OH is the resultant of

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right),\quad \Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right),\quad \Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right).$$

Now D'Alembert's principle asserts that the whole effective forces of a system are together equivalent to the impressed forces.

Hence whatever co-ordinates may be used, if X and L be the resolved parts and moment of the impressed moving forces respectively along and about any fixed straight line which we shall call the axis of x, the equations of motion are

$$\frac{dV_x}{dt} = X, \quad \frac{dH_x}{dt} = L.$$

The first of these corresponds to equations (A), the second to equations (B) of Art. 71.

We may notice the following cases.

(1) If no impressed forces act on the system, the two lines OV, OH are absolutely fixed in direction and magnitude throughout the motion.

(2) If all the impressed forces pass through a fixed point, let this point be chosen as the origin, then though OV may be variable, OH is fixed in position and magnitude.

(3) If all the impressed forces be equivalent to a system of couples, then though OH may be variable, OV is fixed in position and magnitude*.

73. The equations of motion of Art. 71 are the general equations of motion of any dynamical system. They are, however, extremely inconvenient in their present form. When the system considered is a rigid body and not merely a finite number of separate particles, the Σ's are all definite integrals. There are also an infinite number of x's, y's and z's all connected together by an infinite number of geometrical equations. It will be necessary, as suggested in Art. 68, to find some quantities which may determine the position of the body in space and express the effective forces in terms of these quantities. These are called the *co-ordinates of the body*†. It is most important in theoretical dynamics to choose these co-ordinates properly. They should be (1) such that a knowledge of them in terms of the time determines the motion of the body in a convenient manner, and (2) such that the dynamical equations when expressed in terms of them may be as little complicated as possible.

74. Let us first enquire how many co-ordinates are necessary to fix the position of a body.

The position of a body in space is given when we know the co-ordinates of some point in it and the angles which two straight lines fixed in the body make with the axes of co-ordinates. There

* In a memoir on the differential coefficients and determinants of lines, Mr Cohen has discussed some of the properties of these resultant lines. *Phil. Trans.* 1862.

† Sir W. Hamilton uses the phrase "marks of position," but subsequent writers have adopted the term co-ordinates. See Cayley's *Report to the Brit. Assoc.*, 1857.

are three geometrical relations existing between these six angles, so that the position of a body may be made to depend on *six* independent variables, viz. three co-ordinates and three angles. These might be taken as the co-ordinates of the body. By the term "co-ordinates of a body" is meant any quantities which determine the position of the body in space.

It is evident that we may express the co-ordinates (x, y, z) of any particle m of a body in terms of the co-ordinates of that body and quantities which are known and remain constant during the motion. First, let us suppose the system to consist only of a single body, then if we substitute these expressions for x, y, z in the equations (A) and (B) of Art. 71, we shall have six equations to determine the six co-ordinates of the body in terms of the time. Thus the motion will be found. If the system consist of several bodies, we shall, by considering each separately, have six equations for each body. If there be any unknown reactions between the bodies, these will be included in X, Y, Z. For each reaction there will be a corresponding geometrical relation connecting the motion of those bodies. Thus on the whole we shall have sufficient equations to determine the motion of the system.

When the motion is in two dimensions these six co-ordinates become *three*. These are the two co-ordinates of the fixed point in the body, and the angle some straight line fixed in the body makes with a straight line fixed in space.

75. Let us next consider how the equations of motion formed by resolution can be simplified by a proper choice of co-ordinates. We must find the resolved part of the momentum and the resolved part of the effective forces of a system in any direction.

Let the given direction be taken as the axis of x. Let (x, y, z) be the co-ordinates of any particle whose mass is m. The resolved part of its momentum in the given direction is $m\dfrac{dx}{dt}$. Hence the resolved part of the momentum of the whole system is $\Sigma m\dfrac{dx}{dt}$. Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centre of gravity of the system and M the whole mass. Then $M\bar{x} = \Sigma mx$;

$$\therefore\ M\frac{d\bar{x}}{dt} = \Sigma m\frac{dx}{dt}.$$

Hence the resolved part of the momentum of a system in any direction is equal to the whole mass multiplied into the resolved part of the velocity of the centre of gravity.

That is, the linear momentum of a system is the same as if the whole mass were collected into its centre of gravity.

In the same way, the resolved part of the effective forces of a system in any direction is equal to the whole mass multiplied into the resolved part of the acceleration of the centre of gravity.

It appears from this proposition that it will be convenient to take the co-ordinates of the centre of gravity of each rigid body in the system as three of the co-ordinates of that body. We can then express in a simple form the resolved part of the effective forces in any direction.

76. Lastly, let us consider how the equations of motion formed by taking moments can be simplified by a proper choice of the three remaining co-ordinates. We must find the moment of the momentum and the moment of the effective forces about any straight line.

Let the given straight line be taken as the axis of x, then following the same notation as before, the moment of the momentum about the axis of x is

$$\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right).$$

Now this is an expression of the second degree. If, then, we substitute $y=\bar{y}+y'$, $z=\bar{z}+z'$, we get by Art. 14

$$\Sigma m\left(y'\frac{dz'}{dt}-z'\frac{dy'}{dt}\right)+M\left(\bar{y}\frac{d\bar{z}}{dt}-\bar{z}\frac{d\bar{y}}{dt}\right),$$

where M is the mass of the system or body under consideration.

The second term of this expression is the moment about the axis of x of the momentum of a mass M moving with the centre of gravity.

The first term is the moment about a straight line parallel to the axis of x, not of the actual momenta of all the several particles but of their momenta *relatively* to that of the centre of gravity. In the case of any particular body it therefore depends only on the motion of the body relatively to its centre of gravity. In finding its value we shall suppose the centre of gravity reduced to rest by applying to every particle of the system a velocity equal and opposite to that of the centre of gravity. Hence we infer that

The moment of the momentum of a system about any straight line is equal to the moment of the momentum of the whole mass supposed collected at its centre of gravity and moving with it, together with the moment of the momentum of the system relative to its centre of gravity about a straight line drawn parallel to the given straight line through the centre of gravity.

In the same way, this proposition will be also true if for the "momentum" of the system we substitute "effective force."

By taking the axis Ox through the centre of gravity, we see that the moment of the *relative* momenta about any straight line through the centre of gravity is equal to that of the *actual* momenta.

77. It appears from the preceding article that it will be convenient to refer the angular motion of a body to a system of co-ordinate axes meeting at the centre of gravity. A general expression for the moment of the effective forces about any straight line through the centre of gravity cannot be conveniently investigated at this stage. Different expressions will be found advantageous under different circumstances. There are three cases to which attention should be particularly directed: (1) when the body is turning about an axis fixed in the body and fixed in space; (2) when the motion is in two dimensions, and (3) Euler's expression when the body is turning about a fixed point. These will be found at the beginnings of the third and fourth chapters and in the fifth chapter respectively.

78. The quantity $\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$ expresses the moment of the momentum about the axis of z. It is then called the *angular momentum* of the system about the axis of z. There is another interpretation which can be given to it. If we transform to polar co-ordinates, we have

$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}.$$

Now $\frac{1}{2}r^2d\theta$ is the elementary area described round the origin in the time dt by the projection of the particle on the plane of xy. If twice this polar area be multiplied by the mass of the particle, it is called the *area conserved* by the particle in the time dt round the axis of z. Hence

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

is called the *area conserved* by the system in a unit of time, or more simply the area conserved.

79. We may now enunciate two important propositions, which follow at once from the preceding results. It will, however, be more useful to deduce them from first principles.

(1) *The motion of the centre of gravity of a system acted on by any forces is the same as if all the mass were collected at the centre of gravity and all the forces were applied at that point parallel to their former directions.*

(2) *The motion of a body, acted on by any forces, about its centre of gravity is the same as if the centre of gravity were fixed and the same forces acted on the body.*

Taking any one of the equations (A) we have

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma m X.$$

If $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity, then $\bar{x}\Sigma m = \Sigma m x$;

$$\therefore \frac{d^2\bar{x}}{dt^2}\Sigma m = \Sigma m X,$$

and the other equations may be treated in a similar manner.

But these are the equations which give the motion of a mass Σm acted on by forces $\Sigma m X$, &c. Hence the first principle is proved.

Taking any one of equations (B) we have

$$\Sigma m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right) = \Sigma m\,(xY - yX).$$

Let $x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$, then by Art. 14 this equation becomes

$$\Sigma m \left(x' \frac{d^2y'}{dt^2} - y' \frac{d^2x'}{dt^2}\right) + \left(\bar{x} \frac{d^2\bar{y}}{dt^2} - \bar{y} \frac{d^2\bar{x}}{dt^2}\right) \Sigma m = \Sigma m(xY - yX).$$

Now the axes of co-ordinates are quite arbitrary, let them be so chosen that the centre of gravity is passing through the origin at the moment under consideration. Then $\bar{x} = 0$, $\bar{y} = 0$, but $\frac{d\bar{x}}{dt}, \frac{d\bar{y}}{dt}$ are not necessarily zero. The equation then becomes

$$\Sigma m \left(x' \frac{d^2y'}{dt^2} - y' \frac{d^2x'}{dt^2}\right) = \Sigma m\,(x'Y - y'X).$$

This equation does not contain the co-ordinates of the centre of gravity and holds at every separate instant of the motion and therefore is always true. But this and the two similar equations obtained from the other two equations of (B) are exactly the equations of moments we should have had if we had regarded the centre of gravity as a fixed point and taken it as the origin of moments.

80. These two important propositions are called respectively the principles of the conservation of the motions of translation and rotation. The first was given by Newton in the fourth corollary to the third law of motion, and was afterwards generalized by D'Alembert and Montucla. The second is more recent and seems to have been discovered about the same time by Euler, Bernoulli and the Chevalier d'Arcy.

81. By the first principle the problem of finding the motion of the centre of gravity of a system, however complex the system

may be, is reduced to the problem of finding the motion of a single particle. By the second the problem of finding the angular motion of a free body in space is reduced to that of determining the motion of that body about a fixed point.

In using the first principle it should be noticed that the impressed forces are to be applied at the centre of gravity *parallel* to their former directions. Thus, if a rigid body be moving under the influence of a central force, the motion of the centre of gravity is *not* generally the same as if the whole mass were collected at the centre of gravity and it were *then* acted on by the same central force. What the principle asserts is, that, if the attraction of the central force on each element of the body be found, the motion of the centre of gravity is the same as if *these* forces were applied at the centre of gravity parallel to their original directions.

If the impressed forces act always parallel to a fixed straight line, or if they tend to fixed centres and vary as the distance from those centres, the magnitude and direction of their resultant are the same whether we suppose the body collected into its centre of gravity or not. But in most cases care must be taken to find the resultant of the impressed forces as they really act on the body before it has been collected into its centre of gravity.

82. From this proposition we may infer the independence of the motions of translation and rotation. The motion of the centre of gravity is the same as if the whole mass were collected at that point, and is therefore quite independent of the rotation. The motion round the centre of gravity is the same as if that point were fixed, and is therefore independent of the motion of that point.

83. We may now collect together for reference the results of the preceding articles.

Let u, v, w be the velocities of the centre of gravity of any rigid body of mass M resolved parallel to any three fixed rectangular axes, let h_1, h_2, h_3 be the three moments of the momentum relative to the centre of gravity about three rectangular axes fixed in direction and meeting at the centre of gravity. Then the effective forces of the body are equivalent to the three effective forces $M\frac{du}{dt}$, $M\frac{dv}{dt}$, $M\frac{dw}{dt}$ acting at the centre of gravity parallel to the directions into which the velocities have been resolved, and to the three effective couples $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$, $\frac{dh_3}{dt}$ about the axes meeting at the centre of gravity about which the moments were taken. The effective forces of all the other bodies of the system may be expressed in a similar manner.

Then all these effective forces and couples, being reversed, will be in equilibrium with the impressed forces. The equations of equilibrium may then be found by resolving in such directions and taking moments about such straight lines as may be most convenient. Instead of reversing the effective forces it is usually found more convenient to write the impressed and effective forces on opposite sides of the equations.

Application of D'Alembert's Principle to impulsive forces.

84. If a force F act on a particle of mass m always in the same direction, the equation of motion is

$$m\frac{dv}{dt} = F,$$

where v is the velocity of the particle at the time t. Let T be the interval during which the force acts, and let v, v' be the velocities at the beginning and end of that interval. Then

$$m(v' - v) = \int_0^T F dt.$$

Now suppose the force F to increase without limit while the interval T decreases without limit. Then the integral may have a finite limit. Let this limit be P. Then the equation becomes $m(v' - v) = P$.

The velocity in the interval T has increased or decreased from v to v'. Supposing the velocity to have remained finite, let V be its greatest value during this interval. Then the space described is less than VT. But in the limit this vanishes. Hence the particle has not moved during the action of the force F. It has not had time to move but its velocity is suddenly changed from v to v'.

We may consider that a proper measure has been found for a force when from that measure we can deduce all the effects of the force. In the case of finite forces we have to determine both the change of place and the change in the velocity of the particle. It is therefore necessary to divide the whole time of action into elementary times and determine the effect of the force during each of these. But in the case of infinite forces which act for an indefinitely short time, the change of place is zero, and the change of velocity is the only element to be determined. It is therefore more convenient to collect the whole force expended into one measure. Such a force is called an impulse. It may be defined as the limit of a force which is infinitely great, but acts only during an infinitely short time. There are of course no such

forces in nature, but there are forces which are very great, and act only during a very short time. The blow of a hammer is a force of this kind. They may be treated as if they were impulses, and the results will be more or less correct according to the magnitude of the force and the shortness of the time of action. They may also be treated as if they were finite forces, and the displacement of the body during the time of action of the force may be found.

The quantity P may be taken as the measure of the force. An impulsive force is measured by the whole momentum generated by the impulse.

85. *In determining the effect of an impulse on a body, the effect of all finite forces which act on the body at the same time may be omitted.*

For let a finite force f act on a body at the same time as an impulsive force F. Then as before we have

$$v' - v = \frac{\int_0^T F dt}{m} + \frac{\int_0^T f dt}{m} = \frac{P}{m} + \frac{fT}{m}.$$

But in the limit fT vanishes. Similarly the force f may be omitted in the equation of moments.

86. *To obtain the general equations of motion of a system acted on by any number of impulses at once.*

Let u, v, w, u', v', w' be the velocities of a particle of mass m parallel to the axes just before and just after the action of the impulses. Let X', Y', Z' be the resolved parts of the impulse on m parallel to the axes.

Taking the same notation as before, we have the equation

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma m X,$$

or integrating

$$\Sigma m (u' - u) = \Sigma m \int_0^T X dt = \Sigma X' \qquad \text{.....................(1).}$$

Similarly we have the equations

$$\Sigma m (v' - v) = \Sigma Y' \qquad \text{...........................(2),}$$

$$\Sigma m (w' - w) = \Sigma Z' \qquad \text{...........................(3).}$$

Again the equation

$$\Sigma m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right) = \Sigma m (xY - yX)$$

becomes on integration

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\Sigma m\left(x\int Ydt-y\int Xdt\right),$$

or taken between limits,

$$\Sigma m\{x(v'-v)-y(u'-u)\}=\Sigma(xY'-yX')\text{............}(4),$$

and the other two equations become

$$\Sigma m\{y(w'-w)-z(v'-v)\}=\Sigma(yZ'-zY')\text{............}(5),$$

$$\Sigma m\{z(u'-u)-x(w'-w)\}=\Sigma(zX'-xZ')\text{............}(6).$$

In all the following investigations it will be found convenient to use accented letters to denote the states of motion after impact which correspond to those denoted by the same letters unaccented before the action of the impulse. Since the changes in direction and magnitude of the velocities of the several particles of the bodies are the only objects of investigation, it will be more convenient to express the equations of motion in terms of these velocities, and to avoid the introduction of such symbols as $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

87. In applying D'Alembert's Principle to impulsive forces the only change which must be made is in the mode of measuring the effective forces. If (u, v, w), (u', v', w') be the resolved parts of the velocity of any particle, just before and just after the impulse, and if m be its mass, the effective forces will be measured by $m(u'-u)$, $m(v'-v)$, and $m(w'-w)$. The quantity mf in Art. 67 is to be regarded as the measure of the impulsive force which, if the particle were separated from the rest of the body, would produce these changes of momentum.

In this case, if we follow the notation of Arts. 75 and 76, the resolved part of the effective force in the direction of the axis of z is the difference of the values of $\Sigma m\frac{dz}{dt}$ just before and just after the action of the impulses, and this is the same as the difference of the values of $M\frac{d\bar{z}}{dt}$ at the same instants. In the same way the moment of the effective forces about the axis of z will be the difference of the values of

$$\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

just before and just after the action of the impulses.

We may therefore extend the general proposition of Art. 83 to impulsive forces in the following manner.

Let (u, v, w), (u', v', w') be the velocities of the centre of gravity of any rigid body of mass M just before and just after the action

of the impulses resolved parallel to any three fixed rectangular axes. Let (h_1, h_2, h_3), (h_1', h_2', h_3') be the three moments of the momentum relative to the centre of gravity about three rectangular axes fixed in direction and meeting at the centre of gravity, the moments being taken just before and just after the impulses. Then the effective forces of the body are equivalent to the three effective forces $M(u'-u)$, $M(v'-v)$, $M(w'-w)$ acting at the centre of gravity parallel to the rectangular axes together with the three effective couples $(h_1'-h_1)$, $(h_2'-h_2)$, $(h_3'-h_3)$ about those axes.

These effective forces and couples being reversed will be in equilibrium with the impressed forces. The equations of equilibrium may then be formed according to the rules of Statics.

Ex. 1. Two particles moving in the same plane are projected in parallel but opposite directions with velocities inversely proportional to their masses. Find the motion of their centre of gravity.

Ex. 2. A person is placed on a perfectly smooth table, show how he may get off.

Ex. 3. Explain how a person sitting on a chair, is able to move the chair across the room by a series of jerks, without touching the ground with his feet.

Ex. 4. A person is placed at one end of a perfectly rough board which rests on a smooth table. Supposing he walks to the other end of the board, determine how much the board has moved. If he stepped off the board, show how to determine its subsequent motion.

Ex. 5. The motion of the centre of gravity of a shell shot from a gun in vacuo is a parabola, and its motion is unaffected by the bursting of the shell.

Ex. 6. A rod revolving uniformly in a horizontal plane round a pivot at its extremity suddenly snaps in two: determine the motion of each part.

Ex. 7. A cube slides down a perfectly smooth inclined plane with four of its edges horizontal. The middle point of the lowest edge comes in contact with a small fixed obstacle and is reduced to rest. Determine if the cube is also reduced to rest, and show that the resultant impulsive action along the edge will not in general act along the inclined plane.

Ex. 8. Two persons A and B are situated on a perfectly smooth horizontal plane at a distance a from each other. A throws a ball to B which reaches B after a time t. Show that A will begin to slide along the plane with a velocity $\frac{ma}{Mt}$ where M is his own mass and m that of the ball. If the plane were perfectly rough explain in general terms the nature of the pressures between A's feet and the plane which would prevent him from sliding. Would these pressures have a single resultant?

Ex. 9. A cannon rests on an imperfectly rough horizontal plane and is fired with such a charge that the relative velocity of the ball and cannon at the moment when the ball leaves the cannon is V. If M be the mass of the cannon, m that of

the ball and μ the coefficient of friction, show that the cannon will recoil a distance $\left(\frac{mV}{M+m}\right)^2 \frac{1}{2\mu g}$ on the plane.

Ex. 10. A spherical cavity of radius a is cut out of a cubical mass so that the centre of gravity of the remaining mass is in the vertical through the centre of the cavity. The cubical mass rests on a perfectly smooth horizontal plane, but the interior of the cavity is perfectly rough. A sphere of mass m, and radius b, rolls down the side of the cavity starting from rest with its centre on a level with the centre of the cavity. Show that when the sphere next comes to rest, the cubical mass has moved through a space $\frac{2m(a-b)}{M+m}$ where M is the mass of the remaining portion of the cube. Will the result be the same if the cavity were imperfectly rough or smooth?

Ex. 11. Two railway engines drawing the same train are connected by a loose chain and come several times in succession into collision with each other; the leading engine being a little top-heavy and the buffers of both rather low. The fore-wheels of the first engine are observed to jump up and down. What dynamical explanation can be given of this rocking motion? At what level should the buffers be placed that it may not occur? *Camb. Transac.* Vol. VII.

Ex. 12. Sir C. Lyell in his account of the earthquake in Calabria in 1783, mentions two obelisks each of which was constructed of three great stones laid on top of each other. After the earthquake, the pedestal of each obelisk was found to be in its original place, but the separate stones above were turned partially round and removed several inches from their position without falling. The shock which agitated the building was therefore described as having been horizontal and vorticose. Show that such a displacement would be produced by a simple rectilinear shock, if the resultant blow on each stone did not pass through its centre of gravity. See *Mallet's Dynamics of Earthquakes.*

CHAPTER III.

MOTION ABOUT A FIXED AXIS.

88. *A rigid body can turn freely about an axis fixed in the body and in space, to find the moment of the effective forces about the axis of rotation.*

Let any plane passing through the axis and fixed in space be taken as a plane of reference, and let θ be the angle which any other plane through the axis and fixed in the body makes with the first plane. Let m be the mass of any element of the body, r its distance from the axis, let ϕ be the angle a plane through the axis and the element m makes with the plane of reference.

The velocity of the particle m is $r\dfrac{d\phi}{dt}$ in a direction perpendicular to the plane containing the axis and the particle. The moment of the momentum of this particle about the axis is clearly $mr^2\dfrac{d\phi}{dt}$. Hence the moment of the momenta of all the particles is $\Sigma\left(mr^2\dfrac{d\phi}{dt}\right)$. Since the particles of the body are rigidly connected with each other, it is obvious that $\dfrac{d\phi}{dt}$ is the same for every particle, and equal to $\dfrac{d\theta}{dt}$. Hence the moment of the momenta of all the particles of the body about the axis is $\Sigma mr^2\dfrac{d\theta}{dt}$, *i.e.* the moment of inertia of the body about the axis multiplied into the angular velocity.

The accelerations of the particle m are $r\dfrac{d^2\phi}{dt^2}$ and $-r\left(\dfrac{d\phi}{dt}\right)^2$ perpendicular to, and along the directions in which r is measured, the moment of the moving forces of m about the axis is $mr^2\dfrac{d^2\phi}{dt^2}$, hence the moment of the moving forces of all the particles of the body about the axis is $\Sigma\left(mr^2\dfrac{d^2\phi}{dt^2}\right)$. By the same reasoning as

before this is equal to $\Sigma mr^2 \dfrac{d^2\theta}{dt^2}$, *i.e.* the moment of inertia of the body about the axis into the angular acceleration.

89. *To determine the motion of a body about a fixed axis under the action of any forces.*

By D'Alembert's principle the effective forces when reversed will be in equilibrium with the impressed forces. To avoid introducing the unknown reactions at the axis, let us take moments about the axis.

First, let the forces be impulsive. Let ω, ω' be the angular velocities of the body just before and just after the action of the forces. Then, following the notation of the last article,

$$\omega' . \Sigma mr^2 - \omega . \Sigma mr^2 = L,$$

where L is the moment of the impressed forces about the axis;

$$\therefore \omega' - \omega = \frac{\text{moment of forces about axis}}{\text{moment of inertia about axis}}.$$

This equation will determine the change in the angular velocity produced by the action of the forces.

Secondly, let the forces be finite. Then taking moments about the axis, we have

$$\frac{d^2\theta}{dt^2} . \Sigma mr^2 = L;$$

$$\therefore \frac{d^2\theta}{dt^2} = \frac{\text{moment of forces about axis}}{\text{moment of inertia about axis}}.$$

This equation when integrated will give the values of θ and $\dfrac{d\theta}{dt}$ at any time. Two undetermined constants will make their appearance in the course of the solution. These are to be determined from the given initial values of θ and $\dfrac{d\theta}{dt}$. Thus the whole motion can be found.

90. It appears from this proposition that the motion of a rigid body about a fixed axis depends on (1) the moment of the forces about that axis and (2) the moment of inertia of the body about the axis. Let Mk^2 be this moment of inertia, so that k is the radius of gyration of the body. Then if the whole mass of the body were collected into a particle and attached to the fixed axis by a rod without inertia, whose length is the radius of gyration k, and if this system be acted on by forces having the same moment as before, and be set in motion with the same initial

values of θ and $\frac{d\theta}{dt}$, then the whole subsequent angular or gyratory motion of the rod will be the same as that of the body. We may say briefly, that a body turning about a fixed axis is dynamically given, when we know its mass and radius of gyration.

91. Ex. *A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through its centre. A man whose weight is equal to that of the board walks on and round it at the edge: when he has completed the circuit what will be his position in space?*

Let a be the radius of the board, Mk^2 its moment of inertia about the vertical axis. Let ω be the angular velocity of the board, ω' that of the man about the vertical axis at any time. And let F be the action between the feet of the man and the board.

The equation of motion of the board is by Art. 89,

$$Mk^2\frac{d\omega}{dt} = -Fa \quad\quad (1).$$

The equation of motion of the man is by Art. 79,

$$Ma\frac{d\omega'}{dt} = F \quad\quad (2).$$

Eliminating F and integrating, we get

$$k^2\omega + a^2\omega' = 0,$$

the constant being zero, because the man and the board start from rest. Let θ, θ' be the angles described by the board and man round the vertical axis. Then $\omega = \frac{d\theta}{dt}$, $\omega' = \frac{d\theta'}{dt}$, and $k^2\theta + a^2\theta' = 0$. Hence, when $\theta' - \theta = 2\pi$, we have $\theta' = \frac{k^2}{k^2+a^2}2\pi$. This gives the angle in space described by the man. If $k^2 = \frac{a^2}{2}$ we have $\theta' = \frac{2}{3}\pi$.

Let V be the mean relative velocity with which the man walks along the board. Then $\omega' - \omega = \frac{V}{a}$; $\therefore\ \omega = -\frac{Va}{k^2+a^2} = -\frac{2}{3}\frac{V}{a}$. This gives the mean angular velocity of the board.

On the Pendulum.

92. *A body moves about a fixed horizontal axis acted on by gravity only, to determine the motion.*

Take the vertical plane through the axis as the plane of reference, and the plane through the axis and the centre of gravity as the plane fixed in the body. Then the equation of motion is

$$\frac{d^2\theta}{dt^2} = \frac{\text{moment of forces}}{\text{moment of inertia}} \quad\quad (1)$$

$$= -\frac{Mgh\sin\theta}{M(k^2+h^2)},$$

where h is the distance of the centre of gravity from the axis and Mk^2 is the moment of inertia of the body about an axis through the centre of gravity parallel to the fixed axis. Hence

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2 + h^2} \sin\theta = 0 \quad \text{.....................} \quad (2).$$

The equation (2) cannot be integrated in finite terms, but if the oscillations be small, we may reject the cubes and higher powers of θ and the equation will become

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2 + h^2}\theta = 0.$$

Hence the time of a complete oscillation is $2\pi\sqrt{\dfrac{k^2 + h^2}{gh}}$. If h and k be measured in feet and $g = 32{\cdot}18$, this formula gives the time in seconds.

The equation of motion of a particle of any mass suspended by a string l is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}.\sin\theta = 0 \quad \text{.......................} \quad (3),$$

which may be deduced from equation (2) by putting $k = 0$ and $h = l$. Hence the angular motions of the string and the body under the same initial conditions will be identical if

$$l = \frac{k^2 + h^2}{h} \quad \text{...................................} \quad (4).$$

This length is called the *length of the simple equivalent pendulum.*

Through G, the centre of gravity of the body, draw a perpendicular to the axis of revolution cutting it in C. Then C is called the *centre of suspension.* Produce CG to O so that $CO = l$. Then O is called the *centre of oscillation.* If the whole mass of the body were collected at the centre of oscillation and suspended by a thread to the centre of suspension, its angular motion and time of oscillation would be the same as that of the body under the same initial circumstances.

The equation (4) may be put under another form. Since $CG = h$ and $OG = l - h$, we have

$$GC.GO = (\text{rad.})^2 \text{ of gyration about } G,$$

$$CG.CO = (\text{rad.})^2 \text{ of gyration about } C,$$

$$OG.OC = (\text{rad.})^2 \text{ of gyration about } O.$$

Any of these equations show that if O be made the centre of suspension, the axis being parallel to the axis about which k was taken, then C will be the centre of oscillation. Thus *the centres*

of oscillation and suspension are convertible and the time of oscillation about each is the same.

If the time of oscillation be given, l is given and the equation (4) will give two values of h. Let these values be h_1, h_2. Let two cylinders be described with that straight line as axis about which the radius of gyration k was taken, and let the radii of these cylinders be h_1, h_2. Then the times of oscillation of the body about any generating lines of these cylinders are the same, and are approximately equal to $2\pi\sqrt{\frac{l}{g}}$ *.

With the same axis describe a third cylinder whose radius is k. Then $l = 2k + \frac{(h-k)^2}{h}$, hence l is always greater than $2k$, and decreases continually as h decreases and approaches the value k. Thus the length of the equivalent pendulum continually decreases as the axis of suspension approaches from without to the circumference of this third cylinder. When the axis of suspension is a generating line of the cylinder the length of the equivalent pendulum is $2k$. When the axis of suspension is within the cylinder and approaching the centre of gravity the length of the equivalent pendulum continually increases and becomes infinite when the axis passes through the centre of gravity.

The time of oscillation is therefore least when the axis is a generating line of the circular cylinder whose radius is k. But the time about the axis thus found is not an absolute minimum. It is a minimum for all axes drawn parallel to a given straight line in the body. To find the axis about which the time is absolutely a minimum we must find the axis about which k is a minimum. Now it is proved in Art. 23, that of all axes through G the

* The position of the centre of oscillation of a body was first correctly determined by Huyghens in his *Horologium Oscillatorium* published at Paris in 1673. The most important of the theorems given in the text were discovered by him. As D'Alembert's principle was not known at that time, Huyghens had to discover some principle for himself. The hypothesis was, that when several weights are put in motion by the force of gravity, in whatever manner they act on each other their centre of gravity cannot be made to mount to a height *greater* than that from which it had descended. Huyghens considers that he assumes here only that a heavy body cannot of itself move upwards. The next step in the argument was, that at any instant the velocities of the particles are such that, if they were separated from each other and properly guided, the centre of gravity could be made to mount to a second position *as high* as its first position. For if not, consider the particles to start from their last positions, to describe the same paths reversed, and then again to be joined together into a pendulum; the centre of gravity would rise to its first position; but if this be higher than the second position, the hypothesis would be contradicted. This principle gives the same equation which the modern principle of Vis Viva would give, and the rest of the solution is not of much interest.

axis about which the moment of inertia is least or greatest is one of the principal axes. Hence the axis about which the time of oscillation is a minimum is parallel to that principal axis through G about which the moment of inertia is least. And if Mk^2 be the moment of inertia about that axis, the axis of suspension is at a distance k measured in any direction from the principal axis.

93. Ex. 1. Find the time of the small oscillations of a cube (1) when one side is fixed, (2) when a diagonal of one of its faces is fixed; the axis in both cases being horizontal.

Result. If $2a$ be a side of the cube, the length of the simple equivalent pendulum is in the first case $\frac{4\sqrt{2}}{3}a$, and in the second case $\frac{5}{3}a$.

Ex. 2. An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation, prove that the eccentricity is $\frac{1}{2}$.

Ex. 3. A circular arc oscillates about an axis through its middle point perpendicular to the plane of the arc. Prove that the length of the simple equivalent pendulum is independent of the length of the arc, and is equal to twice the radius.

Ex. 4. The density of a rod varies as the distance from one end, find the axis perpendicular to it about which the time of oscillation is a minimum.

Result. The axis passes through either of the two points whose distance from the centre of gravity is $\frac{\sqrt{2}}{6}a$, where a is the length of the rod.

Ex. 5. Find what axis in the area of an ellipse must be fixed that the time of a small oscillation may be a minimum.

Result. The axis must be parallel to the major axis, and bisect the semi-minor axis.

Ex. 6. A uniform stick hangs freely by one end, the other end being close to the ground. An angular velocity in a vertical plane is then communicated to the stick, and when it has risen through an angle of 90°, the end by which it was hanging is loosed. What must be the initial angular velocity so that on falling to the ground it may pitch in an upright position?

Result. The required angular velocity ω is given by

$$\omega^2 = \frac{3g}{2a} + \frac{g}{2a}\cdot\frac{\left\{(2n+1)\frac{\pi}{2}\right\}^2}{(2n+1)\frac{\pi}{2}+1},$$

where n is any integer; and $2a$ is the length of the rod.

Ex. 7. Two bodies can move freely and independently under the action of gravity about the same horizontal axis; their masses are m, m', and the distances of their centres of gravity from the axis are h, h'. If the lengths of their simple equivalent pendulums be L, L', prove that when fastened together the length of the equivalent pendulum will be $\frac{mhL+m'h'L'}{mh+m'h'}$.

Ex. 8. When it is required to regulate a clock without stopping the pendulum, it is usual to add or subtract some small weight from a platform attached to the pendulum. Show that in order to make a given alteration in the going of the clock by the addition of the least possible weight, the platform must be placed at a distance from the point of suspension equal to half the simple equivalent pendulum. Show also that a slight error in the position of the platform will not affect the weight required to be added.

Ex. 9. A circular table centre O is supported by three legs AA', BB', CC' which rest on a perfectly rough horizontal floor, and a heavy particle P is placed on the table. Suddenly one leg CC' gives way, show that the table and the particle will immediately separate if pc be greater than κ^2; where p and c are the distances of P and O respectively from the line AB joining the tops of the legs, and κ is the radius of gyration of the table and legs about the line $A'B'$ joining the points where the legs rest on the floor.

The condition of separation is that the initial normal acceleration of the point of the table at P should be greater than the normal acceleration of the particle itself.

Ex. 10. A string without weight is placed round a fixed ellipse whose plane is vertical, and the two ends are fastened together. The length of the string is greater than the perimeter of the ellipse. A heavy particle can slide freely on the string and performs small oscillations under the action of gravity. Prove that the simple equivalent pendulum is the radius of curvature of the confocal ellipse passing through the position of equilibrium of the particle.

94. In a clock which is regulated by a pendulum, it is necessary that the time of oscillation should be invariable. As all substances expand and contract with every alteration of temperature, it is clear that the distance of the centre of gravity of the pendulum from the axis and the moment of inertia about that axis will be continually altering. The length of the simple equivalent pendulum does not however depend on either of these elements simply, but on their ratio. If then we can construct a pendulum such that the expansion or contraction of its different parts does not alter this ratio, the time of oscillation will be unaffected by any changes of temperature. For an account of the various methods of accomplishing this which have been suggested, we refer the reader to any treatise* on clocks. We shall here only notice for the sake of illustration one simple construction, which has been generally used. It was invented by George Graham about the year 1715.

Some heavy fluid, such as mercury, is enclosed in a cast-iron cylindrical jar into the top of which an iron rod is screwed. This rod is then suspended in the usual manner from a fixed point. The downward expansion of the iron on any increase of temperature tends to lower the centre of oscillation, but the upward expansion of the mercury tends on the contrary to raise it. It is required to deter-

* Reid on *Clocks*; Denison's treatise on *Clocks and Clockmaking* in Weale's Series, 1867; Captain Kater's treatise on *Mechanics* in Lardner's *Cyclopædia*, 1830.

mine the condition that the position of the centre of oscillation may on the whole be unaltered.

Let Mk^2 be the moment of inertia of the iron jar and rod about the axis of suspension, c the distance of their common centre of gravity from that axis. Let l be the length of the pendulum from the point of suspension to the bottom of the jar, a the internal radius of the jar. Let nM be the mass of the mercury, h the height it occupies in the jar.

The moment of inertia of the cylinder of mercury about a straight line through its centre of gravity perpendicular to its axis is by Art. 18, Ex. 8, $nM\left(\frac{h^2}{12}+\frac{a^2}{4}\right)$. Hence the moment of inertia of the whole body about the axis of suspension is

$$Mn\left\{\frac{h^2}{12}+\frac{a^2}{4}+\left(l-\frac{h}{2}\right)^2\right\}+Mk^2,$$

and the moment of the whole mass collected at its centre of gravity is

$$Mn\left(l-\frac{h}{2}\right)+Mc.$$

The length L of the simple equivalent pendulum is the ratio of these two, and on reduction we have

$$L=\frac{n\left(\frac{h^2}{3}-lh+l^2+\frac{a^2}{4}\right)+k^2}{n\left(l-\frac{h}{2}\right)+c} \quad\text{.............................. (1).}$$

Let the linear expansion of the substance which forms the rod and jar be denoted by α and that of mercury by β for each degree of the thermometer. If the thermometer used be Fahrenheit's, we have $\alpha=\cdot0000065668$, $\beta=\cdot00003336$, according to some experiments of Dulong and Petit. Thus we see that α and β are so small that their squares may be neglected. In calculating the height of the mercury it must be remembered that the jar expands laterally, and thus the relative vertical expansion of the mercury is $3\beta-2\alpha$, which we shall represent by γ.

If then the temperature of every part be increased t^0, we have a, l, k, c, all increased in the ratio $1+\alpha t : 1$, while h is increased in the ratio $1+\gamma t : 1$. Since L is to be unaltered, we have

$$\left(\frac{dL}{da}a+\frac{dL}{dl}l+\frac{dL}{dk}k+\frac{dL}{dc}c\right)\alpha+\frac{dL}{dh}h\gamma=0.$$

But L is a homogeneous function of one dimension, hence

$$\frac{dL}{da}a+\frac{dL}{dl}l+\frac{dL}{dk}k+\frac{dL}{dc}c+\frac{dL}{dh}h=L.$$

The condition becomes therefore by substitution

$$\frac{\alpha}{\alpha-\gamma}=\frac{h}{L}\frac{dL}{dh}.$$

Let A, B be the numerator and denominator of the expression for L given by equation (1). Then taking the logarithmic differential

$$\frac{1}{L}\frac{dL}{dh}=\frac{n\left(\frac{2}{3}h-l\right)}{A}+\frac{\frac{1}{2}n}{B}=\frac{n}{B}\left(\frac{\frac{2}{3}h-l}{L}+\frac{1}{2}\right).$$

Hence the required condition is

$$\frac{a}{3(\beta-a)}=\frac{h}{l-\frac{h}{2}+\frac{c}{n}}\cdot\left(\frac{l-\frac{2}{3}h}{L}-\frac{1}{2}\right).$$

This calculation is of more theoretical than practical importance, for the numerical values of a and β depend a good deal on the purity of the metals and on the mode in which they have been worked. The adjustment must therefore be finally made by experiment.

In the investigation we have supposed a and β to be absolutely constant, but this is only a very near approximation. Thus a change of 80° Fah. would alter β by less than a fiftieth of its value.

When the adjustment is made the compensation is not strictly correct, for the iron jar and mercury have been supposed to be of uniform temperature. Now the different materials of which the pendulum is composed absorb heat at different rates and therefore while the temperature is changing there will be some slight error in the clock.

95. Another cause of error in a clock pendulum is the buoyancy of the air. This produces an upward force acting at the centre of gravity of the volume of the pendulum equal to the weight of the air displaced. A very slight modification of the fundamental investigation in Art. 92 will enable us to take this into account. Let V be the volume of the pendulum, D the density of the air; h_1, h_2, the distances of the centres of gravity of the mass and volume respectively from the axis of suspension, Mk^2 the moment of inertia of the mass about the axis of suspension. Let us also suppose the pendulum to be symmetrical about a plane through the axis and either centre of gravity.

The equation of motion is then

$$Mk^2\frac{d^2\theta}{dt^2}=-Mgh_1\sin\theta+VDgh_2\sin\theta \;\ldots\ldots\ldots\; (1).$$

By the same reasoning as before we infer that if l be the length of the equivalent pendulum

$$\frac{k^2}{l}=h_1-h_2\frac{VD}{M} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

The density of the air is continually changing, the changes being indicated by variations in the height of the barometer. Let h be the value of $h_1-h_2\frac{VD}{M}$ for any standard density D. Suppose the actual density to be $D+\delta D$ and let $l+\delta l$ be the corresponding length of the seconds pendulum, then we have by differentiation $\frac{k^2\delta l}{l^2}=h_2\frac{V\delta D}{M}$, and therefore

$$\frac{\delta l}{l}=\frac{h_2}{h}\frac{VD}{M}\frac{\delta D}{D}.$$

If T be the time of oscillation, we have

$$T = 2\pi\sqrt{\frac{l}{g}}, \text{ and } \therefore \frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l}.$$

96. Ex. 1. If the centres of gravity of the mass and volume were very nearly coincident and the weight of the air displaced were $\frac{1}{7100}$ of the weight of the pendulum, show that a rise of one inch in the barometer would cause an error in the seconds pendulum of nearly ·2 sec. per day.

Ex. 2. If we affix to the pendulum rod produced upwards a body of the same volume as the pendulum bob but of very small weight, so that the centre of gravity of the volume lies in the axis of suspension, show that the correction for buoyancy vanishes. This method was suggested in 1871 by the Astronomer Royal, but he remarks that this construction would probably be inconvenient in practice.

Ex. 3. If a barometer be attached to the pendulum show that the rise or fall of the mercury as the density of the air changed could be so arranged as to keep the time of vibration unaltered. This method was suggested first by Dr Robinson of Armagh in 1831 in the fifth volume of the memoirs of the Astronomical Society, and afterwards by Mr Denison in the *Astronomical Notices* for Jan. 1873. In the *Armagh Places of Stars* published in 1859, Dr Robinson describes the difficulties he found in practice before he was satisfied with the working of the clock.

The theory of this construction is that in differentiating equation (2) we are to suppose k^2 and h_1 variable and l constant. This gives $\frac{\delta(Mk^2)}{l} = \delta(Mh_1) - \delta(h_2VD)$. Let r be the rise of the barometer in the glass tube, r' the fall in the cistern, then $r' = mr$, where m is a known fraction depending on the dimensions of the barometer. Let a and b be the depths of the mercury in the tube and cistern below the axis of suspension, $2c$ the diameter of the tube, ρ the density of the mercury. Since $\pi c^2\rho r$ is the quantity of mercury added to the top of the mercury in the tube and taken away from the cistern, we have

$$\delta(Mk^2) = \pi c^2\rho r\left\{\left(a - \frac{r}{2}\right)^2 - \left(b + \frac{r'}{2}\right)^2\right\},$$

$$\delta(Mh_1) = \pi c^2\rho r\left\{\left(a - \frac{r}{2}\right) - \left(b + \frac{r'}{2}\right)\right\}.$$

These are accurate if the barometer be merely a bent tube so that the cylinders transferred are *similar* as well as equal; in this case $m=1$. If the area of the cistern be greater than that of the tube we have here neglected the difference of the moments of inertia of the two cylinders about axes through their centre of gravity. As r is seldom more than one inch, we may write these

$$\delta(Mk^2) = \pi c^2\rho r\,(a^2 - b^2),$$

$$\delta(Mh_1) = \pi c^2\rho r\,(a - b).$$

Since D is very small, we may neglect the variations of Vh_2 when multiplied by D. Thus we have

$$\frac{\delta D}{D} = \frac{\pi c^2 H\rho}{VDh_2}\,\frac{a+b-l}{l}\,r,$$

where $H=b-a$ is the height of the barometer. If the temperature of the air be unaltered we have $\frac{\delta D}{D}=\frac{\delta H}{H}$ and $r(1+m)=\delta H$. The required condition is therefore

$$\frac{\pi c^2 H\rho}{VD}\frac{H}{h_2}\frac{a+b-l}{l}=1+m.$$

It is clearly necessary that $a+b>l$. The jar of mercury in Graham's mercurial pendulum might be used as the cistern of the barometer, as Mr Denison remarks. The height of the barometer being 30 inches this would hardly be effective unless the pendulum was longer than the seconds pendulum, which is about 39 inches.

Prof. Rankine read a paper to the British Association in 1853 in which he proposed to use a clock with a centrifugal or revolving pendulum, part of which should consist of a siphon barometer. The rising and falling of the barometer would affect the rate of going of the clock and thence the *mean* height of the mercurial column during any long period would register itself.

Ex. 4. If the pendulum be supposed to drag a quantity of air with it which bears a constant ratio to the density D of the surrounding air and adds γD to the moment of inertia of the pendulum without increasing the moving power, show that the change produced in the simple equivalent pendulum by a change of density δD is given by $\delta l=\gamma\frac{\delta D}{Mh_1}$. Show that this might be included in Dr Robinson's mode of correcting for buoyancy.

97. In many experimental investigations it is necessary to determine the moment of inertia of the body experimented on about some axis. If the body be of regular shape and be so far homogeneous that the errors thus produced are of the order to be neglected, we can determine the moment of inertia by calculation. But sometimes this cannot be done. If we can make the body oscillate under gravity about any axis parallel to the given axis placed in a horizontal position, we can determine by equation (4) of Art. 92 the radius of gyration about a parallel axis through the centre of gravity. This requires however that the distances of the centre of gravity from the axes should be very accurately found. Sometimes it is more convenient to attach the body to a pendulum of known mass whose radius of gyration about a fixed horizontal axis has been previously found by observing the time of oscillation. Then by a new determination of the time of oscillation, the moment of inertia of the compound body, and therefore of the given body, may be found, the masses being known.

If the body be a lamina, we may thus find the radii of gyration about three axes passing through the centre of gravity. By measuring three lengths along these axes inversely proportional to these radii of gyration, we have three points on a momental ellipse at the centre of gravity. The ellipse may then be easily constructed. The directions of its principal diameters are the principal axes, and the reciprocals of their lengths represent on the same scale as before the principal radii of gyration.

If the body be a solid, six observed radii of gyration will determine the principal axes and moments at the centre of gravity. But in most cases some of the other circumstances of the particular problem under consideration will simplify the process.

On the length of the Seconds Pendulum.

98. The oscillations of a rigid body may be used to determine the numerical value of the accelerating force of gravity. Let τ be the half time of a small oscillation of a body made in vacuo about a horizontal axis, h the distance of the centre of gravity from the axis, k the radius of gyration about a parallel axis through the centre of gravity. Then we have by Art. 92,

$$k^2 + h^2 = \lambda h \tau^2 \quad\text{.......................... (1),}$$

where $\lambda = \frac{g}{\pi^2}$ so that λ is the length of the simple pendulum whose complete time of oscillation is two seconds.

We might apply this formula to any regular body for which k^2 and h could be found by calculation. Experiments have thus been made with a rectangular bar, drawn as a wire and suspended from one end. In this case $\frac{k^2 + h^2}{h}$ which is the length of the simple equivalent pendulum is easily seen to be two-thirds of the length of the rod. The preceding formula then gives λ or g as soon as the time of oscillation has been observed. By inverting the rod and taking the mean of the results in each position any error arising from want of uniformity in density or figure may be partially obviated. It has, however, been found impracticable to obtain a rod sufficiently uniform to give results in accordance with each other.

99. If we make a body oscillate about two parallel axes in succession not at the same distance from the centre of gravity, we get two equations similar to (1), viz.

$$\left.\begin{aligned} k^2 + h^2 &= \lambda h \tau^2 \\ k^2 + h'^2 &= \lambda h' \tau'^2 \end{aligned}\right\} \quad\text{.............................. (2).}$$

Between these two we may now eliminate k^2, thus

$$\frac{h^2 - h'^2}{\lambda} = h\tau^2 - h'\tau'^2 \quad\text{.......................... (3).}$$

This equation gives λ. Since k^2 has disappeared, the form and structure of the body is now a matter of no importance. Let a body be constructed with two apertures into which knife edges

can be fixed. By means of these resting either on a horizontal plane or in two triangular apertures to prevent slipping, the body can be made to oscillate through small arcs. The perpendicular distances h, h' of the centre of gravity from the axes must then be measured with great care. The formula will then give λ.

100. In Capt. Kater's method the body has a sliding weight in the form of a ring which can be moved up and down by means of a screw. The body itself has the form of a bar and the apertures are so placed that the centre of gravity lies between them. The ring weight is then moved until the two times of oscillation are *exactly equal*. The equation (3) then becomes

$$\frac{h+h'}{\lambda} = \tau^2 \quad \text{.................................} (4),$$

which determines λ. The advantage of this construction is that the position of the centre of gravity, which is very difficult to find by experiment, is not required. All we want is $h+h'$, the exact distance between the knife edges. The disadvantage is that the ring weight has to be moved until two times of oscillation, each of which it is difficult to observe, are made equal.

101. The equation (3) can be written in the form

$$\frac{h+h'}{\lambda} = \frac{\tau^2+\tau'^2}{2} + \tfrac{1}{2}\frac{h+h'}{h-h'}(\tau^2-\tau'^2).$$

We now see that if the body be so constructed that the times of oscillation about the two axes of suspension are very nearly equal $\tau^2-\tau'^2$ will be small, and therefore it will be sufficient in the last term to substitute for h and h' their *approximate* values. The position of the centre of gravity is of course to be found as accurately as possible, but any small error in its position is of no very great consequence, for these errors are multiplied by the small quantity $\tau^2-\tau'^2$. The advantage of this construction over Kater's is that the ring weight may be dispensed with and yet the only element which must be measured with extreme accuracy is $h+h'$, the distance between the knife edges.

102. In order to measure the distance between the knife edges, Captain Kater first compared the different standards of length then in use, in terms of each of which he expressed the length of his pendulum. Since then a much more complete comparison of these and other standards has been made under the direction of the Commission appointed for that purpose in 1843. *Phil. Trans.* 1857.

Having settled his unit of length, Captain Kater proceeded to measure the distance between the knife edges by means of micro-

scopes. Two different methods were used, which however cannot be described here. As an illustration of the extreme care necessary in these measurements, the following fact may be mentioned. Though the images of the knife edges were always perfectly sharp and well defined, their distance when seen on a black ground was ·000572 of an inch less than when seen on a white ground. This difference appeared to be the same whatever the relative illumination of the object and ground might be so long as the difference of character was preserved. Three sets of measurements were taken, two at the beginning of the experiments, and the third after some time. The object of these last was to ascertain if the knife edges had suffered from use. The mean results of these three differed by less than a ten-thousandth of an inch from each other, the distance to be measured being 39·44085 inches.

103. The time of a single vibration cannot be observed directly, because this would require the fraction of a second of time as shown by the clock to be estimated either by the eye or ear. The difficulty may be overcome by observing the time, say of a thousand vibrations, and thus the error of the time of a single vibration is divided by a thousand. The labour of so much counting may however be avoided by the use of "the method of coincidences." The pendulum is placed in front of a clock pendulum whose time of vibration is slightly different. Certain marks made on the two pendulums are observed by a telescope at the lowest point of their arcs of vibration. The field of view is limited by a diaphragm to a narrow aperture across which the marks are seen to pass. At each succeeding vibration one pendulum follows the other more closely, and at last its mark is completely covered by the other during their passage across the field of view of the telescope. After a few vibrations it appears again preceding the other. In the interval from one disappearance to the next, one pendulum has made, as nearly as possible, one *complete* oscillation more than the other. In this manner 530 half-vibrations of a clock pendulum, each equal to a second, were found to correspond to 532 of Captain Kater's pendulum. The advantage of this method of observation is such, that an error of one second in noting the interval between two coincidences would occasion an error of only 0·63 in the number of vibrations in 24 hours. The ratio of the times of vibration of the pendulum and the clock pendulum may thus be calculated with extreme accuracy. The rate of going of the clock must then be found by astronomical means.

104. The time of vibration thus obtained will require several corrections which are called "reductions." For instance, if the oscillation be not so small that we can put $\sin\theta = \theta$ in Art. 92, we must make a reduction to infinitely small arcs. The general method of effecting this will be considered in the chapter on Small

Oscillations. Another reduction is necessary if we wish to reduce the result to what it would have been at the level of the sea. The attraction of the intervening land may be allowed for by Dr Young's rule (*Phil. Trans.* 1819). We may thus obtain the force of gravity at the level of the sea, supposing all the land above this level were cut off and the sea constrained to keep its present level. As the level of the sea is altered by the attraction of the land, further corrections are still necessary if we wish to reduce the result to the surface of that spheroid which most nearly represents the earth. See *Camb. Phil. Trans.* Vol. x.

M. Baily gives as the length of the pendulum vibrating in half time a mean solar second in the open air in this latitude 39·133 inches, and the length of a similar pendulum vibrating sidereal seconds 38·919 inches.

105. The observations must be made in the air. To correct for this we have to make a reduction to a vacuum. This reduction consists of three parts: (1) The correction for buoyancy, (2) Du Buat's correction for the air dragged along by the pendulum, (3) The resistance of the air.

Let V be the volume of the pendulum which may be found by measuring the dimensions of the body. As the "reduction to a vacuum" is only a correction, any small unavoidable errors in calculating the dimensions will produce an effect only of the second order on the value of λ. Let ρ be the density of the air when the body is oscillating about one knife edge, ρ' the density when oscillating about the other. If the observation be made within an hour or two hours, we may put $\rho=\rho'$. The effect of buoyancy is allowed for by supposing a force $V\rho g$ to act upwards at the centre of gravity of the *volume* of the body. If the body be made as nearly as possible symmetrical about the two knife edges this centre of gravity will be half way between the knife edges.

Du Buat discovered by experiment that a pendulum drags with it to and fro a certain mass of air which increases the inertia of the body without adding to the moving force of gravity. This result has been confirmed by theory. The mass dragged bears to the mass of air displaced by the body a ratio which depends on the external shape of the body. Let us represent it by $\mu V\rho$. If the body be symmetrical about the knife edges, so that the external shape is the same whichever edge is made the axis of suspension, μ will be the same for each oscillation. Since this mass is to be collected at the centre of gravity of the volume, we must add to the k^2 of equation (1) in Art. 92, and therefore also in Art. 98, the term $\mu V\rho\left(\dfrac{h+h'}{2}\right)^2$.

Taking these two corrections the equation (1) of Art. 98 will now become

$$k^2+h^2+\frac{\mu V\rho}{m}\left(\frac{h+h'}{2}\right)^2=\lambda\tau^2\left(h-\frac{V\rho}{m}\frac{h+h'}{2}\right),$$

where m is the mass of the pendulum. Similarly for the oscillation about the other knife edge,

$$k^2+h'^2+\frac{\mu V\rho'}{m}\left(\frac{h+h'}{2}\right)^2=\lambda\tau'^2\left(h'-\frac{V\rho'}{m}\frac{h+h'}{2}\right).$$

We must eliminate k^2 as before. If the observations about the two knife

edges succeed each other at a short interval we may put $\rho=\rho'$, and then Du Buat's correction will disappear. This is of course a very great advantage. We then have*

$$\frac{h+h'}{\lambda}=\frac{\tau^2+\tau'^2}{2}+\frac{1}{2}\frac{h+h'}{h-h'}(\tau^2-\tau'^2)\left(1-\frac{V\rho}{m}\right),$$

the last term being very small because τ and τ' are nearly equal.

The resistance of the air will be some function of the angular velocity $\frac{d\theta}{dt}$ of the pendulum. Since $\frac{d\theta}{dt}$ is very small we may expand this function and take only the first power. Supposing Maclaurin's theorem not to fail, and that no coefficient of a higher power than the first is very great, this gives a resistance proportional to $\frac{d\theta}{dt}$. The equation of motion will therefore take the form

$$\frac{d^2\theta}{dt^2}+n^2\theta=-2f\frac{d\theta}{dt},$$

where $\frac{2\pi}{n}$ is the time of a complete oscillation in a vacuum and the term on the right-hand side is that due to the resistance of the air. The discussion of this equation will be found in the chapter on Small Oscillations.

106. In constructing a reversible pendulum to measure the force of gravity, the following are points of importance.

1. The axes of suspension, or knife edges, must not be at the same distance from the centre of gravity of the mass. They should be parallel to each other.

2. The times of oscillation about the two knife edges should be nearly equal.

3. The external form of the body must be symmetrical, and the same about the two axes of suspension.

4. The pendulum must be of such a regular shape that the dimensions of all the parts can be readily calculated.

These conditions are satisfied if the pendulum be of a rectangular shape with two cylinders placed one at each end. The external forms of these cylinders are to be equal and similar, but one is to be solid and the other hollow, and such that by *calculation* of moments of inertia the distance between the knife edges is to be as nearly as possible equal to the length of the simple equivalent pendulum.

5. The pendulum should be made, as far as possible, of one metal, so that as the temperature changes it may be always similar to itself. In this case since the times of oscillations of similar bodies vary as the square root of their linear dimensions, it is easy to reduce the observed time of oscillation to a standard tem-

* This formula was mentioned to the author as the one used in the late experiments by Capt. Heaviside to determine the length of the seconds pendulum.

perature. The knife edges however must be made of some strong substance not likely to be easily injured.

107. Ex. 1. If the knife edges be not perfectly sharp, let r be the *difference* of their radii of curvature, show that

$$\frac{h^2 - h'^2 + (h+h')\,r}{\lambda} = h\tau^2 - h'\tau'^2$$

very nearly when the pendulum vibrates in vacuo. It appears that the correction vanishes if the knife edges be only equally sharp. By interchanging the knife edges we have the same equation with the sign of r changed. By making a few observations we may thus determine r. A proposition similar to this has been ascribed to Laplace by Dr Young.

Ex. 2. A heavy spherical ball is suspended successively by a very fine wire from two points of support A and B whose vertical distance b has been carefully measured, thus forming two pendulums. The lowest point of the ball is, on each suspension, made to be as exactly as possible on the same level, which level is approximately at depths a and a' below A and B respectively. If r be the radius of the ball, which is small compared with a or a', and l, l' the lengths of the simple equivalent pendulum, prove that $\frac{l-l'}{b} = 1 - \frac{2}{5}\frac{r^2}{(a-r)(a'-r)}$ very nearly. By counting the number of oscillations performed in a given time by each pendulum, show how to find ratio $\frac{l}{l'}$. Thence show how to find g and point out which lengths must be most carefully measured and which need only be approximately found, so as to render this method effective. This method is mentioned in Grant's *History of Physical Astronomy*, page 155, as having been used by Bessel.

108. The length of the seconds pendulum has been used as a national standard of length. By an Act of Parliament passed in 1824, it was declared that the distance between the centres of the two points in the gold studs in the straight brass rod then in the custody of the clerk of the House of Commons, whereon the words and figures "standard yard, 1760" were engraved, shall be the original and genuine standard of length called a yard, the brass being at the temperature of 62° Fah. And as it was expedient that the said standard yard if injured should be restored of the same length by reference to some invariable natural standard, it was enacted, that the new standard yard should be of such length that the pendulum, vibrating seconds of mean time in the latitude of London in a vacuum at the level of the sea, should be 39·1393 inches.

On Oct. 16, 1834, occurred the fire at the Houses of Parliament, in which the standards were destroyed. The bar of 1760 was recovered, but one of its gold pins bearing a point was melted out and the bar was otherwise injured.

In 1838 a commission was appointed to report to the government on the course best to be pursued under the peculiar circumstances of the case.

In 1841 the commission reported that they were of opinion that the definition by which the standard yard is declared to be a certain brass rod is the best which it is possible to adopt. With respect to the provision for restoration they did not recommend a reference to the length of the seconds pendulum. "Since the passing of the act of 1824 it has been ascertained that several elements of reduction of the pendulum experiments therein referred to are doubtful or erroneous: thus it was shown by Dr Young, *Phil. Trans.* 1819, that the reduction to the level of the sea was doubtful; by Bessel, *Astron. Nachr.* No. 128, and by Sabine, *Phil. Trans.* 1829, that the reduction for the weight of air was erroneous; by Baily, *Phil. Trans.* 1832, that the specific gravity of the pendulum was erroneously estimated and that the faults of the agate planes introduced some elements of doubt; by Kater, *Phil. Trans.* 1830, and by Baily, *Astron. Soc. Memoirs*, Vol. IX., that very sensible errors were introduced in the operation of comparing the length of the pendulum with Shuckburgh's scale used as a representative of the legal standard. It is evident, therefore, that the course prescribed by the act would not necessarily reproduce the length of the original yard."

The commission stated that there were several measures which had been formerly accurately compared with the original standard yard, and by the use of these the length of the original yard could be determined without sensible error.

In 1843 another commission was appointed to compare all the existing measures and construct from them a new Parliamentary standard. Unexpected difficulties occurred in the course of the comparison, which cannot be described here. A full account of the proceedings of the commission will be found in a paper contributed by Sir G. Airy to the Royal Society in 1857.

Oscillation of a Watch Balance.

109. A rod $B'CB$ can turn freely about its centre of gravity C which is fixed, and is acted on by a very fine spiral spring CPB. The spring has one end C fixed in position in such a manner that the tangent at C is also fixed, and has the other end B attached to the rod so that the tangent at B makes a constant angle with the rod. The rod being turned through any angle, it is required to find the time of oscillation. This is the construction used in watches, just as the pendulum is used in clocks, to regulate the motion.

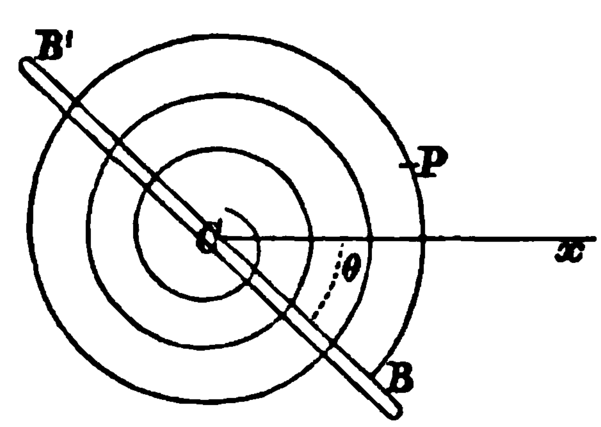
Let Cx be the position of the rod when in equilibrium, and let θ be the angle the rod makes with Cx at any time t, Mk^2 the moment of inertia of the rod about C. Let ρ be the radius of

curvature at any point P of the spring, ρ_0 the value of ρ when in equilibrium. Let (x, y) be the co-ordinates of P referred to C as origin and Cx as axis of x. Let us consider the forces which act on the rod and the portion BP of the spring. The forces on the rod are X, Y the resolved parts of the reaction at C parallel to the axes of co-ordinates, and the reversed effective forces which are equivalent to a couple $Mk^2\frac{d^2\theta}{dt^2}$. The forces on the spring are, the reversed effective forces which are so small that they may be neglected, and the resultant action across the section of the spring at P. This resultant action is produced by the tensions of the innumerable fibres which make up the spring, and these are equivalent to a force at P and a couple. When an elastic spring is bent so that its curvature is changed, it is proved both by experiment and theory that this couple is proportional to the change of curvature at P. We may therefore represent it by $E\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)$, where E depends only on the material of which the spring is made and on the form of its section.

Taking moments about P to avoid introducing the unknown force at P, we have

$$Mk^2\frac{d^2\theta}{dt^2} = -E\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right) - Xy + Yx.$$

This equation is true whatever point P may be chosen. Considering the left side constant at any moment and (x, y) variable, this becomes the intrinsic equation to the form of the spring.



Let $BP = s$, multiply this equation by ds and integrate along the whole length l of the spiral spring, we have

$$Mk^2\frac{d^2\theta}{dt^2}l = -E\int\left(\frac{ds}{\rho}-\frac{ds}{\rho_0}\right) + Y\int xds - X\int yds.$$

Now $\frac{ds}{\rho}$ is the angle between two consecutive normals, hence $\int\frac{ds}{\rho}$ is the angle between the extreme normals. Now at A the normal to the spring is fixed throughout the motion, therefore

$\int\left(\frac{ds}{\rho}-\frac{ds}{\rho_0}\right)$ is the angle between the normals at B in the two positions in which $\theta=\theta$ and $\theta=0$. But since the normal at B makes a constant angle with the rod, this angle is the angle θ which the rod makes with its position of equilibrium. Also if $\bar{x}, \bar{y}$ be the co-ordinates of the centre of gravity of the spring at the time t, we have $\int x ds=\bar{x}l$, $\int y ds=\bar{y}l$. Hence the equation of motion becomes

$$Mk^2\frac{d^2\theta}{dt^2}=-\frac{E}{l}\theta+Y\bar{x}-X\bar{y}.$$

Let us suppose that in the position of equilibrium there is no pressure on the axis C, then X and Y will, throughout the motion, be small quantities of the order θ. Let us also suppose that the fulcrum C is placed over the centre of gravity of the spring when at rest. Then if the number of spiral turns of the spring be numerous and if each turn be nearly circular, the centre of gravity will never deviate far from C. So that the terms $Y\bar{x}$ and $X\bar{y}$ are each the product of two small quantities, and are therefore at least of the second order. Neglecting these terms we have

$$Mk^2\frac{d^2\theta}{dt^2}=-\frac{E}{l}\theta.$$

Hence the time of oscillation is $2\pi\sqrt{\frac{Mk^2l}{E}}$.

It appears that to a first approximation the time of oscillation is independent of the form of the spring in equilibrium, and depends only on its length and on the form of its section.

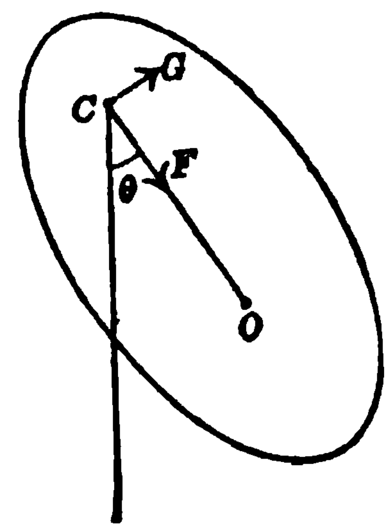
This brief discussion of the motion of a watch balance is taken from a memoir presented to the Academy of Sciences. The reader is referred to an article in Liouville's *Journal*, 1860, for a further investigation of the conditions necessary for isochronism and for a determination of the best forms for the spring.

Pressures on the fixed axis.

110. *A body moves about a fixed axis under the action of any forces, to find the pressures on the axis.*

First. Suppose the body and the forces to be symmetrical about the plane through the centre of gravity perpendicular to the axis. Then it is evident that the pressures on the axis are reducible to a single force at C the centre of suspension.

Let F, G be the actions of the point of support on the body resolved along and perpendicular to CO, where O is the centre



of gravity. Let X, Y be the sum of the resolved parts of the impressed forces in the same directions, and L their moment round C.

Let $CO = h$ and $\theta =$ angle which CO makes with any straight line fixed in space.

Taking moments about C, we have

$$\frac{d^2\theta}{dt^2} = \frac{L}{M(k^2+h^2)} \quad \text{......................... (1).}$$

The motion of the centre of gravity is the same as if all the forces acted at that point. Now it describes a circle round C; hence, taking the tangential and normal resolutions, we have

$$h\frac{d^2\theta}{dt^2} = \frac{Y+G}{M} \quad \text{.........................(2),}$$

$$-h\left(\frac{d\theta}{dt}\right)^2 = \frac{X+F}{M} \quad \text{.........................(3).}$$

Equation (1) gives the values of $\frac{d^2\theta}{dt^2}$ and $\frac{d\theta}{dt}$, and then the pressures may be found by equations (2) and (3).

If the only force acting on the body be that of gravity, let θ be measured from the vertical. If the body start from rest in that position which makes CO horizontal, we have

$$X = Mg\cos\theta, \quad Y = -Mg\sin\theta, \quad L = -Mgh\sin\theta;$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2+h^2}\sin\theta:$$

integrating, we have

$$\left(\frac{d\theta}{dt}\right)^2 = C + \frac{2gh}{k^2+h^2}\cos\theta,$$

but when $\theta = \frac{\pi}{2}$, $\frac{d\theta}{dt}$ vanishes, therefore $C = 0$; substituting these values (2) and (3), we get

$$\left.\begin{aligned} -F &= Mg\cos\theta \cdot \frac{k^2+3h^2}{k^2+h^2} \\ G &= Mg\sin\theta \cdot \frac{k^2}{k^2+h^2} \end{aligned}\right\},$$

where θ is the angle which CO makes with the vertical.

Let ψ be the angle the direction of the pressure at C makes with the line CO, the angle being measured from CO downwards to the left, then

$$\cot\psi = \left(1 + 3\frac{h^2}{k^2}\right)\cot\theta,$$

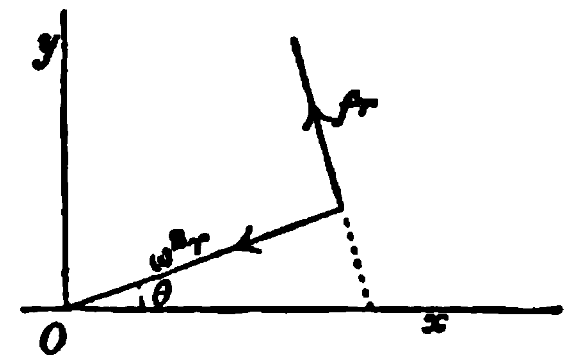
which is a convenient formula to determine the direction of the pressure*.

111. *Secondly.* Suppose either the body or the forces not to be symmetrical.

Let the fixed axis be taken as the axis of z with any origin and plane of xz. These we shall afterwards so choose as to simplify our process as much as possible. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity at the time t.

Let ω be the angular velocity of the body, f the angular acceleration, so that $f = \dfrac{d\omega}{dt}$.

Now every element m of the body describes a circle about the axis, hence its accelerations along and perpendicular to the radius



vector r from the axis are $-\omega^2 r$ and fr. Let θ be the angle

* Let $M.R$ be the resultant of F and G, and let $a = g\dfrac{k^2+3h^2}{k^2+h^2}$ and $b = g\dfrac{k^2}{k^2+h^2}$, then $\dfrac{\cos^2\psi}{a^2} + \dfrac{\sin^2\psi}{b^2} = \dfrac{1}{R^2}$. Construct an ellipse with C for centre and axes equal to a and b measured along and perpendicular to CO. Then the resultant pressure varies as the diameter along which it acts. And the direction may be found thus; let the auxiliary circle cut the vertical in V, and let the perpendicular from V on CO cut the ellipse in R. Then CR is the direction of the pressure.

which r makes with the plane of xz at any time, then from the resolution of forces it is clear that

$$\frac{d^2x}{dt^2} = -\omega^2 r \cos\theta - fr \sin\theta = -\omega^2 x - fy,$$

similarly $$\frac{d^2y}{dt^2} = -\omega^2 y + fx.$$

These equations may also be obtained by differentiating the equations $x = r\cos\theta$, $y = r\sin\theta$ twice, remembering that r is constant.

Conceive the body to be fixed to the axis at two points, distant a and a' from the origin, and let the reactions of the points on the body resolved parallel to the axes be respectively F, G, H; F', G', H'.

The equations of motion of Art. 71 then give

$$\begin{aligned}\Sigma mX + F + F' = \Sigma m\frac{d^2x}{dt^2} &= \Sigma m(-\omega^2 x - fy)\\ &= -\omega^2 M\bar{x} - fM\bar{y} \quad\ldots\ldots\ldots\ldots\ldots (1),\end{aligned}$$

$$\begin{aligned}\Sigma mY + G + G' = \Sigma m\frac{d^2y}{dt^2} &= \Sigma m(-\omega^2 y + fx)\\ &= -\omega^2 M\bar{y} + fM\bar{x} \quad\ldots\ldots\ldots\ldots\ldots (2),\end{aligned}$$

$$\Sigma mZ + H + H' = \Sigma m\frac{d^2z}{dt^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Taking moments about the axes, we have

$$\begin{aligned}\Sigma m(yZ - zY) - Ga - G'a' &= \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)\\ &= -\Sigma m\left(z\frac{d^2y}{dt^2}\right)\\ &= \omega^2\Sigma myz - f\Sigma mxz \quad\ldots\ldots\ldots\ldots\ldots\ldots (4):\end{aligned}$$

by merely introducing z into the results in (2),

$$\begin{aligned}\Sigma m(zX - xZ) + Fa + F'a' &= \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right)\\ &= -\omega^2\Sigma mxz - f\Sigma myz \quad\ldots\ldots\ldots\ldots\ldots (5),\end{aligned}$$

$$\begin{aligned}\Sigma m(xY - yX) &= \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\\ &= \Sigma mr^2 . \frac{d\omega}{dt}\\ &= Mk'^2 . f \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).\end{aligned}$$

Equation (6) serves to determine f and ω, and equations (1), (2), (4), (5) then determine F, G, F', G'; H and H' are indeterminate, but their sum is given by equation (3).

Looking at these equations, we see that they would be greatly simplified in two cases.

First, if the axis of z be a principal axis at the origin,

$$\Sigma mxz = 0, \quad \Sigma myz = 0,$$

and the calculation of the right-hand sides of equations (4) and (5) would only be so much superfluous labour. Hence, in attempting a problem of this kind, we should, when possible, so choose the origin that the axis of revolution is a principal axis of the body at that point.

Secondly, except the determination of f and ω by integrating equation (6), the whole process is merely an algebraic substitution of f and ω in the remaining equations. Hence our results will still be correct if we choose the plane of xz to contain the centre of gravity at the moment under consideration; this will make $\bar{y} = 0$, and thus equations (1) and (2) will be simplified.

112. If the forces which act on the body be impulsive, the equations will require some alterations.

Let ω, ω' be the angular velocities of the body just before and just after the action of the impulses. In the case in which the body and forces are symmetrical, the equations (1), (2), (3) of Art. 110 become respectively

$$\omega' - \omega = \frac{L}{M(k^2 + h^2)} \text{..................} (1),$$

$$h(\omega' - \omega) = \frac{Y + G}{M} \text{......................} (2),$$

$$0 = \frac{X + F}{M} \text{........................} (3),$$

where all the letters have the same meaning as before, except that F, G, X, Y are now impulsive instead of finite forces.

Let us next consider the case in which the forces on the body are not symmetrical. Let u, v, w, u', v', w' be the velocities resolved parallel to the axes of any element m whose co-ordinates are x, y, z. Then $u = -y\omega$, $u' = -y\omega'$, $v = x\omega$, $v' = x\omega'$, and w, w' are both zero.

The several equations of Art. 111 will then be replaced by the following:

$$\Sigma X + F + F' = \Sigma m(u' - u) = -\Sigma my(\omega' - \omega)$$
$$= -M\bar{y}(\omega' - \omega) \text{...............} (1),$$

$$\Sigma Y + G + G' = \Sigma m (v' - v) = \Sigma mx (\omega' - \omega)$$
$$= M\bar{x} (\omega' - \omega) \dots\dots\dots\dots\dots\dots (2),$$

$$\Sigma Z + H + H' = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3),$$

$$\Sigma (yZ - zY) - Ga - G'a' = \Sigma m \{y (w' - w) - z (v' - v)\}$$
$$= - \Sigma mxz . (\omega' - \omega) \dots\dots\dots\dots (4),$$

$$\Sigma (zX - xZ) + Fa + F'a' = \Sigma m \{z (u' - u) - x (w' - w)\}$$
$$= - \Sigma myz . (\omega' - \omega) \dots\dots\dots\dots (5),$$

$$\Sigma (xY - yX) \qquad = \Sigma m (x^2 + y^2) . (\omega' - \omega) \dots\dots\dots (6).$$

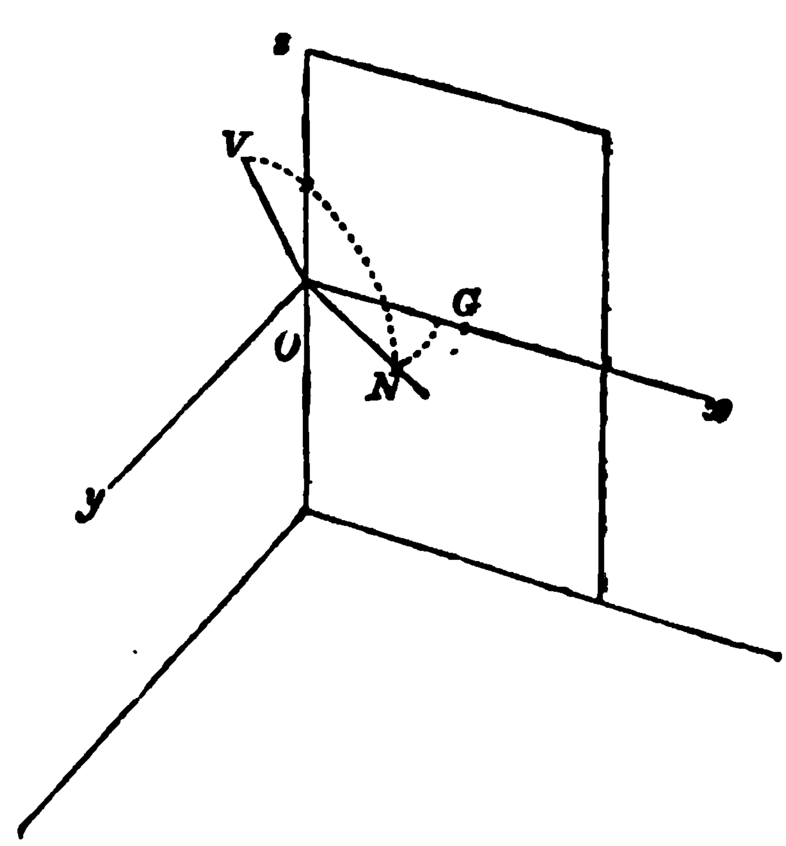
These six equations are sufficient to determine ω', F, F', G, G' and the sum $H + H'$ of the two pressures along the axis.

These equations admit of simplification when the origin can be so chosen that the axis of rotation is a principal axis at that point. In this case the right-hand sides of equations (4) and (5) vanish. Also if the plane of xz be chosen to pass through the centre of gravity of the body, we have $\bar{y} = 0$, and the right-hand side of equation (1) vanishes.

113. **Ex.** *A door is suspended by two hinges from a fixed axis making an angle α with the vertical. Find the motion and pressures on the hinges.*

Since the fixed axis is evidently a principal axis at the middle point, we shall take this point for origin. Also we shall take the plane of xz so that it contains the centre of gravity of the door at the moment under consideration.

The only force acting on the door is gravity, which may be supposed to act at the centre of gravity. We must first resolve this parallel to the axes. Let ϕ be



the angle the plane of the door makes with a vertical plane through the axis of suspension. If we draw a plane ZON such that its trace ON on the plane of XOY makes an angle ϕ with the axis of x, this will be the vertical plane through the

axis; and if we draw OV in this plane making $ZOV=\alpha$, OV will be vertical. Hence the resolved parts of gravity are

$$X=g\sin\alpha\cos\phi,\quad Y=g\sin\alpha\sin\phi,\quad Z=-g\cos\alpha.$$

Since the resolved parts of the effective forces are the same as if the whole mass were collected at the centre of gravity, the six equations of motion are

$$Mg\sin\alpha\cos\phi+F+F'=-\omega^2M\bar{x}\ldots\ldots(1),$$

$$Mg\sin\alpha\sin\phi+G+G'=fM\bar{x}\ldots\ldots(2),$$

$$-Mg\cos\alpha+H+H'=0\ldots\ldots(3),$$

$$-Ga+G'a=0\ldots\ldots(4),$$

$$Mg\cos\alpha\bar{x}+Fa-F'a=0\ldots\ldots(5),$$

because the fixed axis is a principal axis at the origin,

$$-Mg\sin\alpha\sin\phi\,.\,\bar{x}=Mk'^2\,.\,\frac{d^2\phi}{dt^2}\ldots\ldots(6).$$

Integrating the last equation, we have

$$C+2g\sin\alpha\cos\phi\bar{x}=k'^2\omega^2.$$

Suppose the door to be initially placed at rest, with its plane making an angle β with the vertical plane through the axis; then when $\phi=\beta$, $\omega=0$; hence

$$\left.\begin{aligned}k'^2\omega^2&=2g\bar{x}\sin\alpha(\cos\phi-\cos\beta)\\ \text{and}\quad k'^2f&=-g\sin\alpha\sin\phi\,.\,\bar{x}\end{aligned}\right\}.$$

By substitution in the first four equations F, F', G, G', may be found.

114. It should be noticed that these equations do not depend on the form of the body, but only on its moments and products of inertia. We may therefore replace the body by any equimomental body that may be convenient for our purpose.

This consideration will often enable us to reduce the complicated forms of Art. 111 to the simpler ones given in Art. 110. For though the body may not be symmetrical about a plane through its centre of gravity perpendicular to the axis of suspension, yet if the momental ellipsoid at the centre of gravity be symmetrical about this plane we may treat the body as if it were really symmetrical. Such a body may be said to be *Dynamically Symmetrical*. If at the same time the forces be symmetrical about the same plane, and this will always be the case if the axis of suspension be horizontal and gravity be the only force acting, we know that the pressures on the axis must certainly reduce to a single pressure, which may be found by Art. 110.

115. Ex. 1. A uniform heavy lamina in the form of a sector of a circle is suspended by a horizontal axis parallel to the radius which bisects the arc, and oscillates under the action of gravity. Show that the pressures on the axis are equivalent to a single force, and find its magnitude.

Ex. 2. An equilateral triangle oscillates about any horizontal axis situated in its own plane, show that the pressures are equivalent to a single force and find its magnitude.

116. If a body be set in rotation about any axis which is a principal axis at some point O in its length, and if there be no impressed forces acting on the body, it follows at once from these conditions that the pressures on the axis are equivalent to a single resultant force acting at O. Hence if O be fixed in space, the body will continue to rotate about that axis as if it also were fixed in space. Such an axis is called a *permanent axis of rotation* at the point O.

If the body be entirely free and yet turning about an axis of rotation which does not alter its position in space, we may suppose any point we please in the axis to be fixed. In this case the axis must be a principal axis at every point of its length. It must therefore by Art. 49 pass through the centre of gravity.

The existence of principal axes was first established by Segner in the work *Specimen Theoriæ Turbinum*. His course of investigation is the opposite of that pursued in this treatise. He defines a principal axis to be such that when a body revolves round it the forces arising from the rotation have no tendency to alter the position of the axis. From this dynamical definition he deduces the geometrical properties of these axes. The reader may consult Prof. Cayley's report to the British Association on the special problems of Dynamics, 1862, and Bossut, *Histoire de Mathématique*, Tome II.

117. Suppose the body to start from rest and to be acted on by a couple, let us discover the necessary conditions that the pressures on the fixed axis may be reduced to a single resultant pressure. Supposing such a single resultant pressure to exist, we can take as origin that point of the axis at which it is intersected by the single resultant. Then the moments of the two pressures on the axis of rotation about the co-ordinate axes will vanish. Hence since $\omega = 0$ the equations (4), (5), and (6) of Art. 112 become

$$L = -f\Sigma mxz, \quad M = -f\Sigma myz, \quad N = Mk'^2 f,$$

where we have written L, M, N for the three moments $\Sigma m(yZ - zY)$, &c. of the impressed forces about the co-ordinate axes.

The plane of the couple whose resolved parts about the axes are L, M, N, is known by Statics to be

$$LX + MY + NZ = 0,$$

or in our case,

$$-\Sigma mxzX - \Sigma myzY + Mk'^2 Z = 0 \quad \text{.................} \quad (1).$$

Let the momental ellipsoid at the fixed point be constructed, and let its equation be

$$AX^2 + BY^2 + CZ^2 - 2DYZ - 2EZX - 2FXY = \epsilon^4.$$

The equation to the diametral plane of the axis of Z is

$$-EX - DY + CZ = 0 \quad \text{.......................} \quad (2).$$

Comparing (1) and (2) we see that the plane of the resultant couple must be the diametral plane of the axis of revolution.

Since the pressures on the axis are equivalent to a single resultant force acting at some point O of the axis, we may suppose this point alone to be fixed and the axis of rotation to be otherwise free. If then a body at rest with one point fixed be acted on by any couple, it will begin to rotate about the diametral line of the plane of the couple with regard to the momental ellipsoid at the fixed point.

Thus the body will begin to rotate about a perpendicular to the plane of the couple only when the plane of the couple is parallel to a principal plane of the body at the fixed point.

If the acting couple be an impulsive couple, the equations of motion, by Art. 112, will be the same as those obtained above when ω is put zero and ω' written for f. Hence the same conclusion will follow.

The body will not in general continue to rotate about the diametral line.

118. Ex. 1. If a body at rest have one point O fixed and be acted on by any couple whose axis is a radius vector OP of the ellipsoid of gyration at O, the body will begin to turn about a perpendicular from O on the tangent plane at P.

Ex. 2. A solid ellipsoid is fixed at its centre, and is acted on by a couple in a plane whose direction-cosines referred to the principal diameters are (l, m, n). Prove that the direction-cosines of the initial axis of rotation are proportional to $\dfrac{l}{b^2+c^2}$, $\dfrac{m}{c^2+a^2}$ and $\dfrac{n}{a^2+b^2}$.

Ex. 3. Any plane section being taken of the momental ellipsoid of a body at a fixed point, the body may be made to rotate about either of the principal diameters of this section by the application of a couple of the proper magnitude whose axis is the other principal diameter.

For assume the body to be turning uniformly about the axis of z. Then the couples which must act on the body to produce this motion are $L=\omega^2\Sigma myz$, $M=-\omega^2\Sigma mxz$, $N=0$. Then by taking the axis of x such that $\Sigma mxz=0$ we see that the axis of the couple must be the axis of x and the magnitude of the couple will be $L=\omega^2\Sigma myz$.

Ex. 4. A body having one point O fixed in space is made to rotate about any proposed straight line by the application of the proper couple. The position of the axis of rotation when the magnitude of the couple is a maximum, has been called an axis of *maximum reluctance*. Show that there are six axes of maximum reluctance, two in each principal plane, each two bisecting the angles between the principal axes in the plane in which they are.

Let the axes of reference be the principal axes of the body at the fixed point, let (l, m, n) be the direction-cosines of the axis of rotation, (λ, μ, ν) those of the axis

of the couple G. Then by the last question and the fifth and sixth examples of Art. 33, we have

$$\frac{\lambda}{(B-C)\,mn}=\frac{\mu}{(C-A)\,nl}=\frac{\nu}{(A-B)\,lm},$$

$$G^2=(A-B)^2l^2m^2+(B-C)^2m^2n^2+(C-A)^2n^2l^2.$$

We have then to make G a maximum by variation of (lmn) subject to the condition $l^2+m^2+n^2=1$. The positions of these axes were first investigated by Mr Walton in the *Quarterly Journal of Mathematics*, 1865.

The Centre of Percussion.

119. When the fixed axis is given and the body can be so struck that there is no impulsive pressure on the axis, any point in the line of action of the force is called a *centre of percussion.*

When the line of action of the blow is given, the axis about which the body begins to turn is called the *axis of spontaneous rotation.* It obviously coincides with the position of the fixed axis in the first case.

PROP. *A body is capable of turning freely about a fixed axis. To determine the conditions that there shall be a centre of percussion and to find its position.*

Take the fixed axis as the axis of z, and let the plane of xz pass through the centre of gravity of the body. Let X, Y, Z be the resolved parts of the impulse, and let ξ, η, ζ be the co-ordinates of any point in its line of action. Let Mk'^2 be the moment of inertia of the body about the fixed axis. Then since $\bar{y}=0$, the equations of motion are, by Art. 61,

$$\left.\begin{aligned}X&=0\\ Y&=M\bar{x}\,(\omega'-\omega)\\ Z&=0\end{aligned}\right\}\text{ .... (1)},$$

$$\left.\begin{aligned}\eta Z-\zeta Y&=-(\omega'-\omega)\,\Sigma mxz\\ \zeta X-\xi Z&=-(\omega'-\omega)\,\Sigma myz\\ \xi Y-\eta X&=(\omega'-\omega)\,.\,Mk'^2\end{aligned}\right\}\text{.................. (2)}.$$

The impulsive pressures on the fixed axis are omitted because by hypothesis they do not exist.

From these equations we may deduce the following conditions.

I. From (1) we see that $X=0$, $Z=0$, and therefore the force must act perpendicular to the plane containing the axis and the centre of gravity.

II. Substituting from (1) in the first two equations of (2) we have $\Sigma myz=0$ and $\zeta=\dfrac{\Sigma mxz}{M\bar{x}}$. Since the origin may be taken

anywhere in the axis of rotation, let it be so chosen that $\Sigma mxz = 0$. Then the axis of z must be a principal axis at the point where a plane passing through the line of action of the blow perpendicular to the axis cuts the axis. So that there can be no centre of percussion unless the axis be a principal axis at some point in its length.

III. Substituting from (1) in the last equation of (2) we have $\xi = \frac{k'^2}{h}$. By Art. 92 this is the equation to determine the centre of oscillation of the body about the fixed axis treated as an axis of suspension. Hence the perpendicular distance between the line of action of the impulse and the fixed axis must be equal to the distance of the centre of oscillation from the axis.

If the fixed axis be parallel to a principal axis at the centre of gravity, the line of action of the blow will pass through the centre of oscillation.

The Ballistic Pendulum.

120. It is a matter of considerable importance in the Theory of Gunnery to determine the velocity of a bullet as it issues from the mouth of a gun. By means of it we obtain a complete test of any theory we have reason to form concerning the motion of the bullet in the gun; or we may find by experiment the separate effects produced by varying the length of the gun, the charge of powder, or the weight of the ball. By determining the velocity of a bullet at different distances from the gun we may discover the laws which govern the resistance of the air.

It was to determine this initial velocity that Mr Robins about 1743 invented the *Ballistic Pendulum*. Before his time but little progress had been made in the true theory of military projectiles. His *New Principles of Gunnery* was soon translated into several languages, and Euler added to his translation of it into German an extensive commentary; the work of Euler's being again translated into English in 1784. The experiments of Robins were all conducted with musket balls of about an ounce weight, but they were afterwards continued during several years by Dr Hutton, who used cannon balls of from one to nearly three pounds in weight. These last experiments are still regarded as some of the most trustworthy on smooth-bore guns.

There are two methods of applying the ballistic pendulum, both of which were used by Robins. In the first method, the gun is attached to a very heavy pendulum; when the gun is fired the recoil causes the pendulum to turn round its axis and to oscillate

through an arc which can be measured. The velocity of the bullet can be deduced from the magnitude of this arc. In the second method, the bullet is fired into a heavy pendulum. The velocity of the bullet is itself too great to be measured directly, but the angular velocity communicated to the pendulum may be made as small as we please by increasing its bulk. The arc of oscillation being measured, the velocity of the bullet can be found by calculation.

The initial velocity of small bullets may also be determined by the use of some rotational apparatus. Two circular discs of paper are attached perpendicularly to the straight line joining their centres, and are made to rotate about this straight line with a great but known angular velocity. Instead of two discs, a cylinder of paper might be used. The bullet being fired through at least two of the moving surfaces, its velocity can be calculated when the situations of the two small holes made by the bullet have been observed. This was originally an Italian invention, but it was much improved and used by Olinthus Gregory in the early part of this century.

121. *A rifle is attached in a horizontal position to a large block of wood which can turn freely about a horizontal axis. The rifle being fired, the recoil causes the pendulum to turn round its axis, until brought to rest by the action of gravity. A piece of tape is attached to the pendulum, and is drawn out of a reel during the backward motion of the pendulum, and thus serves to measure the amount of the angle of recoil. It is required to find the velocity of the bullet.*

The initial velocity of the bullet is so much greater than that of the pendulum that we may suppose the ball to have left the rifle before the pendulum has sensibly moved from its initial position. The initial momentum of the bullet may be taken as a measure of the impulse communicated to the pendulum.

Let h be the distance of the centre of gravity from the axis of suspension; f the distance from the axis of the rifle to the axis of suspension; c the distance from the axis of suspension to the point of attachment of the tape, m the mass of the bullet; M that of the pendulum and rifle, and n the ratio of M to m; b the chord of the arc of the recoil which is measured by the tape. Let k' be the radius of gyration of the rifle and pendulum about the axis of suspension, v the initial velocity of the bullet.

The explosion of the gunpowder generates an equal impulsive action on the bullet and on the rifle. Since the initial velocity of the bullet is v, this action is measured by mv. The initial angular velocity generated in the pendulum by this impulse is by Art. 89

$\omega = \dfrac{mvf}{Mk'^2}$. The subsequent motion is given (Art. 92) by the equation

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k'^2}\sin\theta\,;$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = C + \frac{2gh}{k'^2}\cos\theta:$$

when $\theta = 0$ we have $\dfrac{d\theta}{dt} = \omega$, and if α be the angle of recoil, when $\theta = \alpha$, $\dfrac{d\theta}{dt} = 0$. Hence $\omega^2 = \dfrac{2gh}{k'^2}(1 - \cos\alpha)$. Eliminating ω we have $v = \dfrac{nk'}{f} \cdot 2\sin\dfrac{\alpha}{2}\sqrt{gh}$. But the chord of the arc of the recoil is $b = 2c\sin\dfrac{\alpha}{2}$;

$$\therefore v = \frac{nbk'}{cf}\sqrt{gh}.$$

The magnitude of k' may be found experimentally by observing the time of a small oscillation of the pendulum and rifle. If T be a half-time we have $T = \pi\sqrt{\dfrac{k'^2}{gh}}$. (Art. 92.)

This is the formula given by Poisson in the second volume of his *Mécanique*. The reader will find in the *Philosophical Magazine* for June 1854, an account of some experiments conducted by *Dr S. Haughton* from which, by the use of this formula, the initial velocities of rifle bullets were calculated.

The formula must however be regarded as only a first approximation, for the recoil of the pendulum when the gun is fired without a ball has been altogether neglected. In Dr Haughton's experiments the charge of powder was comparatively small, and this assumption was nearly correct. But in some of Dr Hutton's experiments, where comparatively large charges of powder were used, the recoil without a ball was found to be very considerable.

To allow for this Dr Hutton, following Mr Robins, assumed that the effect of the charge of powder on the recoil of the gun is the same either with or without a ball. If p be the momentum generated by the powder, the whole momentum generated in the pendulum will be $mv + p$ instead of mv. Proceeding as before, we find

$$mv + p = \frac{Mbk'}{cf}\sqrt{gh}.$$

If we now repeat the experiment, with an equal charge without a ball, we have $p = \dfrac{Mb_0k'}{cf}\sqrt{gh}$, where b_0 is the chord measured by the tape. Subtracting one result from the other, we have

$$v = \frac{M}{m}\frac{(b - b_0)k'}{cf}\sqrt{gh}.$$

Thus Dr Hutton's formula differs from Poisson's in this respect, that the chord of vibration is first found for any charge without a ball and then for an equal charge with a ball: the difference of these chords is regarded as the chord which is due to the recoil of the ball.

When the magnitude of the charge of powder is small, the two methods of using the ballistic pendulum give nearly the same result. With large charges Dr Hutton found that the difference was very considerable, a less velocity being indicated by the method of observing the recoil than by that of firing the ball into the pendulum. He therefore inferred that the effect of the charge of powder on the recoil of the gun is not the same when it is fired without a ball as when it is fired with one.

We may in some measure understand the reason of this discrepancy if we consider separately the effects of the inflamed powder while the ball is in the gun and after it has left the barrel. Supposing, merely as an approximation, that the gas urging the ball forward is of uniform density; its centre of gravity, at the moment when the ball is leaving the gun, will be at the middle point of the barrel and moving relatively to the gun with half the relative velocity of the ball. If μ be the mass of the powder, the angular velocity ω' communicated to the pendulum will be given approximately by $Mk'^2\omega' = \left(m + \frac{\mu}{2}\right) vf$. After the ball has left the gun, the inflamed powder escapes from the mouth and continues to exert some pressure tending to increase the recoil. The determination of this motion is a problem in Hydrodynamics which has not yet been properly solved and which cannot be discussed here. We may, however, suppose that Robins' principle applies more nearly to this part of the motion than to the whole. If so, the momentum generated by the issuing gas, considered as an impulse, is nearly the same for a given charge and a given gun, whatever the magnitude of the ball may have been.

If p' be the momentum thus generated we have

$$\left(m + \frac{\mu}{2}\right) v + p' = \frac{Mbk'}{cf} \sqrt{gh}.$$

If v_0 and b_0 be the values of v and b when the gun is fired without a ball, we have

$$v - \frac{\mu}{2m}(v_0 - v) = \frac{M}{m} \frac{(b - b_0) k'}{cf} \sqrt{gh}.$$

Since v_0 is greater than v, this equation would show that, for considerable charges, Dr Hutton's formula will give too small a value for v. The value of v_0 is however very imperfectly known.

122. *A gun is placed in front of a heavy pendulum, which can turn freely about a horizontal axis. The ball strikes the pendulum horizontally at a distance* i *from the axis of suspension. It penetrates into the wood a short distance and communicates a momentum to the pendulum. The chord of the arc being measured as before by a piece of tape, find the velocity of the bullet.*

The time, which the bullet takes to penetrate, is so short that we may suppose it completed before the pendulum has sensibly moved from its initial position. If we follow the same notation as before, the moment of inertia of the pendulum and ball about

the axis of suspension will be $Mk'^2 + mi^2$, and the distance of the centre of gravity will be $\dfrac{Mh+mi}{M+m}$. Following the same reasoning, we find

$$v = \frac{b\sqrt{g}}{ci}\,\frac{(Mk'^2+mi^2)^{\frac{1}{2}}\,(Mh+mi)^{\frac{1}{2}}}{m}.$$

If the gun be placed as nearly as possible opposite the centre of gravity of the pendulum, we may put $h = i$ in the small terms, and since M is large compared with m the formula takes the simple form

$$v = \frac{M+m}{m}\cdot\frac{bh}{ci}\sqrt{gl},$$

where l is the distance of the centre of oscillation of the pendulum and ball from the axis of suspension.

The inconvenience of this construction as compared with the former is that the balls remain in the pendulum during the time of making one whole set of experiments. The weight, and the positions of the centres of gravity and oscillation, will be changed by the addition of each ball which is lodged in the wood. Even then the changes produced in the pendulum itself by each blow are omitted. A great improvement was made by the French in conducting their experiments at Metz in 1839, and at L'Orient in 1842. Instead of a mass of wood, requiring frequent renewals, as in the English pendulum, a permanent *récepteur* was substituted. This receiver is shaped within as a truncated cone, which is sufficiently long to prevent the shot from passing entirely through the sand with which it is filled. The front is covered with a thin sheet of lead to prevent the sand from being shaken out. This sheet is marked by a horizontal and by a vertical line, the intersection corresponding to the axial line of the cone, so that the actual position of the shot when entering the receiver can be readily determined by these lines.

Ex. 1. Show that after each bullet has been fired into a ballistic pendulum constructed on the English plan, h must be increased by $\dfrac{m}{M}(i-h)$ and l by $\dfrac{m}{M}(i-l)$ nearly in order to prepare the formula for the next shot.

Ex. 2. Dr Haughton found that, for rifles fired with a constant charge, the initial velocity of the bullet varies as the square root of the mass of the bullet inversely and as the square root of the length of the gun directly. Show from this, that the force developed by the explosion of the powder diminished by the friction of the barrel is constant as the ball traverses the rifle.

Dr Hutton found that in smooth bores the velocity increases in a ratio somewhat less than the square root of the length of the gun, but greater than the cube root of the length. Show that this might be expected from the decreased friction in a smooth bore as compared with a rifle.

Ex. 3. If the velocity of a bullet issuing from the mouth of a gun 30 inches long be 1000 feet per second, show that the time the bullet took to traverse the gun was about $\frac{1}{200}$ of a second.

Ex. 4. It has been found by experiment that if a bullet be fired into a large fixed block of wood, the penetration of the bullet into the wood varies nearly as the square of the velocity, though as the velocity is very much increased the depth of penetration falls short of that given by this rule. Assuming this rule, show that the resistance to penetration is constant and that the time of penetration is the ratio of twice the space to the initial velocity of the bullet. In an experiment of Dr Hutton's a ball fired with a velocity of 1500 feet per second was found to penetrate about 14 inches into a block of sound dry elm: show that the time of penetration was $\frac{1}{643}$ of a second.

CHAPTER IV.

MOTION IN TWO DIMENSIONS.

On the Equations of Motion.

123. THE position of a body in space of two dimensions may be determined by the co-ordinates of its centre of gravity, and the angle some straight line fixed in the body makes with some straight line fixed in space. These three have been called the co-ordinates of the body, and it is our object to determine them in terms of the time.

It will be necessary to express the effective forces of the body in terms of these co-ordinates. The resolved parts of these effective forces parallel to the axes have been already found in Art. 79, all that is now necessary is to find their moment about the centre of gravity. If (x', y') be the co-ordinates of any particle of mass m referred to rectangular axes meeting at the centre of gravity and parallel to axes fixed in space, this moment has been shown in Art. 72 to be equal to $\dfrac{dh}{dt}$, where

$$h = \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right).$$

Let θ be the "angular co-ordinate" of the body, *i.e.* the angle some straight line fixed in the body makes with some straight line fixed in space. Let (r', ϕ') be the polar co-ordinates of any particle m referred to the centre of gravity of the body as origin. Then r' is constant throughout the motion, and $\dfrac{d\phi'}{dt}$ is the same for every particle of the body and equal to $\dfrac{d\theta}{dt}$. Thus the angular momentum h, exactly as in Art. 88, is

$$\begin{aligned} h = \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) &= \Sigma m \left(r'^2 \frac{d\phi'}{dt} \right) \\ &= (\Sigma m r'^2) \frac{d\phi'}{dt} \\ &= Mk^2 \frac{d\theta}{dt}, \end{aligned}$$

where Mk^2 is the moment of inertia of the body about its centre of gravity.

The angle θ is the angle some straight line fixed in the body makes with a straight line fixed in space. Whatever straight lines are chosen $\frac{d\theta}{dt}$ is the same. If this be not obvious, it may be shown thus. Let OA, $O'A'$ be any two straight lines fixed in the body inclined at an angle α to each other. Let OB, $O'B'$ be two straight lines fixed in space inclined at an angle β to each other. Let $AOB = \theta$, $A'O'B' = \theta'$, then $\theta' + \beta = \theta + \alpha$. Since α and β are independent of the time, $\frac{d\theta}{dt} = \frac{d\theta'}{dt}$. By this proposition we learn that the angular velocities of a body in two dimensions are the same about all points.

The general method of proceeding will be as follows.

Let (x, y) be the co-ordinates of the centre of gravity of any body of the system referred to rectangular axes fixed in space, M the mass of the body. Then the effective forces of the body are together equivalent to two forces measured by $M\frac{d^2x}{dt^2}$, $M\frac{d^2y}{dt^2}$ acting at the centre of gravity and parallel to the axes of co-ordinates, together with a couple measured by $Mk^2\frac{d^2\theta}{dt^2}$ tending to turn the body about its centre of gravity in the direction in which θ is measured. By D'Alembert's principle the effective forces of all the bodies, if reversed, will be in equilibrium with the impressed forces. The dynamical equations may then be formed according to the ordinary rules of Statics.

For example, if we took moments about a point whose co-ordinates are (p, q) we should have an equation of the form

$$M\left\{(x-p)\frac{d^2y}{dt^2} - (y-q)\frac{d^2x}{dt^2}\right\} + Mk^2\frac{d^2\theta}{dt^2} = L,$$

where L is the moment of the impressed forces and the other letters have the same meaning as before. In this equation (p, q) may be the co-ordinates of any point whatever, whether fixed or moving. Just as in a statical problem, the solution of the equations may frequently be much simplified by a proper choice of the point about which to take moments. Thus if we wished to avoid the introduction into our equations of some unknown reaction, we might take moments about the point of application or use the principle of virtual velocities. So again in resolving

our forces we might replace the Cartesian expressions $M\dfrac{d^2x}{dt^2}$, $M\dfrac{d^2y}{dt^2}$ by the polar forms

$$M\left\{\frac{d^2r}{dt}-r\left(\frac{d\phi}{dt}\right)^2\right\} \text{ and } M\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)$$

for the resolved parts parallel and perpendicular to the radius vector. If v be the velocity of the centre of gravity, ρ the radius of curvature of its path, we may sometimes also use with advantage the forms $M\dfrac{dv}{dt}$ and $M\dfrac{v^2}{\rho}$ for the resolved parts of the effective forces along the tangent and radius of curvature of the path of the centre of gravity.

124. As we shall have so frequently to use the equation formed by taking moments, it is important to consider other forms into which it may be put. Let the point about which we are to take moments be *fixed in space*, so that it may be chosen as the origin of co-ordinates. Then the moment of the effective forces on the body M is

$$\frac{d}{dt}\left\{M\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)+Mk^2\frac{d\theta}{dt}\right\}=L.$$

The attention of the reader is directed to the meaning of the several parts of this expression. We see that, as explained in Art. 72, the moment of the effective forces is the differential coefficient of the moment of the momentum about the same point. The moment of the momentum by Art. 76 is the same as the moment about the centre of gravity together with the moment of the whole mass collected at the centre of gravity, and moving with the velocity of the centre of gravity. The moment round the centre of gravity is by the first Article either of Chap. III. or Chap. IV. equal to $Mk^2\dfrac{d\theta}{dt}$ and the moment of the collected mass is $M\left(x\dfrac{dy}{dt}-y\dfrac{dx}{dt}\right)$, where (x, y) are the co-ordinates of the centre of gravity. Hence in space of two dimensions we have for any body of mass M

$$\left.\begin{array}{c}\text{angular momentum round}\\ \text{the origin}\end{array}\right\}=M\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)+Mk^2\frac{d\theta}{dt}.$$

If we prefer to use polar co-ordinates, we can put this into another form. Let (r, ϕ) be the polar co-ordinates of the centre of gravity, then,

$$\left.\begin{array}{c}\text{angular momentum round}\\ \text{the origin}\end{array}\right\}=Mr^2\frac{d\phi}{dt}+Mk^2\frac{d\theta}{dt}.$$

If v be the velocity of the centre of gravity, and p the perpendicular from the origin on the tangent to its direction of

motion, the moment of momentum of the mass collected at the centre of gravity is Mvp, so that we also have

$$\left.\begin{array}{c}\text{angular momentum round}\\ \text{the origin}\end{array}\right\} = Mvp + Mk^2\frac{d\theta}{dt}.$$

It is clear from Art. 76 that this is the instantaneous angular momentum of the body about the origin, whether it is fixed or moveable, though in the latter case its differential coefficient with regard to t is not the moment of the effective forces.

Since the instantaneous centre of rotation may be regarded as a fixed point, when we have to deal only with the coordinates and with their first differential coefficients with regard to the time, we have

$$\left.\begin{array}{c}\text{angular momentum round the}\\ \text{instantaneous centre}\end{array}\right\} = M(r^2 + k^2)\frac{d\theta}{dt}.$$

If Mk'^2 be the moment of inertia about the instantaneous centre, this last moment may be written $Mk'^2\dfrac{d\theta}{dt}$.

In taking moments about any point whether it be the centre of gravity or not, it should be noticed that the Mk^2 in all these formulæ is the moment of inertia with regard to the centre of gravity, and not with regard to the point about which we are taking moments. It is only when we are taking moments about the instantaneous centre or about a fixed point that we can use the moment of inertia about that point instead of the moment of inertia about the centre of gravity, and in these cases our expression for the angular momentum includes the angular momentum of the mass collected at the centre of gravity.

125. Suppose we form the equations of motion of each body by resolving parallel to the axes of co-ordinates and by taking moments about the centre of gravity. We shall get three equations for each body of the form

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= F\cos\phi + R\cos\psi + \dots \\ M\frac{d^2y}{dt^2} &= F\sin\phi + R\sin\psi + \dots \\ Mk^2\frac{d^2\theta}{dt^2} &= \quad Fp \quad + Rq \quad\quad + \dots \end{aligned}\right\}\dots\dots\dots\dots\dots(1),$$

where F is any one of the impressed forces acting on the body, whose resolved parts are $F\cos\phi$, $F\sin\phi$, and whose moment about the centre of gravity is Fp, and R is any one of the reactions. These we shall call the *Dynamical equations* of the body.

Besides these there will be certain geometrical equations expressing the connections of the system. As every such forced connection is accompanied by a reaction and every reaction by some forced connection, the number of geometrical equations will be the same as the number of unknown reactions in the system.

Having obtained the proper number of equations of motion we proceed to their solution. Two general methods have been proposed.

First Method. Differentiate the geometrical equations twice with respect to t, and substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, from the dynamical equations. We shall then have a sufficient number of equations to determine the reactions. This method will be of great advantage whenever the geometrical equations are of the form

$$Ax + By + C\theta = D \qquad \text{(2)},$$

where A, B, C, D are constants. Suppose also that the dynamical equations are such that when written in the form (1) they contain only the reactions and constants on the right-hand side without any x, y, or θ. Then, when we substitute in the equation

$$A\frac{d^2x}{dt^2} + B\frac{d^2y}{dt^2} + C\frac{d^2\theta}{dt^2} = 0,$$

obtained by differentiating (1), we have an equation containing only the reactions and constants. This being true for all the geometrical relations, it is evident that all the reactions will be constant throughout the motion and their values may be found. Hence when these values are substituted in the dynamical equations (1), their right-hand members will all be constants and the values of x, y, and θ may be found by an easy integration.

If however the geometrical equations are not of the form (2), this method of solution will usually fail. For suppose any geometrical equation took the form

$$x^2 + y^2 = c^2,$$

containing *squares* instead of *first* powers, then its second differential equation will be

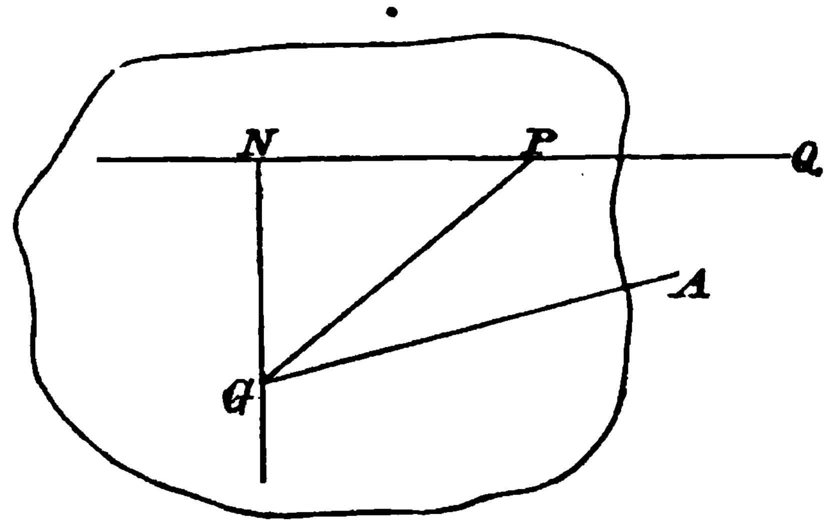
$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 0;$$

and though we can substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, we cannot, in general, eliminate the terms $\left(\frac{dx}{dt}\right)^2$ and $\left(\frac{dy}{dt}\right)^2$.

126. The reactions in a dynamical problem are in many cases produced by the pressures of some smooth fixed obstacles which are touched by the moving bodies. Such obstacles can only *push*, and therefore if the equation showed that such a reaction changes sign at any instant, it is clear that the body will leave the obstacle at that instant. This will occasionally introduce discontinuity into our equations. At first the system moves under certain constraints, and our equations are found on that supposition. At some instant which may be determined by the vanishing of some reaction, one of the bodies leaves its constraints and the equations of motion have to be changed by the omission of this reaction. Similar remarks apply if the reactions be produced by the pressure of one body against another.

It is important to notice that when this first method of solution applies, the reactions are constant throughout the motion, so that this kind of discontinuity can never occur. If a moving body be in contact with another, they will either separate at the beginning of the motion or will always continue in contact.

127. Suppose that in a dynamical system we have two bodies which press on each other with a reaction R; let us consider how we should form the corresponding geometrical equation. We have clearly to express the fact that the velocities of the points of contact of the two bodies resolved along the direction of R are equal. The following proposition will be often useful. Let a body be turning about a point G with an angular velocity $\frac{d\theta}{dt} = \omega$ in a direction opposite to the hands of a watch, and let G be moving in the direction GA with a velocity V. It is required to find the velocity of any point P resolved in any



direction PQ, making an angle ϕ with GA. In the time dt the whole body, and therefore also the point P, is moved through a space Vdt parallel to GA, and during the same time P is moved perpendicular to GP through a space $\omega . GP . dt$. Resolving parallel to PQ, the whole displacement of P

$$= (V \cos \phi - \omega . GP \sin GPN)\, dt.$$

If $GN = p$ be the perpendicular from G on PQ, we see that the velocity of P parallel to PQ is $= V\cos\phi - \omega p$.

It should be noticed that this is independent of the position of P on the straight line PQ. It follows that the velocities of all points in any straight line PQ resolved along PQ are the same. In practice, therefore, we only use that point in the direction of PQ which is most convenient, and this is generally the foot of the perpendicular from the centre of gravity.

If (x, y, θ), (x', y', θ') be the co-ordinates of the two bodies, q, q' the perpendiculars from the points (x, y), (x', y') on the direction of any reaction R, ψ the angle the direction of R makes with the axis of x, the required geometrical equation will be

$$\frac{dx}{dt}\cos\psi + \frac{dy}{dt}\sin\psi + \frac{d\theta}{dt}q = \frac{dx'}{dt}\cos\psi + \frac{dy'}{dt}\sin\psi + \frac{d\theta'}{dt}q'.$$

If the bodies be perfectly rough and roll on each other without sliding, there will be *two* reactions at the point of contact, one normal and the other tangential to the common surface of the touching bodies. For each of these we shall have an equation similar to that just found. But if there be any *sliding* friction this reasoning will not apply. This case will be considered a little further on.

128. *Second Method of Solution.* Suppose in a dynamical system two bodies of masses M, M' are pressing on each other with a reaction R. Let the equations of motion of M be those marked (1) in Art. 125, and let those of M' be obtained from these by accenting all the letters except R, ψ and t, and writing $-R$ for R, ψ and t being of course unaltered. Let us multiply the equation of M by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{d\theta}{dt}$ respectively, and those of M' by corresponding quantities. Adding all these six equations, we get

$$\begin{aligned}
2M\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt} + k^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right) &+ \&c. \\
= 2F\left(\cos\phi\frac{dx}{dt} + \sin\phi\frac{dy}{dt} + p\frac{d\theta}{dt}\right) &+ \&c. \\
+ 2R\left(\cos\psi\frac{dx}{dt} + \sin\psi\frac{dy}{dt} + q\frac{d\theta}{dt}\right)& \\
- 2R\left(\cos\psi\frac{dx'}{dt} + \sin\psi\frac{dy'}{dt} + q'\frac{d\theta'}{dt}\right)&.
\end{aligned}$$

The coefficient of R will vanish by virtue of the geometrical equation obtained in the last Article. And this reasoning will apply to all the reactions between each two of the moving bodies.

Suppose the body M to press against some external fixed obstacle, then in this case R acts only on the body M, and its coefficient will be restricted to the part included in the first bracket. But the velocity of the point of contact resolved along the direction of R must vanish, and therefore the coefficient of R is again zero.

Let A be the point of application of the impressed force F, and let $\frac{df}{dt}$ be the velocity of A resolved along the direction of action of F. Then we see that the coefficient of $2F$ is $\frac{df}{dt}$. It also follows from the definition of $\frac{df}{dt}$ that Fdf is what is called in Statics the virtual moment of the force F.

We have thus a general method of obtaining an equation free from the unknown reactions of perfectly smooth or perfectly rough bodies. The rule is, Multiply the equations having $M\frac{d^2x}{dt^2}$, $M\frac{d^2y}{dt^2}$, $Mk^2\frac{d^2\theta}{dt^2}$, &c. on their left-hand sides by $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d\theta}{dt}$, &c., and add together all the resulting equations for all the bodies. The coefficients of all the unknown reactions will be found to be zero by virtue of the geometrical equations.

The left-hand side of the equation thus obtained is clearly a perfect differential. Integrating we get

$$M\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2\right\}+\text{\&c.}=C+2\int Fdf+\ldots\ldots$$

where C is the constant of integration.

In practice it is usual to omit all the intermediate steps and write down the resulting equation in the following manner:

$$\Sigma M\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2\right\}=C+2U,$$

where U is the integral of the virtual moment of the forces.

This is called the equation of *Vis Viva*. Another proof will be given in the chapter under that heading.

129. The left-hand side of this equation is called the ***vis viva*** of the whole system. Taking any one body M, we may say that

$$\text{vis viva of } M=M\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2\right\}.$$

If the whole mass were collected into its centre of gravity and were to move with the velocity of the centre of gravity, k would be

zero, and the vis viva would be reduced to the two first terms. These terms are therefore together called the *vis viva of translation*, and the last term is called the *vis viva of rotation.*

If v be the velocity of the centre of gravity, we may write this equation

$$\text{vis viva of } M = Mv^2 + Mk^2 \left(\frac{d\theta}{dt}\right)^2.$$

If we wish to use polar co-ordinates, we have

$$\text{vis viva of } M = M\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 + k^2\left(\frac{d\theta}{dt}\right)^2\right\}$$

where (r, ϕ) are the polar co-ordinates of the centre of gravity.

If ρ be the distance of the centre of gravity from the instantaneous centre of rotation of the body, $\rho\frac{d\theta}{dt}$ is clearly the velocity of the centre of gravity, and therefore

$$\text{vis viva of } M = M(\rho^2 + k^2)\left(\frac{d\theta}{dt}\right)^2.$$

The right-hand side of the equation of vis viva, after division by 2, is called sometimes the *force function* of the forces and sometimes the *work* of the forces. It may always be obtained by writing down the virtual moment of the forces according to the rules of Statics, integrating the result and adding a constant.

Frequently it is convenient to avoid introducing the unknown constant C by taking the integral between limits. We then subtract from the left side the initial vis viva, and from the right side the initial value of the force function.

130. If there is only one way in which the system can move, that motion will be determined by the equation of vis viva. But if there be more than one possible motion, we must find another integral of the equations of the second order. What should be done will depend on the special case under consideration. The discovery of the proper treatment of the equations is often a matter of great difficulty. The difficulty will be increased, if in forming the equations care has not been taken that they should have the simplest possible forms.

131. In many cases a great simplification of the equations will be effected by a proper choice of the direction in which to resolve the forces, or of the points about which we take moments.

First we should search if there be any direction in which the resolved part of the impressed forces vanishes. By resolving in this direction we get an equation which can be immediately

integrated. Suppose the axis of x to be taken in this direction; let M, M', &c. be the masses of the several bodies, x, x', &c. the abscissæ of their centres of gravity, then by Art. 123 we have

$$M\frac{d^2x}{dt^2}+M'\frac{d^2x'}{dt^2}+\ldots=0,$$

which by integration gives

$$M\frac{dx}{dt}+M'\frac{dx'}{dt}+\ldots\ldots=C,$$

where C is some constant to be found from the initial conditions. This equation may also be again integrated if required.

This result might have been derived from the general principles of the conservation of the translation of the centre of gravity laid down in Art. 79. For since there is no impressed force parallel to the axis of x, the velocity of the centre of gravity of the *whole system* resolved in that direction is constant.

132. Next we should search if there be any point about which the moment of the impressed forces vanishes. By taking moments about that point we again have an equation which admits of immediate integration. Suppose this point to be taken as origin, and the letters to have their usual meaning, then by the first article of this chapter we have

$$\Sigma\left\{M\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)+Mk^2\frac{d^2\theta}{dt^2}\right\}=0,$$

the Σ referring to summation for all the bodies of the system Integrating as in Art. 124 we have

$$\Sigma\left\{M\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)+Mk^2\frac{d\theta}{dt}\right\}=C,$$

where C is some constant to be determined by the initial conditions of the question.

This equation expresses that if the impressed forces have no moment about any point, the angular momentum about that point is constant throughout the motion. This result follows at once from the reasoning in Chap. II.

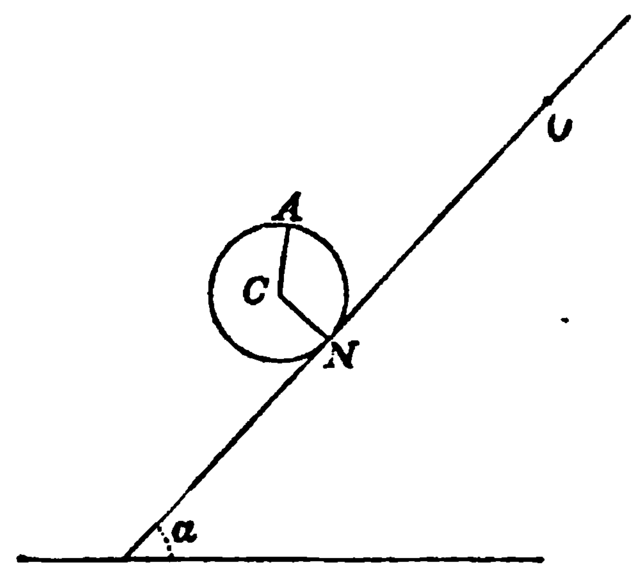
133. *A homogeneous sphere rolls directly down a perfectly rough inclined plane under the action of gravity. Find the motion.*

Let α be the inclination of the plane to the horizon, a the radius of the sphere, mk^2 its moment of inertia about a horizontal diameter.

Let O be that point of the inclined plane which was initially touched by the sphere, and N the point of contact at the time t. Then it is obviously convenient to choose O for origin and ON for the axis of x.

The forces which act on the sphere are first the reaction R perpendicular to ON, secondly, F the friction acting at N along NO and mg acting vertically at C the centre.

The effective forces are $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$ acting at C parallel to the axes of x and y and a couple $mk^2\frac{d^2\theta}{dt^2}$ tending to turn the sphere round C in the direction NA.



Here θ is the angle any fixed straight line in the body makes with a fixed straight line in space. We shall take the fixed straight line in the body to be the radius CA, and the fixed straight line in space the normal to the inclined plane. Then θ is the angle turned through by the sphere.

Resolving along and perpendicular to the inclined plane we have

$$m\frac{d^2x}{dt^2} = mg\sin\alpha - F \quad\text{.................................... (1)},$$

$$m\frac{d^2y}{dt^2} = -mg\cos\alpha + R \quad\text{.................................. (2)}.$$

Taking moments about N to avoid the reactions, we have

$$ma\frac{d^2x}{dt^2} + mk^2\frac{d^2\theta}{dt^2} = mga\sin\alpha \quad\text{.......................... (3)}.$$

Since there are two unknown reactions F and R, we shall require two geometrical relations. Because there is no slipping at N, we have

$$x = a\theta \quad\text{... (4)}.$$

Also because there is no jumping $y = a$ .... (5).

Both these equations are of the form described in the first method. Differentiating (4) we get $\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}$. Joining this to (3) we have

$$\frac{d^2x}{dt^2} = \frac{a^2}{a^2+k^2}g\sin\alpha \quad\text{.................................... (6)}.$$

Since the sphere is homogeneous, $k^2 = \frac{2}{5}a^2$, and we have

$$\frac{d^2x}{dt^2} = \frac{5}{7}g\sin\alpha.$$

If the sphere had been sliding down a *smooth* plane, the equation of motion would have been

$$\frac{d^2x}{dt^2} = g\sin\alpha,$$

so that two-sevenths of gravity is used in turning the sphere, and five-sevenths in urging the sphere downwards. Supposing the sphere to start from rest we have clearly

$$x = \frac{1}{2} \cdot \frac{5}{7} g \sin \alpha \,.\, t^2,$$

and the whole motion is determined.

In the above solutions, only a few of the equations of motion have been used, and if only the motion had been required it would have been unnecessary to write down any equations except (3) and (4). If the reactions also be required, we must use the remaining equations. From (1) we have

$$F = \frac{2}{7} mg \sin \alpha.$$

From (2) and (5) we have

$$R = mg \cos \alpha.$$

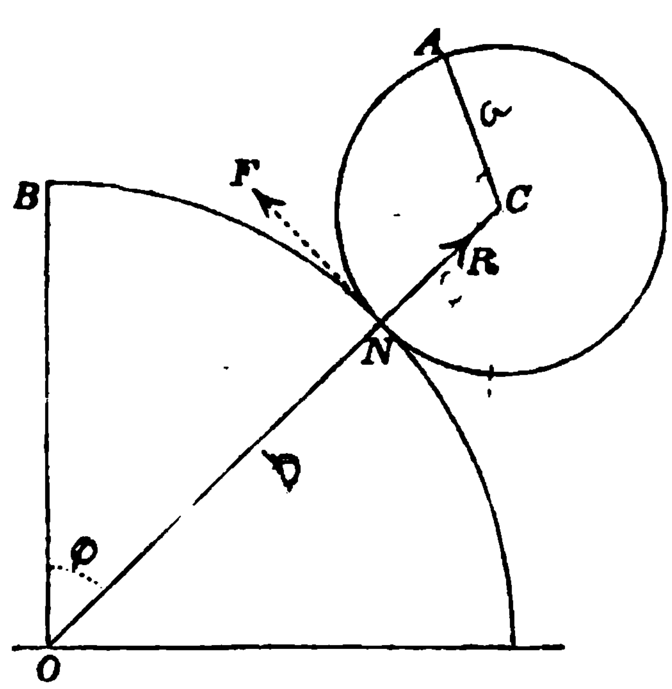
It is usual to delay the substitution of the value of k^2 in the equations until the end of the investigation, for this value is often very complicated. But there is another advantage. It serves as a verification of the signs in our original equations, for if equation (6) had been

$$\frac{d^2x}{dt^2} = \frac{a^2}{a^2 - k^2} g \sin \alpha,$$

we should have expected some error to exist in the solution. For it seems clear that the acceleration could not be made infinite by any alteration of the internal structure of the sphere.

Ex. If the plane were imperfectly rough with a coefficient of friction μ less than $\frac{2}{7}\sin\alpha$, show that the angular velocity of the sphere after a time t from rest would be $\frac{5\mu}{2}\frac{g\cos\alpha}{a}t$.

184. *A homogeneous sphere rolls down another perfectly rough fixed sphere. Find the motion.*



Let a and b be the radii of the moving and fixed spheres, respectively, C and O the two centres.

Let OB be the vertical radius of the fixed sphere, and $\phi = \angle BOC$. Let F and R be the friction and the normal reaction at N. Then resolving tangentially and normally to the path of C we have

$$(a+b)\frac{d^2\phi}{dt^2}=g\sin\phi-\frac{F}{m}\quad\text{.................................(1)},$$

$$(a+b)\left(\frac{d\phi}{dt}\right)^2=g\cos\phi-\frac{R}{m}\quad\text{.................................(2)}.$$

Let A be that point of the moving sphere which originally coincided with B. Then if θ be the angle which any fixed line, as CA, in the body makes with any fixed line in space, as the vertical, we have by taking moments about C

$$\frac{d^2\theta}{dt^2}=\frac{Fa}{mk^2}\quad\text{...(3)}.$$

It should be observed that we cannot take θ as the angle ACO because, though CA is fixed in the body, CO is not fixed in space.

The geometrical equation is clearly

$$a(\theta-\phi)=b\phi\quad\text{.......................................(4)}.$$

No other is wanted, since in forming equations (1) and (2) the constancy of the distance CO has been already supposed.

The form of equation (4) shows that we can apply the first method. We thus obtain

$$F=\frac{k^2}{k^2+a^2}mg\sin\phi,$$

and we are finally led to the equation

$$(a+b)\frac{d^2\phi}{dt^2}=\frac{5}{7}g\sin\phi.$$

By multiplying by $2\dfrac{d\phi}{dt}$ and integrating we get after determining the constant

$$\left(\frac{d\phi}{dt}\right)^2=\frac{10}{7}\frac{g}{a+b}(1-\cos\phi),$$

the rolling body being supposed to start from rest at a point indefinitely near B.

This result might also have been deduced from the equation of vis viva. The vis viva of the sphere is $m\left\{v^2+k^2\left(\dfrac{d\theta}{dt}\right)^2\right\}$ and $v=(a+b)\dfrac{d\phi}{dt}$. The force function is $m\int g\,dy=mgy$ if y be the vertical space descended by the centre. We thus have

$$(a+b)^2\left(\frac{d\phi}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2=2g(a+b)(1-\cos\phi),$$

which is easily seen to lead, by help of (4), to the same result.

To find where the body leaves the sphere we must put $R=0$. This gives by (2) $(a+b)\left(\dfrac{d\phi}{dt}\right)^2=g\cos\phi$; $\therefore\ \dfrac{10}{7}g(1-\cos\phi)=g\cos\phi$; $\therefore\ \cos\phi=\dfrac{10}{17}$. It may be remarked that this result is independent of the magnitudes of the spheres.

Ex. 1. If the spheres had been smooth the upper sphere would have left the lower sphere when $\cos\phi=\frac{2}{3}$.

Ex. 2. A rod rests with one extremity on a smooth horizontal plane and the other on a smooth vertical wall at an inclination α to the horizon. If it then slips down, show that it will leave the wall when its inclination is $\sin^{-1}(\frac{2}{3}\sin\alpha)$.

Ex. 3. A beam is rotating on a smooth horizontal plane about one extremity, which is fixed, under the action of no forces except the resistance of the atmosphere. Supposing the retarding effect of the resistance on a small element of the beam of length ϵ to be $A\epsilon$ (vel.)^2, then the angular velocity at the time t is given by

$$\frac{1}{\omega}-\frac{1}{\Omega}=\frac{A\alpha^4}{4M\kappa^2}t. \quad \text{[Queens' Coll.]}$$

Ex. 4. An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the inclined plane, x the part of the plane rolled over by the sphere, prove that

$$(M+m)\,x'=mx\cos\alpha,$$

$$\tfrac{7}{5}x-\cos\alpha x'=\tfrac{1}{2}g\sin\alpha t^2,$$

where α is the inclination of the plane to the horizon.

Ex. 5. Two equal perfectly rough spheres are placed in unstable equilibrium, one on the top of the other; the lower sphere resting on a perfectly smooth table. The slightest disturbance being given to the system, shew that the spheres will continue to touch each other at the same points and if θ be the inclination to the vertical of the straight line joining the centres,

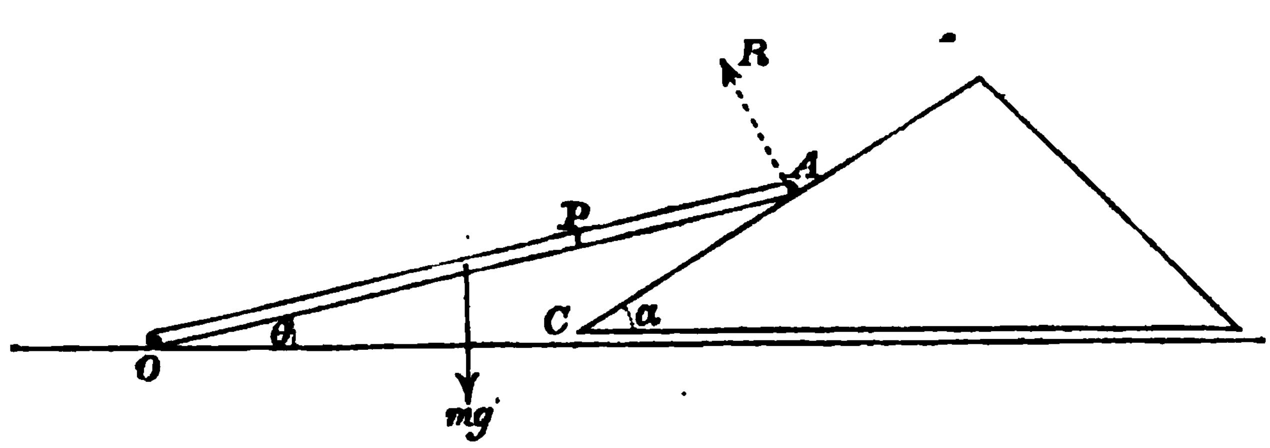
$$(k^2+a^2+a^2\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2=2ga\,(1-\cos\theta).$$

Ex. 6. Two unequal perfectly smooth spheres are placed in unstable equilibrium one on the top of the other; the lower sphere resting on a perfectly smooth table. A very slight disturbance being given to the system, shew that the spheres will separate when the straight line joining the centres makes an angle ϕ with the vertical, given by the equation $\dfrac{m}{M+m}\cos^3\phi-3\cos\phi+2=0$, where M is the mass of the lower and m of the upper sphere.

Ex. 7. A sphere of mass M and radius a is constrained to roll on a perfectly rough curve of any form and initially the velocity of its centre of gravity is V. If the initial velocity were changed to V', shew that the normal reaction would be increased by $M\dfrac{V'^2-V^2}{\rho-a}$ and that the friction would be unaltered, ρ being the radius of curvature of the curve at the point of contact.

135. *A rod OA can turn about a hinge at O, while the end A rests on a smooth wedge which can slide along a smooth horizontal plane through O. Find the motion.*

Let α = the inclination of the wedge, M = its mass and $x = OC$.



Let l = the length of the beam, m = its mass and $\theta = AOC$.

Let R = the reaction at A. Then we have

the dynamical equations,

$$\frac{d^2x}{dt^2} = \frac{R \sin \alpha}{M} \quad \text{...} (1),$$

$$\frac{d^2\theta}{dt^2} = \frac{Rl \,.\cos(\alpha - \theta) - mg\dfrac{l}{2}\cos\theta}{mk^2} \quad \text{.........................} (2),$$

and the geometrical equation,

$$x = \frac{l}{\sin \alpha} \,.\sin(\alpha - \theta) \quad \text{.....................................} (3).$$

It is obvious we must apply the second method of solution. Hence

$$2M\frac{dx}{dt}\frac{d^2x}{dt^2} + 2mk^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -mgl\cos\theta\frac{d\theta}{dt} + 2R\left\{\sin\alpha\frac{dx}{dt} + l\cos(\alpha-\theta)\frac{d\theta}{dt}\right\}.$$

The coefficient of R is seen to vanish by differentiating equation (3). Integrating we have

$$M\left(\frac{dx}{dt}\right)^2 + mk^2\left(\frac{d\theta}{dt}\right)^2 = C - mgl\sin\theta.$$

This result might have been written down at once by the principle of vis viva. For the vis viva of the wedge is clearly $M\left(\frac{dx}{dt}\right)^2$ and that of the rod $Mk^2\left(\frac{d\theta}{dt}\right)^2$. The virtual moment of the forces is $-mgdy$ where y is the altitude above OC of the centre of gravity of the rod OA, hence twice the force function is $C - 2mgy$. Since $y = \frac{1}{2}l\sin\theta$, this reduces to the result already written down.

Substituting from (3) we have

$$\left\{M\frac{l^2}{\sin^2\alpha}\cos^2(\alpha-\theta) + mk^2\right\}\left(\frac{d\theta}{dt}\right)^2 = C - mgl\sin\theta \quad \text{.................} (4).$$

If the beam start from rest when $\theta = \beta$, then $C = mgl\sin\beta$.

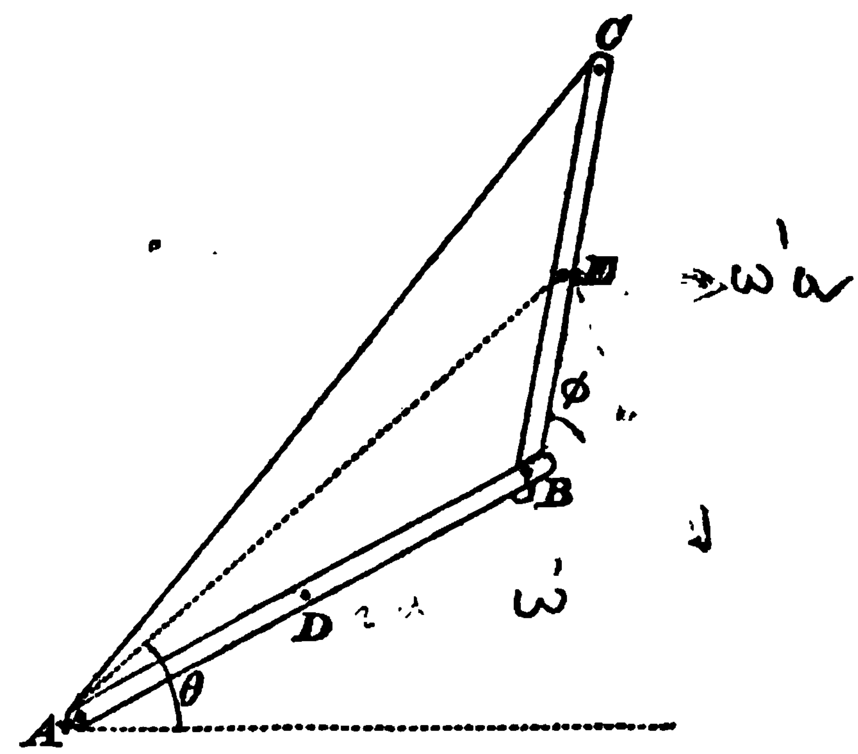
This equation cannot be integrated any further. We cannot therefore find θ in terms of t. But the angular velocity of the beam, and therefore the velocity of the wedge, is given by the above equation.

136. *Two rods AB, BC are hinged together at B and can freely slide on a smooth horizontal plane. The extremity A of the rod AB is attached by another hinge to a fixed point on the table. An elastic string AC, whose unstretched length is equal to AB or BC, joins A to the extremity C of the rod BC. Initially the two rods and the string form an equilateral triangle and the system is started with an angular velocity Ω round A. Find the greatest length of the elastic string during the motion. Find also the angular velocities of the rods when they are at right angles, and the least value of Ω that this may be possible.*

Let the length of either rod be $2a$, mk^2 the moment of inertia of either about its centre of gravity, so that $k^2 = \frac{a^2}{3}$. Let D and E be the middle points of the rods, and let (r, θ) be the polar co-ordinates of E referred to A as origin.

The only forces on the system are the reaction of the hinge at A and the tension of the elastic string AC. If we search for any direction in which the sum of the resolved parts of these vanishes, we can find none, since the direction of the

reaction is at present unknown. But since the lines of action of both these forces pass through A, their moments about A vanish, and therefore, by Art. 132, the angular momentum about A is constant throughout the motion and equal to its initial value. Let ω, ω' be the angular velocities of AB, BC at any instant t. The



angular momentum of BC about A is by Art. 124 $m\left(r^2\dfrac{d\theta}{dt}+k^2\omega'\right)$. The angular momentum of AB is by the same article $m(k^2+a^2)\,\omega$, since AB is turning about A as a fixed point. The initial values of these are respectively $m(3a^2\Omega+k^2\Omega)$, and $m(k^2+a^2)\,\Omega$, since ω, ω' and $\dfrac{d\theta}{dt}$ are each initially equal to Ω, and r is initially equal to the perpendicular from A on the opposite side of the equilateral triangle formed by the system. Hence

$$m(k^2+a^2)\,\omega+mk^2\omega'+mr^2\frac{d\theta}{dt}=m(2k^2+4a^2)\,\Omega\ldots\ldots\ldots\ldots(1).$$

We may obtain another equation by the use of the principle of vis viva. The vis viva of the rod BC is by Art. 129 $m\left\{\left(\dfrac{dr}{dt}\right)^2+r^2\left(\dfrac{d\theta}{dt}\right)^2+k^2\omega'^2\right\}$. The vis viva of AB is by the same article $m(k^2+a^2)\,\omega^2$ since it is turning round A as a fixed point. The initial values of these are respectively $m(3a^2+k^2)\,\Omega^2$ and $m(k^2+a^2)\,\Omega^2$. If T be the tension of the string, ρ its length at time t, the force function of the tension is $\int_{2a}^{\rho}(-T)\,d\rho$. According to the rule given in Statics to calculate virtual moments, the minus sign is given to the tension because it acts to diminish ρ; and the limits are $2a$ to ρ because the string has stretched from its initial length $2a$ to ρ. By Hooke's law $T=E\dfrac{\rho-2a}{2a}$, so that, by integration, the force function $=-E\dfrac{(\rho-2a)^2}{4a}$. The reaction at A does not appear by Art. 128. The equation of vis viva is therefore

$$m(k^2+a^2)\,\omega^2+m\left\{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2+k^2\omega'^2\right\}=m(2k^2+4a^2)\,\Omega^2-E\frac{(\rho-2a)^2}{2a}\ldots\ldots\ldots(2).$$

There are only two possible independent motions of the rods. We can turn AB about A and BC about B, all other motions, not compounded of these, are inconsistent with the geometrical conditions of the question. Two dynamical equations

are sufficient to determine these, and these we have just obtained. All the other equations which may be wanted must be derived from geometrical considerations.

We must now express the geometrical conditions of the question. Let ϕ be the supplement of the angle ABC, then

$$r^2 = 5a^2 + 4a^2 \cos\phi \quad\text{.......................................(3).}$$

Since $\dfrac{d\phi}{dt}$ is the relative angular velocity of the rods BC, AB,

$$\frac{d\phi}{dt} = \omega' - \omega \quad\text{.......................................(4),}$$

$$\therefore\ r\frac{dr}{dt} = -2a^2 \sin\phi\,(\omega' - \omega) \quad\text{...........................(5).}$$

Let ψ be the angle EAB, then

$$\sin\psi = \sin\phi\,\frac{a}{r} \quad\text{.......................................(6),}$$

and since $\dfrac{d\psi}{dt} = \dfrac{d\theta}{dt} - \omega$, we have

$$\cos\psi\left(\frac{d\theta}{dt} - \omega\right) = \left(\frac{a}{r}\cos\phi + \frac{2a^3}{r^3}\sin^2\phi\right)(\omega' - \omega) \quad\text{..................(7).}$$

Also from the triangle ABC

$$\rho^2 + 2a^2 = 2r^2 \quad\text{.......................................(8).}$$

From these eight equations we can eliminate ω, ω', r, $\dfrac{dr}{dt}$, ρ, ψ and $\dfrac{d\theta}{dt}$. We shall then have a differential equation of the first order to solve, containing ϕ and $\dfrac{d\phi}{dt}$.

It is required to find the greatest length of the elastic string during the motion. At the moment when ρ is a maximum, $\dfrac{d\rho}{dt} = 0$ and the whole system is therefore moving as if it were a rigid body. We therefore have for a single moment ω, ω' and $\dfrac{d\theta}{dt}$ all equal to each other and $\dfrac{dr}{dt} = 0$. The two first equations become, when we have substituted for k^2 its value $\dfrac{a^2}{3}$,

$$\left.\begin{aligned}(5a^2 + 3r^2)\,\omega &= 14a^2\Omega\\ (5a^2 + 3r^2)\,\omega^2 &= 14a^2\Omega^2 - \frac{3E}{2am}(\rho - 2a)^2\end{aligned}\right\}.$$

Eliminating ω and substituting for r from (8) we have the cubic

$$(3\rho^2 + 16a^2)(\rho - 2a) = \frac{28m\Omega^2a^3}{E}\cdot(\rho + 2a),$$

which has one positive root greater than $2a$.

It is also required to find the motion at the instant when the rods are at right angles. At this moment $\phi = \dfrac{\pi}{2}$ and hence by (3) $r = a\sqrt{5}$, by (5) $\dfrac{dr}{dt} = -\dfrac{2}{\sqrt{5}}a(\omega' - \omega)$, by (7) $\dfrac{d\theta}{dt} = \dfrac{1}{5}(\omega' + 4\omega)$. Substituting in equations (1) and (2) we get

$$\left.\begin{aligned}4\omega + \omega' &= \frac{7}{2}\Omega\\ 4\omega^2 + \omega'^2 &= \frac{7}{2}\Omega^2 - \frac{E}{ma}\frac{3(\sqrt{2} - 1)^2}{2}\end{aligned}\right\}.$$

From these two equations we may easily find ω and ω'. It is easily seen that the values of ω, ω' will not be real unless $\Omega^2 > \frac{10}{7}\frac{E}{ma}(\sqrt{2}-1)^2$.

We may often save ourselves the trouble of some elimination if we form the equations derived from the principles of angular momentum and vis viva in a slightly different manner. The rod BC is turning round B with an angular velocity ω', while at the same time B is moving perpendicularly to AB with a velocity $2a\omega$. The velocity of E is therefore the resultant of $a\omega'$ perpendicular to BC and $2a\omega$ perpendicular to AB, both velocities, of course, being applied to the point E. When we wish our results to be expressed in terms of ω, ω' we may use these velocities to express the motion of E instead of the polar co-ordinates (r, θ).

Thus in applying the principle of angular momentum, we have to take the moment of the velocity of E about A. Since the velocity $2a\omega$ is perpendicular to AB, the length of the perpendicular from A on its direction is AB together with the projection of BE on AB, which is $2a + a\cos\phi$. Since the velocity $a\omega'$ is perpendicular to BE, the length of the perpendicular from A on its line of action is BE together with the projection of AB on BE, which is $a + 2a\cos\phi$. Hence the angular momentum of the rod BC about A is, by Art. 124,

$$mk^2\omega' + 2ma\omega(2a + a\cos\phi) + ma\omega'(a + 2a\cos\phi).$$

The principle of angular momentum for the two rods gives therefore

$$m(k^2 + 5a^2 + 2a^2\cos\phi)\,\omega + m(k^2 + a^2 + 2a^2\cos\phi)\,\omega' = m(2k^2 + 4a^2)\,\Omega.$$

The right-hand side of this equation, being the initial value of the angular momentum, is derived from the left-hand side by putting $\cos\phi = -\frac{1}{2}$ and $\omega = \omega' = \Omega$.

In applying the principle of vis viva, we require the velocity of E. Regarding it as the resultant of $2a\omega$ and $a\omega'$ we see that, if v be this velocity,

$$v^2 = (2a\omega)^2 + (a\omega')^2 + 2 \,.\, 2a\omega \,.\, a\omega'\cos\phi.$$

The initial value being found, as before, by putting $\cos\phi = -\frac{1}{2}$, $\omega = \omega' = \Omega$, the principle of vis viva gives, by Art. 129,

$$m(k^2 + 5a^2)\,\omega^2 + m(k^2 + a^2)\,\omega'^2 + 4ma^2\omega\omega'\cos\phi = m(2k^2 + 4a^2)\,\Omega^2 - E\,\frac{(\rho - 2a)^2}{2a}.$$

The force function is found in the same manner as before. If we join to this equation (4) given above, and substitute $\rho = 4a\cos\frac{\phi}{2}$, we have just three equations to find ω, ω', and ϕ. If these quantities are all that are required, as in the two cases considered above, this form of solution has the advantage of brevity. When ρ is a maximum, we put $\omega = \omega'$, when the rods are at right angles, we put $\cos\phi = 0$. The equations then lead to the results already given.

137. *The bob of a heavy pendulum contains a spherical cavity which is filled with water. To determine the motion.*

Let O be the point of suspension, G the centre of gravity of the solid part of the pendulum, MK^2 its moment of inertia about O and let $OG = h$. Let C be the centre of the sphere of water, a its radius and $OC = c$. Let m be the mass of the water.

If we suppose the water to be a perfect fluid, the action between it and the case must, by the definition of a fluid, be normal to the spherical boundary. There will therefore be no force tending to turn the fluid round its centre of gravity. As the pendulum oscillates to and fro, the centre of the sphere will partake of its motion, but there will be no rotation of the water.

The effective forces of the water are by Art. 128 equivalent to the effective force of the whole mass collected at its centre of gravity together with a couple $mk^2\frac{d\omega}{dt}$ where ω is the angular velocity of the water, and mk^2 its moment of inertia about a diameter. But ω has just been proved zero, hence this couple may be omitted. It follows that in all problems of this kind where the body does not turn, or turns with uniform angular velocity, we may collect the body into a single particle placed at its centre of gravity.

The pendulum and the collected fluid now form a rigid body turning about a fixed axis, hence if θ be the angle CO a fixed line in the body makes with the vertical, the equation of motion by Art. 88 is

$$(MK^2+mc^2)\frac{d^2\theta}{dt^2}+(Mh+mc)g\sin\theta=0,$$

where in finding the moment of gravity, O, G and C have been supposed to lie in a straight line.

The length L' of the simple equivalent pendulum is, by Art. 92,

$$L'=\frac{MK^2+mc^2}{Mh+mc}.$$

Let mk^2 be the moment of inertia of the sphere of water about a diameter. Then if the water were to become solid and to be rigidly connected with the case, the length L of the simple equivalent pendulum would be, by similar reasoning,

$$L=\frac{MK^2+m(c^2+k^2)}{Mh+mc}.$$

It appears that $L'<L$, so that the time of oscillation is less than when the whole is solid.

138. If we refer to the equations of motion of a body given in Art. 125, we see that the motion depends on (1) the mass of the body, (2) the position of the centre of gravity, (3) the external forces, (4) the moments of inertia of the body about straight lines through the centre of gravity, (5) the geometrical equations. Two bodies, however different they may really be, which have these characteristics the same, will move in the same manner, *i.e.* their centres of gravity will describe the same path, and their angular motions about their centres of gravity will be the same. It is often convenient to use this proposition to change the given body into some other whose motion can be more simply found.

For example, if a sphere have an eccentric spherical cavity filled with fluid of the same density as that of the solid sphere, the motion of the sphere is independent of the position of the cavity, so that, if it be more convenient, we may put the cavity at the centre. To prove this, we may notice that since the sphere of fluid does not rotate, or rotates with uniform angular velocity, the motion is unaltered by collecting the fluid into a particle placed at its centre. This being done, the first, second, third, and fifth characteristics are clearly independent of the position of the cavity. As for the fourth characteristic, let a be the radius of the sphere,

b that of the cavity, c the distance of its centre from the centre of the sphere, D the density, then the moment of inertia of the solid part of the sphere is $\frac{4}{3}\pi a^3 . \frac{2}{5}a^2 - \frac{4}{3}\pi b^3 . (\frac{2}{5}b^2 + c^2)$. The moment of inertia of the fluid collected into its centre is $\frac{4}{3}\pi b^3 . c^2$. When we add these together c disappears, so that the whole moment of inertia is independent of the position of the cavity.

The motion of a uniform triangular area moving under the action of gravity is another example. If we replace the area by three wires forming its perimeter but without weight, the geometrical conditions of the motion will in general be unaltered, and if we also place at the middle points of these wires three weights, each one-third of the mass of the triangle, this body will have all its characteristics the same as that of the real triangle, and may replace it in any problem.

When a string connecting two parts of a dynamical system passes over a rough pulley, it was formerly the custom to consider the inertia of rotation of the pulley by replacing it by another pulley of the same size but without mass and loaded with a particle at its circumference. If a be the radius of the pulley, k its radius of gyration about the centre, m its mass, the mass of the particle is $\frac{k^2}{a^2}m$, so that in a cylindrical pulley the mass of the particle is half that of the pulley. This mass must then be added on to the other particles attached to the string. For example, if two heavy masses M, M' be connected by a string passing over a cylindrical pulley of mass m, which can turn freely about its axis, the equation of motion is

$$\left(M + M' + \frac{m}{2}\right)\frac{dv}{dt} = (M - M')\,g$$

where v is the velocity. Here the inertia of the pulley is taken account of by simply adding $\frac{m}{2}$ to the mass moved. If the pulley be moveable in space as well as free to rotate, its inertia of translation is as usual taken account of by collecting the whole mass into its centre of gravity. As this representation of the inertia of rotation is not often used now, the demonstration of the above remarks, if any be needed, is left to the reader.

Ex. 1. A rod AB whose centre of gravity is at the middle point C of AB has its extremities A and B constrained to move along two straight lines Ox, Oy inclined at right angles and is acted on by any forces. Shew that the motion is the same as if the whole mass were collected into its centre of gravity and all the forces reduced in the ratio $1 + \frac{k^2}{a^2} : 1$ where $2a$ is the length AB and k is the radius of gyration about the centre of gravity.

Ex. 2. A circular disc whose centre of gravity is in its centre rolls on a perfectly rough curve under the action of any forces, shew that the motion of the centre is the same as if the curve were smooth and all the forces were reduced in the ratio $1+\frac{k^2}{a^2}:1$, where a is the radius of the disc and k is the radius of gyration about the centre. But the normal pressures on the curve in the two cases are not the same. In any position of the disc they differ by $X\frac{k^2}{a^2+k^2}$ where X is the force on the disc resolved along the normal to the rough curve.

On the stress at any point of a rod.

139. Suppose a rod OA to be in equilibrium under the action of any forces, it is required to determine the action across any section of the rod at P. This action may be conceived to be the resultant of the tensions positive or negative of the innumerable fibres which form the material of the rod. All these we know by Statics may be compounded into a single force R and a couple G acting at any point Q we may please to choose. Since each portion of the rod is in equilibrium, these must also be the resultants of all the external forces which act on the rod on *one side* of the section at P. If the section be indefinitely small it is usual to take Q in the plane of the section, and these two, the force R and the couple G, will together measure the *stress** at the section.

If the rod be bent by the action of the forces, the fibres on one side will all be stretched and on the other compressed. The rod will begin to break as soon as these fibres have been sufficiently stretched or compressed. Let us compare the tendencies of the force R and the couple G to break the rod. Let A be the area of the section of the rod, then a force F pulling the rod will cause a resultant force $R=F$, and will produce a tension in the fibres which when referred to a unit of area is equal to $\frac{F}{A}$. The same force F acting on the rod at an arm from P whose length is p, will cause a couple $G=Fp$, which must be balanced by the couple formed by the tensions. Let $2a$ be the mean breadth of the rod, then the mean tension referred to a unit of area produced by G is of the order $\frac{F}{A}\cdot\frac{p}{a}$. Now if the section of the rod be very small $\frac{p}{a}$ will be large. It appears therefore that the couple, when it exists, will generally have much more effect in breaking the

* Sir W. Thomson has appropriated the word *strain* to the alterations of volume and figure produced in an elastic body by the forces applied to it, and the word *stress* to the elastic pressures.

rod than the force. This couple is therefore often taken to measure the whole effect of the forces to break the rod. The "*tendency to break*" at any point P of a rod OA of very small section is measured by the moment about P of all the forces which act on either of the sides OP or PA of the rod.

The resolved part of the force R perpendicular to the rod is called the *shear*. This is therefore equal to all the forces which act on either of the sides OP or PA resolved perpendicular to the rod.

If the rod be in motion the same reasoning will, by D'Alembert's principle, be applicable; provided we include the reversed effective forces among the forces which act on the rod.

In most cases the rod will be so little bent that in finding the moment of the impressed forces we may neglect the effects of curvature.

If the section of the rod be not very small, this measure of the "tendency to break" becomes inapplicable. It then becomes necessary to consider both the force and the couple. This does not come within the limits of the present treatise, and the reader is referred to works on Elastic Solids.

In the case of a string the couple vanishes and the force acts along a tangent to the string. The stress at any point is therefore simply measured by the tension.

140. *A rod OA, of length $2a$, and mass m, which can turn freely about one extremity O, falls under the action of gravity in a vertical plane. Find the "tendency to break" at any point P.*

Let du be any element of the rod distant u from P and on the side of P nearer the end A of the rod, and let $OP=x$. Let θ be the angle the rod makes with the vertical at the time t. The effective forces on du are

$$m\frac{du}{2a}(x+u)\frac{d^2\theta}{dt^2} \text{ and } -m\frac{du}{2a}(x+u)\left(\frac{d\theta}{dt}\right)^2$$

respectively perpendicular and along the rod. The impressed force is $m\dfrac{du}{2a}g$ acting vertically downwards. The effective forces being reversed the tendency to break at P is equal to the moment about P of all the forces which act on the part PA of the rod. If this be called L, we have

$$L=\int m\frac{du}{2a}gu\sin\theta-\int m\frac{du}{2a}(x+u)\,u\frac{d^2\theta}{dt^2},$$

the limits being from $u=0$ to $u=2a-x$. Also taking moments about O, the equation of motion is

$$m\frac{4a^2}{3}\frac{d^2\theta}{dt^2}=mga\sin\theta.$$

Hence we easily find

$$L=-\frac{mg\sin\theta}{16a^2}x(2a-x)^2.$$

The meaning of the minus sign is that the forces tend to bend PA round P in the opposite direction to that in which θ has been measured.

To find where the rod supposed equally strong throughout is most likely to break, we must make L a maximum. This gives $\dfrac{dL}{dx}=0$ and therefore $x=\dfrac{2a}{3}$. The point required is at a distance from the fixed end equal to one-third of the length of the rod. This point, it should be noticed, is independent of the initial conditions.

To find the shear at P we must resolve perpendicularly to the rod. If the result be called Y, we have

$$Y=\int m\frac{du}{2a}g\sin\theta-\int m\frac{du}{2a}(x+u)\frac{d^2\theta}{dt^2},$$

the limits being the same as before. This gives

$$Y=\frac{mg\sin\theta}{16a^2}(2a-x)(2a-3x),$$

which vanishes when the tendency to break is a maximum, and is a maximum at a distance from the fixed end equal to two-thirds of the length of the rod.

To find the tension at P we must resolve along the rod. If the result be called X, we have

$$X=-\int m\frac{du}{2a}g\cos\theta+\int m\frac{du}{2a}(x+u)\left(\frac{d\theta}{dt}\right)^2.$$

If the rod start from rest at an inclination α to the vertical, we find, by integrating the equation of motion, $\left(\dfrac{d\theta}{dt}\right)^2=\dfrac{3g}{2a}(\cos\alpha-\cos\theta)$. Hence

$$X=\frac{mg}{8a^2}(2a-x)\{-4a\cos\theta+3(\cos\alpha-\cos\theta)(2a+x)\}.$$

From these equations we may deduce the following results. (1) The magnitudes of the stress couple and of the shear are independent of the initial conditions. (2) The magnitude of either the couple or the shear at any given point of the rod varies as the sine of the inclination of the rod to the vertical. (3) The ratio of the magnitudes of the stress couples at any two given points of the rod is always the same, and the same proposition is also true of the shear. (4) The tension depends on the initial conditions and unless the rod start from rest in the horizontal position, the ratio of the tensions at any two given points varies with the position of the rod.

141. *A rigid hoop completely cracked at one point rolls on a perfectly rough horizontal plane and is acted on by no forces but gravity. Prove that the wrench couple at the point of the hoop most remote from the crack will be a maximum whenever, the crack being lower than the centre, the inclination of the diameter through the crack to the horizon is* $\tan^{-1}\dfrac{2}{\pi}$. [The Math. Tripos, 1864.]

Let ω be the angular velocity of the hoop, a its radius. The velocity of any point P of the hoop is the resultant of a velocity $a\omega$ parallel to the horizontal plane and an equal velocity $a\omega$ along a tangent to the hoop. The first is constant in direction and magnitude and therefore gives nothing to the acceleration of P. The latter is constant in magnitude but variable in direction and gives $a\omega^2$ as the acceleration which is directed along a radius of the hoop. Let A be the cracked point, B the other end of the diameter, C the centre, θ the inclination of ACB to

the horizon. Let PP' be any element on the upper half of the circle, $BCP=\phi$. Then the wrench couple, or tendency to break, at B is proportional to

$$\int_0^\pi [-a\omega^2 a \sin\phi + g\{a\cos\theta - a\cos(\phi+\theta)\}]\, ad\phi = -2a^3\omega^2 + ga^2(\cos\theta\pi + 2\sin\theta).$$

This is a maximum when $\tan\theta = \frac{2}{\pi}$.

Ex. 1. A semicircular wire AB of radius a is rotating on a smooth horizontal plane about one extremity A with a constant angular velocity ω. If $a\phi$ be the arc between the fixed point A and the point where the tendency to break is greatest, prove that $\tan\phi = \pi - \phi$. If the extremity B be suddenly fixed and the extremity A let go, prove that the tendency to break is greatest at a point P where

$$\tfrac{1}{2}\tan PBA = PBA.$$

Ex. 2. Two of the angles of a heavy square lamina, a side of which is a, are connected with two points equally distant from the centre of a rod of length $2a$, so that the square can rotate about the rod. The weight of the square is equal to the weight of the rod, and the rod when supported by its extremities in a horizontal position is on the point of breaking. The rod is then held by its extremities in a vertical position, and an angular velocity ω is then impressed on the square. Shew that it will break if $\omega > \sqrt{\frac{3g}{a}}$. [Coll. Exam.]

Ex. 3. A wire in the form of the portion of the curve $r = a(1+\cos\theta)$ cut off by the initial line rotates about the origin with angular velocity ω. Prove that the tendency to break at the point $\theta = \frac{\pi}{2}$ is measured by $m\frac{12\sqrt{2}}{5}\omega^2 a^3$. [St John's Coll.]

On Friction between Imperfectly Rough Bodies.

142. When one body rolls on another under pressure, the two bodies yield slightly, and are therefore in contact along a small area. At every point of this area there is a mutual action between the bodies. The elements just behind the geometrical point of contact are on the point of separation and may tend to adhere to each other, those in front may tend to resist compression. The whole of the actions across all the elements are equivalent to (1) a component R, normal to the common tangent plane, and usually called the *reaction*; (2) a component F in the tangent plane usually called the *friction*; (3) a couple L about an axis lying in the tangent plane and which we shall call the *couple of rolling friction*; (4) if the bodies have any relative angular velocity about their common normal, a couple N about this normal as axis which may be called the *couple of twisting friction*.

143. These two couples are found by experiment to be in most cases very small and are generally neglected. But in certain cases where the friction forces are also small, it may be necessary to take account of them.

144. When one body presses against another over any small area, the force of friction acts in such a direction and with such a magnitude that it is just sufficient to prevent sliding. Both the magnitude and direction of friction may, therefore, be unknown beforehand, and their determination will be part of the problem under consideration. It is found by experiment that no more than a certain amount of friction can be called into play, and when more is required to keep the bodies from sliding on each other, sliding will begin. This amount is called *limiting friction.* The magnitude of this limit is found to bear a ratio to the normal pressure which is very nearly constant for the same two bodies. Though all experimenters have not entirely agreed with each other as to the accuracy of this result, yet it has been found generally that, if the relative motion of the two bodies be the same at all points of the area of contact, this ratio is nearly independent of the extent of the area and of the relative velocity. If, however, the bodies have remained in contact for some time under pressure in a position of equilibrium, it is found that, for the more compressible bodies, the ratio is a little greater than after motion has begun. This ratio has been called the *coefficient of friction* of the materials of the two bodies. Its constancy is generally assumed by mathematicians. When the friction which can be called into play is insufficient to prevent sliding, the bodies slide on each other. In this case the magnitude of the friction is equal to its limiting value, and the direction of the friction is opposite to that of relative motion.

145. If the bodies be perfectly rough, the coefficient of friction is infinite, and there is no limit to the amount of friction which can be called into play. There can, therefore, be no sliding between the bodies.

146. Discontinuity of motion will often occur when a body moves under the action of friction. Suppose the body rolls on a rough surface, the friction called into play just prevents sliding, and is possibly variable in magnitude and direction. By writing down and solving the equations of motion we can find the ratio of the friction F to the normal pressure R. If this ratio be always less than the coefficient μ of friction, enough friction can always be called into play to make the body roll on the rough surface. In this case we have obtained the true motion. But if at any instant the ratio $\frac{F}{R}$ thus found should be greater than the co-

efficient of friction, the point of contact will begin to slide at that moment. In this case the equations do not represent the true motion. To correct them we must replace the unknown friction F by μR, and remove the geometrical equation which expresses the fact that there is no slipping between the bodies. The equations must now be again solved on this new supposition. It is of course possible that another change may take place. If at any instant the velocities of the points of contact become equal to each other, all the possible friction may not be called into play. At that instant the friction ceases to be equal to μR and becomes again unknown in magnitude and direction.

Discontinuity may also arise in other ways. When, for example, one body is sliding over another, the friction is opposite to the direction of relative motion, and numerically equal to the normal reaction multiplied by the coefficient of friction. If then, during the course of the motion the direction of the normal reaction should change sign, while the direction of motion remains unaltered; or if the direction of motion should change sign while the normal reaction should remain unaltered, the sign of the coefficient of friction must be changed. This may modify the dynamical equations and alter the subsequent motion. The same cause of discontinuity operates when a body moves in a resisting medium, when the law of resistance is an even function of the velocity, or any function which does not change sign when the direction of motion is changed.

In some cases the motion may be rendered indeterminate by the introduction of friction. Thus, we have seen in Art. 111, that when a body swings on two hinges, the pressures on the hinges resolved in the direction of the straight line joining them cannot be found. The sum of these components can be found, but not either of them. But there was no indeterminateness in the motion. If however these hinges were imperfectly rough, there would be two friction couples, one at each hinge, acting on the body. The common axis of these couples would be the straight line joining the hinges. The magnitude of each would be equal to the pressure resolved along its axis multiplied by a constant depending on the roughness of the hinge. If the hinges were unequally rough, the magnitude of the resultant couple would depend on the distribution of the pressure on the two hinges. In such a case the motion of the body would be indeterminate.

147. *A homogeneous sphere is placed at rest on a rough inclined plane, the coefficient of friction being μ, determine whether the sphere will slide or roll.*

Let F be the friction required to make the sphere roll. The problem then becomes the same as that discussed in Art. 133. We have, therefore, $\dfrac{F}{R} = \frac{2}{7} \tan \alpha$, where α is the inclination of the plane to the horizon.

If then $\frac{2}{7}\tan\alpha$ be not greater than μ, the solution given in the article referred to is the correct one. But if $\mu < \frac{2}{7}\tan\alpha$ the sphere will begin to slide on the inclined plane. The subsequent motion will be given by the equations

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= mg\sin\alpha - \mu R \\ 0 &= -mg\cos\alpha + R \\ ma\frac{d^2x}{dt^2} + mk^2\frac{d^2\theta}{dt^2} &= mga\sin\alpha \end{aligned}\right\}$$

whence we have, remembering that $k^2 = \frac{2}{5}a^2$,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= g(\sin\alpha - \mu\cos\alpha) \\ \frac{d^2\theta}{dt^2} &= \tfrac{5}{2}\mu\frac{g}{a}\cos\alpha \end{aligned}\right\}.$$

Since the sphere starts from rest, we have by integration

$$\left.\begin{aligned} x &= \tfrac{1}{2}gt^2(\sin\alpha - \mu\cos\alpha) \\ \theta &= \tfrac{5}{4}\mu\frac{g}{a}t^2\cos\alpha \end{aligned}\right\}.$$

The velocity of the point of the sphere in contact with the plane is

$$\frac{dx}{dt} - a\frac{d\theta}{dt} = gt(\sin\alpha - \tfrac{7}{2}\mu\cos\alpha).$$

But since, by hypothesis, μ is less than $\frac{2}{7}\tan\alpha$, this velocity can never vanish. The friction therefore will never change to rolling friction. The motion has thus been completely determined.

148. *A homogeneous sphere is rotating about a horizontal diameter, and is gently placed on a rough horizontal plane, the coefficient of friction being μ. Determine the subsequent motion.*

Since the velocity of the point of contact with the horizontal plane is not zero, the sphere will evidently begin to slide, and the motion of its centre will be along a straight line perpendicular to the initial axis of rotation. Let this straight line be taken as the axis of x, and let θ be the angle between the vertical and that radius of the sphere which was initially vertical. Let a be the radius of the sphere, mk^2 its moment of inertia about a diameter, and Ω the initial angular velocity. Let R be the normal reaction of the plane. Then the equations of motion are clearly

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \mu R \\ 0 &= mg - R \\ mk^2\frac{d^2\theta}{dt^2} &= -\mu Ra \end{aligned}\right\} \quad \text{.................................... (1)},$$

whence we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mu g \\ \frac{d^2\theta}{dt^2} &= -\tfrac{5}{2}\mu\frac{g}{a} \end{aligned}\right\} \quad \text{.................................... (2)}.$$

Integrating, and remembering that the initial value of $\frac{d\theta}{dt}$ is Ω, we have

$$\left.\begin{aligned} x &= \tfrac{1}{2}\mu gt^2 \\ \theta &= \Omega t - \tfrac{5}{4}\mu\frac{g}{a}t^2 \end{aligned}\right\} \quad \text{.................................... (3)}.$$

But it is evident that these equations cannot represent the whole motion, for they would make $\frac{dx}{dt}$, the velocity of the centre of the sphere, increase continually. This is quite contrary to experience. The velocity of the point of the sphere in contact with the plane is

$$\frac{dx}{dt} - a\frac{d\theta}{dt} = -a\Omega + \tfrac{7}{2}\mu g t.$$

This vanishes at a time $$t_1 = \tfrac{2}{7}\frac{a\Omega}{\mu g} \qquad\qquad (4).$$

At this instant the friction suddenly changes its character. It now becomes only of sufficient magnitude to keep the point of contact of the sphere at rest. Let F be the friction required to effect this. The equations of motion will then be

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= F \\ 0 &= mg - R \\ mk^2\frac{d^2\theta}{dt^2} &= -Fa \end{aligned}\right\} \qquad\qquad (5),$$

and the geometrical equation will be $x = a\theta$.

Differentiating this twice, and substituting from the dynamical equations, we get $F(a^2 + k^2) = 0$, and therefore $F = 0$. That is, no friction is required to keep the point of contact of the sphere at rest, and therefore none will be called into play. The sphere will therefore move uniformly with the velocity which it had at the time t_1. Substituting the value of t_1 in the expression for $\frac{dx}{dt}$ obtained from equations (3) we find that this velocity is $\frac{2}{7}a\Omega$. It appears therefore that the sphere will move with a uniformly increasing velocity for a time $\frac{2}{7}\frac{a\Omega}{\mu g}$ and will then move uniformly with a velocity $\frac{2}{7}a\Omega$. It may be remarked that this velocity is independent of μ.

If the plane be perfectly rough, μ is infinite, and the time t_1 vanishes. The sphere therefore immediately begins to move with a uniform velocity $= \frac{2}{7}a\Omega$.

149. In this investigation the couple of rolling friction has been neglected. Its effect would be to diminish the angular velocity. The velocity of the lowest point of the sphere would then tend to be no longer zero, and thus a small sliding friction will be required to keep that point at rest. Suppose the moment of the friction-couple to be measured by fmg, where f is a constant. Introducing this into the equations (5) the third is changed into

$$mk^2\frac{d^2\theta}{dt^2} = -Fa - fmg,$$

the others remaining unaltered. Solving these as before we find

$$F = -\frac{afmg}{a^2 + k^2}.$$

We see from this that F is negative and retards the sphere. The effect of the couple is to call into play a friction-force which gradually reduces the sphere to rest.

As the sphere moves in the air we may wish to determine the effect of its resistances. The chief part of this resistance may be pretty accurately represented by a

force $m\beta\frac{v^2}{a}$ acting at the centre in the direction opposite to motion, v being the velocity of the sphere and β a constant whose magnitude depends on the density of the air. Besides this there will be also a small friction between the sphere and air whose magnitude is not known so accurately. Let us suppose it to be represented by a couple whose moment is $m\gamma v^2$ where γ is a constant of small magnitude. The equations of motion can be solved without difficulty, and we find

$$\tan^{-1} v\sqrt{\frac{\beta+\gamma}{fg}} - \tan^{-1} V\sqrt{\frac{\beta+\gamma}{fg}} = -\frac{a\sqrt{(\beta+\gamma)fg}}{a^2+k^2}t,$$

where V is the velocity of the sphere at the epoch from which t is measured.

150. In order to determine by experiment the magnitude of rolling friction, let a cylinder of mass M and radius r be placed on a rough horizontal plane. Let two weights whose masses are P and $P+p$ be suspended by a fine thread passing over the cylinder and hanging down through a slit in the horizontal plane. Let F be the force of friction, L the couple at the point of contact A of the cylinder with the horizontal plane. Imagine p to be at first zero, and to be gradually increased until the cylinder just moves. When the cylinder is on the point of motion, we have by resolving horizontally $F=0$ and by taking moments $L=pgr$. Now in the experiments of Coulomb and Morin p was found to vary as the normal pressure directly, and as r inversely. When p was great enough to set the cylinder in motion, Coulomb found that the acceleration of the cylinder was nearly constant, and thence we may conclude that the rolling friction was independent of the velocity. M. Morin found that it was not independent of the length of the cylinder.

The laws which govern the couple of rolling friction are similar to those which govern the force of friction. The magnitude is just sufficient to prevent rolling. But no more than a certain amount can be called into play, and this is called the *limiting rolling couple*. The moment of this couple bears a constant ratio to the magnitude of the normal pressure. This ratio is called the *coefficient of rolling friction*. It depends on the materials in contact, it is independent of the curvatures of the bodies, and, in some cases, of the angular velocity.

No experiments seem to have been made on bodies which touch at one point only and have their curvatures in all directions unequal. But since the magnitude of the couple is independent of the curvature, it seems reasonable to assume that the axis of the rolling couple, when there is no twisting couple, is the instantaneous axis of rotation.

In order to test these laws of friction let us compare the results of the following problem with experiment.

151. *A carriage on* n *pairs of wheels is dragged on a level horizontal plane by a horizontal force* 2P *with uniform motion. Find the magnitude of* P.

Let the radii of the wheels be respectively r_1, r_2, &c., their weights w_1, w_2, &c., and the radii of the axles ρ_1, ρ_2, &c. Let $2W$ be the whole weight of the carriage, $2Q_1$, $2Q_2$, &c. the pressures on the several axles, so that $W=\Sigma Q$. Let the pressures between the wheels and axles be R_1, R_2, &c. and the pressures on the ground R'_1, R'_2, &c. Let C be the common centre of any wheel and axle, P their point of contact, and A the point of contact of the wheel with the ground. Let the angle $ACP=\theta$ supposed positive when P is behind AC. Let μ be the coefficient of the force of sliding friction at P and f the coefficient of the couple of rolling friction at A. The equations of equilibrium for any wheel, found by resolving vertically and taking moments about A, are

$$R'=Q+w \quad\text{.......(1)},$$

$$\mu R(r\cos\theta-\rho)-Rr\sin\theta=fR' \quad\text{.......(2)}.$$

The friction force at A does not appear because we have not resolved horizontally. The equations of equilibrium of the carriage, found by resolving vertically and horizontally, are

$$R\cos\theta+\mu R\sin\theta=Q \quad\text{.......(3)},$$

$$\Sigma(R\sin\theta-\mu R\cos\theta)+P=0 \quad\text{.......(4)}.$$

The effective forces have been omitted because the carriage is supposed to move uniformly, so that the $M\dfrac{dv}{dt}$ of the carriage and the $mk^2\dfrac{d\omega}{dt}$ of the wheel are both zero.

The first three of these equations give by eliminating R and R'

$$\frac{\mu\left(\cos\theta-\dfrac{\rho}{r}\right)-\sin\theta}{\cos\theta+\mu\sin\theta}=\frac{f}{r}\left(1+\frac{w}{Q}\right) \quad\text{.......(5)}.$$

This gives the value of θ. In most wheels $\dfrac{\rho}{r}$ and $\dfrac{w}{Q}$ are both small as well as f. In such a case $\mu\cos\theta-\sin\theta$ is a small quantity. If therefore $\mu=\tan\epsilon$ we have $\theta=\epsilon$ very nearly.

The third and fourth of these equations give by eliminating R

$$P=\Sigma\frac{\mu\cos\theta-\sin\theta}{\mu\sin\theta+\cos\theta}Q$$

$$=\Sigma\left\{\frac{\mu}{\mu\sin\theta+\cos\theta}\frac{\rho}{r}Q+\frac{f}{r}(Q+w)\right\}$$

by equation (5). If $\dfrac{\rho}{r}$ be small, it will be sufficient to substitute for θ in the first term its approximate value ϵ. This gives

$$P=\Sigma\left\{\sin\epsilon\frac{\rho}{r}Q+f\frac{Q+w}{r}\right\} \quad\text{.......(6)}.$$

Here we have neglected terms of the order $\left(\dfrac{\rho}{r}\right)^2 Q$.

If all the wheels are equal and similar we have, since $\Sigma Q=W$,

$$P=\sin\epsilon\frac{\rho}{r}W+f\frac{W+nw}{r} \quad\text{.......(7)}.$$

Thus the force required to drag a carriage of given weight with any constant velocity is very nearly independent of the number of wheels.

In a gig the wheels are usually larger than in a four-wheel carriage, and therefore the force of traction is usually less. In a four-wheel carriage the two fore wheels must be small in order to pass *under* the carriage when turning. This will cause the term $\sin \epsilon \frac{\rho_1}{r_1} Q_1$ in the expression for P containing the radius r_1 of the fore wheel to be large. To diminish the effect of this term, the load should be so adjusted that its centre of gravity is nearly over the axle of the large wheels, the pressure Q_1 in the numerator of this term will then be small.

A variety of experiments were made by a French engineer, M. Morin, at Metz in the years 1837 and 1838, and afterwards at Courbevoie in 1839 and 1841, with a view to determine with the utmost exactness the force necessary to drag carriages of different kinds over the ordinary roads. These experiments were undertaken by order of the French Minister of War, and afterwards under the directions of the Minister of Public Works. The effect of each element was determined separately, thus the same carriage was loaded with different weights to determine the effect of pressure and dragged on the same road in the same state of moisture. Then the weight being the same, wheels of different radii but the same breadth were used, and so on.

The general results were that for carriages on equal wheels, the resistance varied as the pressure directly and the diameter of the wheels inversely, and was independent of the number of wheels. On wet soils the resistance increased as the breadth of the tire was decreased, but on solid roads the resistance was independent of the breadth of the tire. For velocities which varied from a foot pace to a gallop, the resistance on wet soils did not increase sensibly with the velocity, but on solid roads it did increase with the velocity if there were many inequalities on the road. As an approximate result it was found that the resistance might be expressed by a formula of the kind $a + bV$, where a and b are two constants depending on the nature of the road and the stiffness of the carriage, and V is the velocity.

M. Morin's analytical determination of the value of P does not altogether agree with that given here, but it so happens that this does not materially affect the comparison between theory and observation. See his *Notions Fondamentales de Mécanique*, Paris 1855. It is easy to see that M. Morin's experiments tend to confirm the laws of rolling friction stated in a previous article.

Ex. 1. A homogeneous sphere is projected without rotation directly up an imperfectly rough plane, the inclination of which to the horizon is α, and the coefficient of friction μ. Show that the whole time during which the sphere ascends the plane is the same as if the plane were smooth, and that the time during which the sphere slides is to the time during which it rolls as $1 : \frac{2}{7}\frac{\tan \alpha}{\mu}$.

Ex. 2. A homogeneous sphere of mass M is placed on an imperfectly rough table, the coefficient of friction of which is μ. A particle of mass m is attached to the extremity of a horizontal diameter. Show that the sphere will begin to roll or slide according as μ is greater or less than $\frac{5(M+m)m}{7M^2 + 17Mm + 5m^2}$. If μ be equal to this value, show that the sphere will begin to roll.

Ex. 3. A rod AB has two small rings at its extremities which slide on two rough horizontal rods Ox, Oy at right angles. The rod is started with an angular velocity Ω when very nearly coincident with Ox, show that if the coefficient of fric-

tion is less than $\sqrt{2}$, the motion of the rod is given by $\theta = \frac{2-\mu^2}{3\mu} \log\left(1 + \frac{3\mu\Omega t}{2-\mu^2}\right)$ until $\tan\theta = \frac{2}{\mu}$, and that when the rod reaches Oy, its angular velocity is ω, where

$$\Omega^2 e^{-\frac{6\mu}{2-\mu^2}\tan^{-1}\frac{2}{\mu}} = \omega^2 \frac{(2+\mu^2)(4+\mu^2)}{(2-\mu^2)(4-\mu^2)}.$$

What is the motion if $\mu^2 > 2$?

On Impulsive Forces.

152. In the case in which the impressed forces are impulsive the general principle enunciated in Art. 123 of this chapter requires but slight modification.

Let (u, v), (u', v') be the velocities of the centre of gravity of any body of the system resolved parallel to any rectangular axes respectively just before and just after the action of the impulses. Let ω and ω' be the angular velocities of the body about the centre of gravity at the same instants. And let Mk^2 be the moment of inertia of the body about the centre of gravity. Then the effective forces on the body are equivalent to two forces measured by $M(u'-u)$ and $M(v'-v)$ acting at the centre of gravity parallel to the axes of co-ordinates together with a couple measured by $Mk^2(\omega'-\omega)$.

The resultant effective forces of all the bodies of the system may be found by the same rule. By D'Alembert's principle these will be in equilibrium with the impressed forces. The equations of motion may then be found by resolving in such directions and taking moments about such points as may be found most convenient.

In many cases it will be found that by the use of Virtual Velocities the elimination of the unknown reactions may be effected without difficulty.

153. *A string is wound round the circumference of a circular reel, and the free end is attached to a fixed point. The reel is then lifted up and let fall so that at the moment when the string becomes tight it is vertical, and a tangent to the reel. The whole motion being supposed to take place in one plane, determine the effect of the impulse.*

The reel in the first instance falls vertically without rotation. Let v be the velocity of the centre at the moment when the string becomes tight; v', ω' the velocity of the centre and the angular velocity just after the impulse. Let T be the impulsive tension, mk^2 the moment of inertia of the reel about its centre of gravity, a its radius.

In order to avoid introducing the unknown tension into the equations of motion, let us take moments about the point of contact of the string with the reel; we then have

$$m(v'-v)a + mk^2\omega' = 0 \quad\quad (1).$$

Just after the impact the part of the reel in contact with the string has no velocity.

Hence $$v' - a\omega' = 0 \qquad\qquad (2).$$

Solving these we have $\omega' = \dfrac{av}{a^2+k^2}$. If the reel be a homogeneous cylinder $k^2 = \dfrac{a^2}{2}$, and in this case we have $\omega' = \dfrac{2}{3}\dfrac{v}{a}$, $v' = \dfrac{2}{3}v$. If it be required to find the impulsive tension, we have resolving vertically

$$m(v'-v) = -T \qquad\qquad (3).$$

Hence $$T = \frac{1}{3}mv.$$

To find the subsequent motion. The centre of the reel *begins* to descend vertically, and there is no horizontal force on it. Hence it will continue to descend in a vertical straight line, and throughout all the subsequent motion the string is vertical. The motion may therefore be easily investigated as in Art. 133. If we put $\alpha = \dfrac{\pi}{2}$, and let F = the finite tension of the string, it may be shown that F = one-third of the weight, and that the reel descends with a uniform acceleration $= \dfrac{2}{3}g$. The initial velocity of the reel has been found in this article $= v'$, so that the space descended in a time t after the impact is $= v't + \dfrac{1}{2}\cdot\dfrac{2}{3}gt^2$.

Ex. 1. An inelastic sphere of radius a sliding on a smooth horizontal plane impinges on a fixed rough point at a height c above the plane, show that if the velocity of the sphere be $\sqrt{\dfrac{2gc(a^2+k^2)}{(a-c)^2}}$, it will just roll over the point.

Ex. 2. A rectangular parallelopiped of mass $3m$, having a square base $ABCD$, rests on a horizontal plane and is moveable about CD as a hinge. The height of the solid is $3a$ and the side of the base a. A particle m moving with a horizontal velocity v strikes directly the middle of that vertical face which stands on AB and lodges there without penetrating. Show that the solid will not upset unless $v^2 > \dfrac{53}{9}ga$. [King's Coll.]

154. *Four equal rods each of length* 2a *and mass* m *are freely jointed so as to form a rhombus. The system falls from rest with a diagonal vertical under the action of gravity and strikes against a fixed horizontal inelastic plane. Find the subsequent motion.*

Let AB, BC, CD, DA be the rods and let AC be the vertical diagonal impinging on the horizontal plane at A. Let V be the velocity of every point of the rhombus just before impact and let α be the angle any rod makes with the vertical.

Let u, v be the horizontal and vertical velocities of the centre of gravity and ω the angular velocity of either of the upper rods just after impact. Then the effective forces on either rod are equivalent to the force $m(v-V)$ acting vertically and mu horizontally at the centre of gravity and a couple $mk^2\omega$ tending to increase the angle α. Let R be the impulse at C, the direction of which by the rule of symmetry is horizontal. To avoid introducing the reaction at B into our equations, let us take moments for the rod BC about B and we have

$$mk^2\omega + m(v-V)\,a\sin\alpha - mua\cos\alpha = -R\,.\,2a\cos\alpha \qquad\qquad (1).$$

Each of the lower rods will begin to turn round its extremity A as a fixed point. If ω' be its angular velocity just after impact, the moment of the momentum about A just after impact will be $m(k^2+a^2)\,\omega'$ and just before will be $mVa\sin\alpha$. The difference of these two is the moment about A of the effective forces on either of the lower rods. We may now take moments about A for the two rods AB, BC together and we have

$$m(k^2+a^2)\,\omega' - mVa\sin\alpha - mk^2\omega + m(v-V)\,a\sin\alpha + mu\,.\,3a\cos\alpha = R\,.\,4a\cos\alpha \ldots (2).$$

The geometrical equations may be found thus.

Since the two rods must make equal angles with the vertical during the whole motion we have

$$\omega' = \omega \ldots\ldots (3).$$

Again, since the two rods are connected at B the velocities of the extremities of the two rods must be the same in direction and magnitude. Resolving these horizontally and vertically, we have

$$u + a\omega\cos\alpha = 2a\omega'\cos\alpha \ldots\ldots (4),$$

$$v - a\omega\sin\alpha = 2a\omega'\sin\alpha \ldots\ldots (5).$$

These five equations are sufficient to determine the initial motion.

Eliminating R between (1) and (2), substituting for u, v, ω' in terms of ω from the geometrical equations, we find

$$\omega = \frac{3}{2}\cdot\frac{V\sin\alpha}{a(1+3\sin^2\alpha)} \ldots\ldots (6).$$

In this problem we might have avoided the introduction of the unknown reaction R by the use of Virtual Velocities. Suppose we give the system such a displacement that the inclination of each rod to the vertical is increased by the same quantity $\delta\alpha$. Then the principle of Virtual Velocities gives

$$mk^2\omega\delta\alpha - m(v-V)\,\delta(3a\cos\alpha) + mu\delta(a\sin\alpha) + m(k^2+a^2)\,\omega'\delta\alpha + mV\delta(a\cos\alpha) = 0,$$

which reduces to

$$(2k^2+a^2)\,\omega - Va\sin\alpha + 3(v-V)\,a\sin\alpha + ua\cos\alpha = 0,$$

and the solution may be continued as before.

Ex. 1. Prove that the rhombus loses by the impact $\dfrac{1}{1+3\sin^2\alpha}$ of its momentum.

Ex. 2. Show that the direction of the impulsive action at the hinges B or D makes with the horizon an angle whose tangent is $\dfrac{3\sin^2\alpha - 2}{\tan\alpha}$.

To find the subsequent motion. This may be found very easily by the method of *Vis Viva*. But in order to illustrate as many modes of solution as possible, we shall proceed in a different manner. The effective forces on either of the upper rods will be represented by the differential coefficients $m\dfrac{dv}{dt}$, $m\dfrac{du}{dt}$, $mk^2\dfrac{d\omega}{dt}$, and the moment for either of the lower rods will be $m(k^2+a^2)\dfrac{d\omega'}{dt}$. Let θ be the angle any rod makes with the vertical at the time t. Taking moments in the same way as before, we have

$$mk^2\frac{d\omega}{dt} + m\frac{dv}{dt}a\sin\theta - m\frac{du}{dt}a\cos\theta = -R\,.\,2a\cos\theta + mga\sin\theta \ldots\ldots (1)',$$

$$m(k^2+a^2)\frac{d\omega'}{dt} - mk^2\frac{d\omega}{dt} + m\frac{dv}{dt}a\sin\theta + m\frac{du}{dt}\,.\,3a\cos\theta = R\,.\,4a\cos\theta + 2mga\sin\theta \ldots (2)'.$$

The geometrical equations are the same as those given above, with θ written for a.

Eliminating R and substituting for u, v, we get

$$(2k^2+a^2)\frac{d\omega}{dt}+a^2\left\{9\sin\theta\frac{d}{dt}(\omega\sin\theta)+\cos\theta\frac{d}{dt}(\omega\cos\theta)\right\}=4ga\sin\theta;$$

multiplying both sides by $\omega=\frac{d\theta}{dt}$ and integrating, we get

$$\{2(k^2+a^2)+8a^2\sin^2\theta\}\,\omega^2=C-8ga\cos\theta.$$

Initially when $\theta=a$, ω has the value given by equation (6). Hence we find that the angular velocity ω when the inclination of any rod to the vertical is θ is given by

$$(1+3\sin^2\theta)\,\omega^2=\frac{9V^2}{4a^2}\cdot\frac{\sin^2 a}{1+3\sin^2 a}+\frac{3g}{a}(\cos a-\cos\theta).$$

Ex. 1. A square is moving freely about a diagonal with angular velocity ω, when one of the angular points not in that diagonal becomes fixed; determine the impulsive pressure on the fixed point, and show that the instantaneous angular velocity will be $\frac{\omega}{7}$. [Christ's Coll.]

Ex. 2. Three equal rods placed in a straight line are jointed by hinges to one another; they move with a velocity v perpendicular to their lengths; if the middle point of the middle one become suddenly fixed, show that the extremities of the other two will meet in a time $\frac{4\pi a}{9v}$, a being the length of each rod. [Coll. Exam.]

Ex. 3. The points $ABCD$ are the angular points of a square; AB, CD are two equal similar rods connected by the string BC. The point A receives an impulse in the direction AD, show that the initial velocity of A is seven times that of the point D. [Queens' Coll.]

Ex. 4. A series of equal beams AB, BC, CD...... is connected by hinges; the beams are placed on a smooth horizontal plane, each at right angles to the two adjacent, so as to form a figure resembling a set of steps, and an impulse is given at the end A along AB: determine the impulsive action at any hinge. [Math. Tripos.]

Result. If X_n be the impulsive action at the n^{th} angular point, show that $X_{2n+1}-5X_{2n+2}-2X_{2n+3}=0$ and $X_{2n+2}-5X_{2n+1}-2X_{2n}=0$. Thence find X_n.

155. *A free lamina of any form is turning in its own plane about an instantaneous centre of rotation* S *and impinges on an obstacle at* P, *situated in the straight line joining the centre of gravity* G *to* S. *To find the point* P *when the magnitude of the blow is a maximum*.*

First, let the obstacle P *be a fixed point.*

Let $GP=x$, and let R be the force of the blow. Let $SG=h$, and let ω, ω' be the angular velocities about the centre of gravity before and after the impact. Then $h\omega$

* Poinsot, Sur la percussion des corps, *Liouville's Journal*, 1857; translated in the *Annals of Philosophy*, 1858.

is the linear velocity of G just before the impact; let v' be its linear velocity just after the impact. We have the equations

$$\left.\begin{aligned} \omega' - \omega &= \frac{-Rx}{Mk^2} \\ v' - h\omega &= -\frac{R}{M} \end{aligned}\right\} \qquad \text{(1)},$$

and supposing the point of impact to be reduced to rest,

$$v' + x\omega' = 0 \qquad \text{(2)}.$$

Substituting for ω' and v' from (1) in equation (2), we get

$$R = M\omega \,.\, k^2 \frac{x+h}{x^2+k^2}.$$

This is to be made a maximum. Equating to zero its differential coefficient with respect to x, we get

$$x^2 + 2hx - k^2 = 0 \qquad \text{(3)};$$

$$\therefore\ x = -h \pm \sqrt{h^2+k^2}.$$

One of these values of x is positive and the other negative. Both these correspond to *maximum* points of percussion, but opposite in direction. Thus there is a point P with which the body strikes in front and a point P' with which it strikes in rear of its own translation in space more forcibly than with any other point.

Ex. 1. Show that the two points P, P' are equally distant from S, and if O be the centre of oscillation with regard to S as a centre of suspension; $SP^2 = SG \,.\, SO$.

Ex. 2. If P be made a point of suspension, P' is the corresponding centre of oscillation. Also PP' is harmonically divided in G and O.

Ex. 3. The magnitudes of the blows at P, P' are inversely proportional to their distances from G.

Secondly, let the obstacle be a free particle of mass m.

Then, besides the equations (1), we have the equation of motion of the particle m. Let V' be its velocity after impact, $\therefore\ V' = \dfrac{R}{m}$.

The point of impact in the two bodies will have after impact the same velocity, hence instead of equation (2) we have $V' = v' + x\omega'$.

Eliminating ω', v', V', we get

$$R = M\omega \,.\, k^2 \,.\, \frac{m(x+h)}{(M+m)k^2 + mx^2}.$$

This is to be made a maximum. Equating to zero its differential coefficient with respect to x, we find

$$x = -h \pm \sqrt{h^2 + k^2\left(1 + \frac{M}{m}\right)} \qquad \text{(4)}.$$

This point does not coincide with that found when the obstacle was fixed, unless m is infinite. To find when it coincides with the centre of oscillation, we must put $k^2 = xh$. This gives $\dfrac{M}{m} = \dfrac{x+h}{h}$, or if $l = x+h$ be the length of the simple equivalent pendulum, $\dfrac{M}{m} = \dfrac{l}{h}$. Since $V' = \dfrac{R}{m}$, it is evident that when R is a maximum V' is a maximum. Hence the two points found by equation (4) might be called the centres of greatest communicated velocity.

There are other singular points in a moving body whose positions may be found; thus we might inquire at what point a body must impinge against a *fixed* obstacle, that *first* the linear velocity of the centre of gravity might be a maximum, or *secondly*, that the angular velocity might be a maximum. These points, respectively, have been called by Poinsot the centres of maximum Reflexion and Conversion. Referring to equations (1), we see that when v' is a maximum R is either a maximum or a minimum, and hence it may be shown that the first point coincides with the point of greatest impact. When ω' is a maximum, we have to make $\omega - \frac{Rx}{Mk^2} = maximum$. Substituting for R, this gives $x^2 - 2\frac{k^2}{h}x - k^2 = 0$. If O be the centre of oscillation, we have $GO = \frac{k^2}{h}$. Let this length be represented by h'. Then this equation becomes

$$x^2 - 2h'x - k^2 = 0 \qquad\qquad (5).$$

The roots of this equation are the same functions of h' and k that those of equation (3) are of h and k, except that the signs are opposite. Now S and O are on opposite sides of G, hence the positions of the two centres of maximum Conversion bear to O and G the same relation that the positions of the two centres of maximum Reflexion do to S and G. If the point of suspension be changed from S to O, the positions of the centres of maximum Reflexion and Conversion are interchanged.

Ex. A free lamina of any form is turning in its own plane about an instantaneous centre of rotation S and impinges on a fixed obstacle P, situated in the straight line joining the centre of gravity G to S. Find the position of P, *first*, that the centre of gravity may be reduced to rest, *secondly*, that its velocity after impact may be the same as before but reversed in direction.

Result. In the first case, P coincides either with G or with the centre of oscillation. In the second case the points $x = GP$ are found from the equation

$$x^2 - \frac{k^2}{2h}x + \frac{k^2}{2} = 0,$$

where $SG = h$. [Poinsot.]

156. *Two bodies impinge on each other, to explain the nature of the action which takes place between them.*

When two spheres of any hard material impinge on each other, they appear to separate almost immediately, and a finite change of velocity is generated in each by their mutual action. This sudden change of velocity is the characteristic of an impulsive force. Let the centres of gravity of the spheres be moving before impact in the same straight line with velocities u and v. Then after impact they will continue to move in the same straight line, and let u', v' be the velocities. Let m, m' be the masses of the spheres, R the action between them, then we have by Article 152,

$$\left.\begin{aligned} u' - u &= -\frac{R}{m} \\ v' - v &= \frac{R}{m'} \end{aligned}\right\} \qquad\qquad (1).$$

These equations are not sufficient to determine the three quantities u', v', R. To obtain a third equation we must consider what takes place during the impact.

Each of the balls will be slightly compressed by the other, so that they will no longer be perfect spheres. Each also will, in general, tend to return to its original shape, so that there will be a rebound. The period of impact may therefore be divided into two parts. First, the period of compression, while the distance between the centres of gravity of the two bodies is diminishing, and secondly the period of restitution, while the distance between the centres of gravity is increasing. At the termination of this second period the bodies separate.

The arrangement of the particles of a body being disturbed by impact, we ought to determine the relative motions of the several parts of the body. Thus we might regard each body as a collection of free particles connected by their mutual actions. These particles being thus set in motion might continue always in motion oscillating about some mean positions.

It is however usual to assume that the changes of shape and structure are so small that the effect in altering the position of the centre of gravity and the moments of inertia of the body may be neglected, and that the whole time of impact is so short that the motion of the body in that time may be neglected. If for any bodies these assumptions are not true, the effects of their impact must be deduced from the equations of the second order. We may therefore assert that at the moment of greatest compression the centres of gravity of the two spheres are moving with equal velocity.

The ratio of the magnitude of the action between the bodies during the period of restitution to that during compression is found to be different for bodies of different materials. In some cases this ratio is so small that the force during the period of restitution may be neglected. The bodies are then said to be *inelastic*. In this case we have just after the impact $u'=v'$. This gives $R=\dfrac{mm'}{m+m'}(u-v)$, whence $u'=\dfrac{mu+m'v}{m+m'}$.

If the force of restitution cannot be neglected, let R be the whole action between the balls, R_0 the action up to the moment of greatest compression. The magnitude of R must be found by experiment. This may be done by determining the values of u' and v', and thus determining R by means of equations (1). These experiments were made in the first instance by Newton, and the result is that $\dfrac{R}{R_0}$ is a *constant ratio* depending on the material of the balls. Let this constant ratio be called $1+e$. The quantity

e is always less than unity, in the limiting case when $e=1$ the bodies are said to be perfectly elastic.

The value of e being supposed known the velocities after impact may be easily found. The action R_0 must be first calculated as if the bodies were inelastic, then the whole value of R may be found by multiplying this result by $1+e$. This gives

$$R=\frac{mm'}{m+m'}(u-v)(1+e),$$

whence u' and v' may be found by equations (1).

157. As an example, let us consider how the motion of the reel discussed in Art. 153 would be affected if the string were elastic.

Since the point of the reel in contact with the string has no velocity at the moment of greatest compression, the impulsive tension found in the article referred to, measures the whole momentum communicated to the reel from the beginning of the impact up to the moment of greatest compression. By what has been said in the last article, the whole momentum communicated from the beginning to the termination of the period of restitution will be found by multiplying the tension found in Art. 153 by $1+e$, if e be the measure of the elasticity of the string. This gives

$$T=\tfrac{1}{3}mv(1+e).$$

The motion of a reel acted on by this known impulsive force is easily found. Resolving vertically we find

$$m(v'-v)=-\tfrac{1}{3}mv(1+e).$$

Taking moments about the centre of gravity

$$mk^2\omega'=\tfrac{1}{3}mva(1+e),$$

whence v' and ω' may be found.

Ex. A uniform beam is balanced about a horizontal axis through its centre of gravity, and a perfectly elastic ball is let fall from a height h on one extremity; determine the motion of the beam and ball.

Result. Let M, m be the masses of beam and ball, $2a$=length of beam, V, V' the velocities of ball at the moments just before and after impact, ω' the angular velocity of the beam. Then

$$\omega'=\frac{6mV}{(M+3m)a},\quad V'=V\cdot\frac{3m-M}{3m+M}.$$

158. Hitherto we have only considered the impulsive action normal to the common surface of the two bodies. If the bodies are rough there will clearly be an impulsive friction called into play. Since an impulse is only the integral of a very great force acting for a very short time, we might suppose that impulsive friction obeys the laws of ordinary friction. But these laws are founded on experiment, and we cannot be sure that they are correct in the extreme case in which the forces are very great. This point M. Morin undertook to determine by experiment at the express request of Poisson. He found that the frictional

impulse between two bodies which strike and slide bears to the normal impulse the same ratio as in ordinary friction, and that this ratio is independent of the relative velocity of the striking bodies. M. Morin's experiment is described in the following example.

159. A box AB which can be loaded with shot so as to be of any proposed weight has two vertical beams AC, BD erected on its lid; CD is joined by a cross piece and supports a weight equal to mg attached to it by a string. The weight of the loaded box is Mg. A string AEF passes horizontally from the box over a smooth pulley E and supports a weight at F equal to $(M+m)g\mu$. The box can slide on a horizontal plane whose coefficient of friction is μ, and therefore having been once set in motion, it moves in a straight line with a uniform velocity which we will call V. Suddenly the string supporting mg is cut, and this weight falls into the box and immediately becomes fixed to the box. Show that an impulsive friction is called into play between the box and the horizontal plane. Prove that if the velocity of the box immediately after the impulse is again equal to V, the coefficient of impulsive friction is equal to that of finite friction. Find also the whole space passed over by the box in any time which includes the impact.

160. When two inelastic bodies impinge on each other at some point A, the points in contact at the beginning of the impact have a relative velocity both along the common tangent plane at A and also along the normal. Thus two reactions will be called into play, a normal force and a friction, the ratio of these two being μ, the coefficient of friction. As the impact proceeds the relative normal velocity gets destroyed, and is zero at the moment of greatest compression. Let R be the whole momentum transferred normally from one body to the other in this very short time. This force R is an unknown reaction, to determine it we have the geometrical condition that just after impact the normal velocities of the points in contact are equal. This condition must be expressed in the manner explained in Art. 127.

The relative sliding velocity at A is also diminished. If it vanishes before the moment of greatest compression, then during the rest of the impact, only so much friction is called into play, and in such a direction, as is necessary (if any be necessary) to prevent the points in contact at A from sliding, provided that this amount is less than the limiting friction. Let F be the whole momentum transferred tangentially from one body to the other. This reaction F is to be determined by the condition that just after impact the tangential velocities of the points in contact are equal. If, however, the sliding motion does not vanish before the moment of greatest compression, then the whole of the friction is called into play in the direction opposite to that of relative sliding, and we have $F=\mu R$. Generally we may distinguish these two cases in the following manner. In the first case it is necessary that the values of F and R found by solving

the equations of motion should be such that $F<\mu R$. In the second case, the final relative velocity of the points in contact at A must be in the same direction after impact as before. These are however not sufficient conditions, for it is possible that, in more complicated cases, the sliding may change, or tend to change, its direction during the impact. See Art. 164.

161. If the impinging bodies be elastic, there may be both a normal reaction and a friction during the period of restitution. Sometimes we shall have to consider this stage of the motion as a separate problem. The motion of the bodies at the moment of greatest compression having been determined, these are to be regarded as the initial conditions of a new state of motion under different impulses. The friction called into play during restitution must follow the same laws as that during compression. Just as before, two cases will present themselves, either there will be sliding during the whole period of restitution or only during a portion of it. These are to be treated in the manner already explained.

162. There is one very important difference between the periods of compression and restitution. During the compression the normal reaction is unknown. The motion of the body just before compression is given, and we have a geometrical equation expressing the fact that the relative normal velocity of the points in contact is zero at the termination of the period of compression. From this geometrical equation we deduce the force of compression. The motion of the body just before restitution is thus found, but the motion just after is the thing we want to determine. For this, we have no geometrical equation, but the force of restitution bears a given ratio to the force of compression, and is therefore known.

163. *A spherical ball moving without rotation on a smooth horizontal plane impinges with velocity* v *against a rough vertical wall whose coefficient of friction is* μ. *The line of motion of the centre of gravity before incidence making an angle* α *with the normal to the wall, determine the motion just after impact.*

Let u, v be the velocities of the centre of the ball just before impact resolved along and perpendicular to the wall in such directions that they are both positive. Then $u=V\sin\alpha$, $v=V\cos\alpha$. Let u', v' be the velocities of the centre at any instant during the impact, resolved in the same directions, ω' the angular velocity at that instant. Let R be the normal blow from the beginning of the impact up to that instant, F the frictional blow. Then we have

$$\left.\begin{aligned} u'-u &= -\frac{F}{m} \\ v'-v &= -\frac{R}{m} \\ \omega' &= \frac{Fa}{mk^2} \end{aligned}\right\}.$$

If the instant considered be any moment subsequent to that at which the tangential velocity of the point of contact vanishes, we have $u' - a\omega' = 0$. This gives, since $k^2 = \frac{2}{5}a^2$, $F = \frac{2}{7}mV \sin \alpha$. Since F is independent of the moment considered, we see that no friction is called into play after the tangential velocity of the point of contact is reduced to zero.

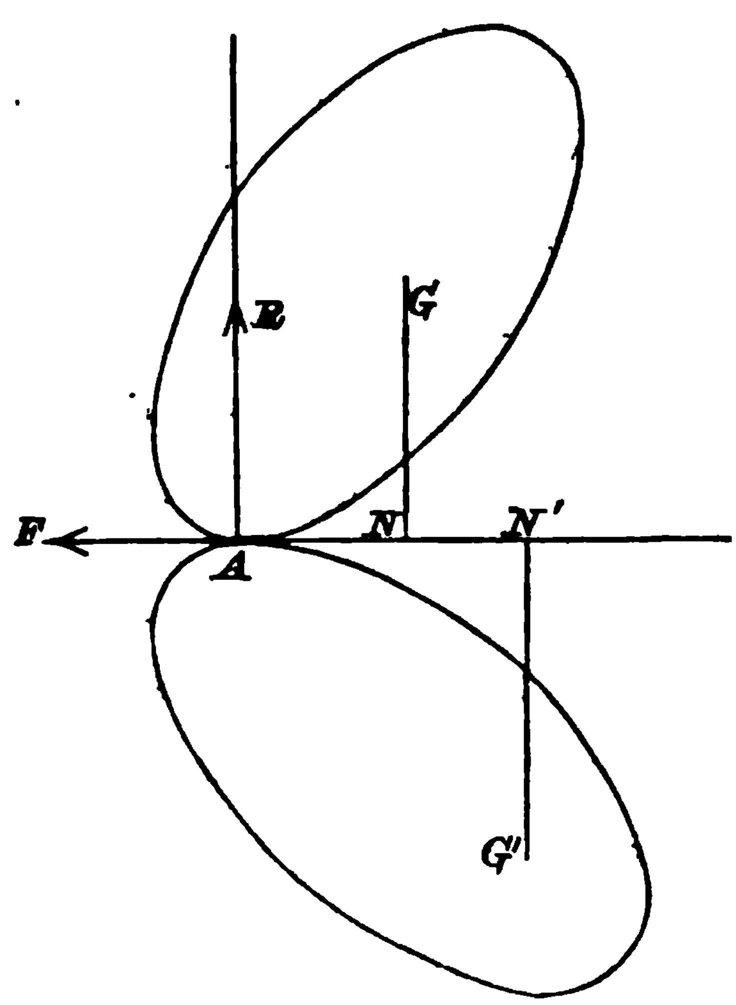
At the moment of greatest compression the normal velocity of the point of contact is zero, hence $V' = 0$ and $\therefore R = mV \cos \alpha$. If F be $< \mu R$, i.e. if $\frac{2}{7} \tan \alpha < \mu$, these will be the proper values of F and R, and by substituting in the equations the motion of the sphere may be found.

But if $\frac{2}{7} \tan \alpha > \mu$, the sphere will separate slightly from the wall before sufficient friction has been called into play to reduce the tangential velocity of the point of contact to zero. In this case we must replace F by μR in the equations. At the moment of greatest compression we have as before $V' = 0$. This gives $R = mV \cos \alpha$. By substituting in the equations the motion of the sphere may be found. The initial velocity of the point of contact is easily seen to be $u' - a\omega' = V(\sin \alpha - \mu \frac{7}{2} \cos \alpha)$. If this were negative, the friction at the end of the impact would be acting in the direction of relative motion, which is impossible. This solution is therefore correct only if $\frac{2}{7} \tan \alpha > \mu$.

If the sphere be imperfectly elastic, a normal force of restitution is called into play equal to $emV \cos \alpha$. If then $\frac{2}{7} mV \sin \alpha$ be $< \mu (1 + e) mV \cos \alpha$, the friction necessary to destroy the tangential velocity of the point of contact is less than the limiting friction. In this case by writing $F = \frac{2}{7} mV \sin \alpha$, $R = (1 + e) mV \cos \alpha$ in the equations of motion, we can find u', V' and ω'. If $\frac{2}{7} mV \sin \alpha$ be $> \mu (1 + e) mV \cos \alpha$, we must put $R = (1 + e) mV \cos \alpha$, $F = \mu (1 + e) mV \cos \alpha$, and the same equations will now give u', V' and ω'.

164. *Two rough bodies of any form impinge on each other in a given manner. It is required to find the motion just after impact.*

Let G, G' be the centres of gravity of the two bodies, A the



point of contact. Let U, V be the resolved velocities of G just before impact, parallel to the tangent and normal respectively at A; u, v the resolved velocities at any time t after the commencement of the impact, but before its termination. Then t is indefinitely small. Let Ω be the angular velocity of the body, whose centre of gravity is G, just before impact, ω the angular velocity after the interval t. Let M be the mass of the body, k its radius of gyration about G. Let GN be a perpendicular from G on the tangent at A, and let $AN=x$, $NG=y$. Let accented letters denote corresponding quantities for the other body.

Let R be the whole momentum communicated to the body M in the time t of the impact by the normal pressure, and let F be the momentum communicated by the frictional pressure. We shall suppose these to act on the body whose mass is M in the directions NG, NA respectively. Then they must be supposed to act in the opposite directions on the body whose mass is M'.

Since R represents the whole momentum communicated to the body M in the direction of the normal, the momentum communicated in the time dt is dR. As the bodies can only push against each other, dR must be positive, and, by Art. 126, when dR vanishes, the bodies separate. Thus the magnitude of R may be taken to measure the progress of the impact. It is zero at the beginning, gradually increases throughout, and is a maximum at the termination of the impact. It will be found more convenient to choose R rather than the time t as the independent variable.

The dynamical equations are by Art. 152

$$\left.\begin{aligned} M(u-U)&=-F\\ M(v-V)&=R\\ Mk^2(\omega-\Omega)&=Fy+Rx \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{aligned} M'(u'-U')&=F\\ M'(v'-V')&=-R\\ M'k'^2(\omega'-\Omega')&=Fy'-Rx' \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The relative velocity of sliding of the points in contact is by Art. 127

$$S=u-y\omega-u'-y'\omega'\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and the relative velocity of compression is by the same article

$$C=v'+x'\omega'-v-x\omega\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Substituting in these equations from the dynamical equations we find

$$S=S_0-aF-bR\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$C = C_0 - bF - a'R \text{ ....(6)},$$

where
$$S_0 = U - y\Omega - U' - y'\Omega' \text{ ....(7)},$$

$$C_0 = V' + x'\Omega' - V - x\Omega \text{ ....(8)},$$

$$a = \frac{1}{M} + \frac{1}{M'} + \frac{y^2}{Mk^2} + \frac{y'^2}{M'k'^2} \text{ ....(9)},$$

$$a' = \frac{1}{M} + \frac{1}{M'} + \frac{x^2}{Mk^2} + \frac{x'^2}{M'k'^2} \text{ ... .(10)},$$

$$b = \frac{xy}{Mk^2} - \frac{x'y'}{M'k'^2} \text{ ....(11)}.$$

These may be called the constants of the impact. The first two S_0, C_0 represent the initial velocities of sliding and compression. These we shall consider to be positive; so that the body M is sliding over the body M' at the beginning of the compression. The other three constants a, a', b are independent of the initial motion of the striking bodies. The constants a and a' are essentially positive, while b may have either sign. It will be found useful to notice that $aa' > b^2$.

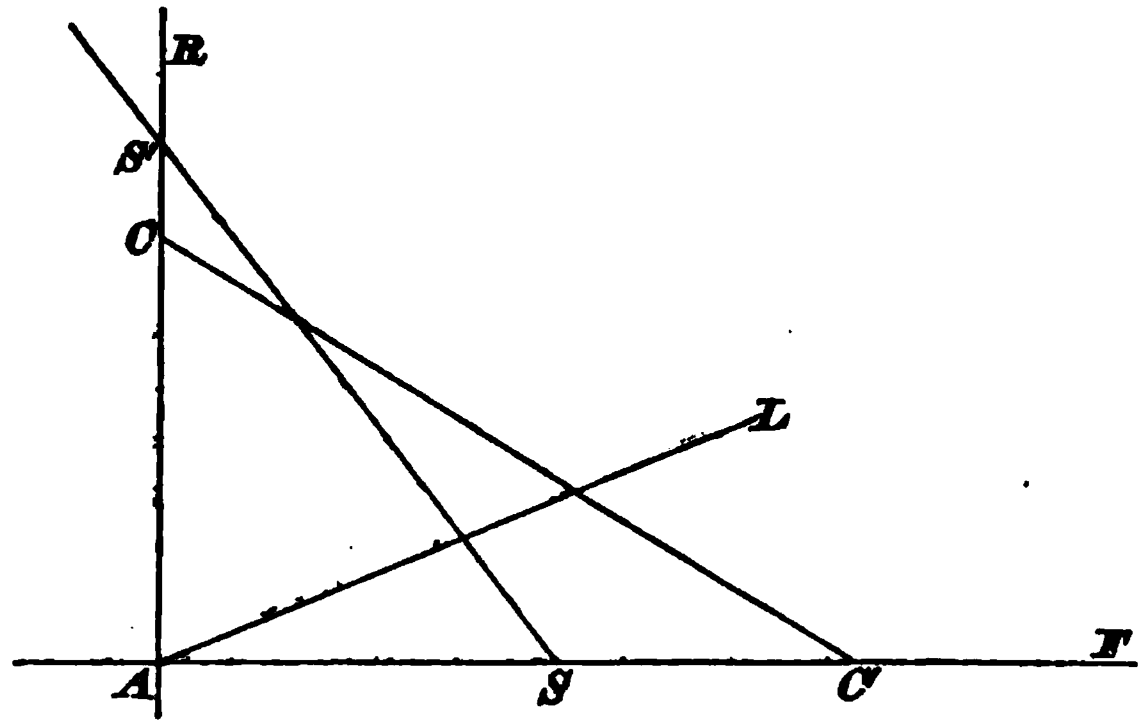
165. When $b = 0$, the discussion of these equations, as in Art. 163, does not present any difficulty, but in the general case it is more easy to follow the changes in the forces, if we adopt a graphical method. Let us draw two lengths AR, AF along the normal and tangent at A in the directions NG, AN respectively, to represent the magnitudes of R and F at any moment of the impact. Then if we consider AR and AF to be the co-ordinates of a point P, referred to AR, AF as axes of R and F, the changes in the position of P will indicate to the eye the changes that take place in the forces during the progress of the impact. It will be convenient to trace the two loci determined by $S = 0$, $C = 0$. By reference to (5) and (6) we see that they are both straight lines. These we shall call the straight lines of *no sliding* and of *greatest compression*. To trace these, we must find their intercepts on the axes of F and R. Take

$$AC = \frac{C_0}{a'},\ AS = \frac{S_0}{a},\ AC' = \frac{C_0}{b},\ AS' = \frac{S_0}{b},$$

then SS', CC' will be these straight lines. Since a and a' are necessarily positive, while b has any sign, we see that their intercepts on the axes of F and R respectively are *positive*, while their intercepts on the axes of R and F must have the *same sign*. Since $aa' > b^2$, the acute angle made by the line of no sliding with the axis of F is greater than that made by the line of greatest compression, i.e. the former line is steeper to the axis of F than the latter. It easily follows that the two straight lines cannot

intersect in the quadrant contained by RA produced and FA produced.

166. In the beginning of the impact the bodies slide over each other, hence, as explained in Art. 144, the whole limiting friction is called into play. The point P therefore moves along a



straight line AL, defined by the equation $F=\mu R$, where μ is the coefficient of friction. The friction will continue to be limiting until P reaches the straight line SS'. If R_0 be the abscissa of this point we find $R_0=\frac{S_0}{a\mu+b}$. This gives the whole normal blow, from the beginning of the impact, until friction can change from sliding to rolling. If R_0 is negative, the straight lines AL and SS' will not intersect on the positive side of the axis of F. In this case the friction will be limiting throughout the impact. If R_0 is positive the representative point P will reach SS'. After this only so much friction is called into play as will suffice to prevent sliding, provided this amount is less than the limiting friction. If the acute angle which SS' makes with the axis of R be less than $\tan^{-1}\mu$, the friction dF necessary to prevent sliding will be less than the limiting friction μdR. Hence P must travel along SS' in such a direction that the abscissa R continues to increase positively. In this case the friction will not again become limiting during the impact.

But if the acute angle which SS' makes with the axis of R be greater than $\tan^{-1}\mu$, the ratio of dF to dR will be numerically greater than μ, and more friction is necessary to prevent sliding than can be called into play. The friction will therefore continue to be limiting, and P, after reaching SS', must travel along a straight line, making the same angle with the axis of R that AL does. But this angle must be measured on the opposite side of the axis of R, for when the point P has crossed SS' the direction

of relative sliding and therefore the direction of friction is changed. In this case it is clear that the friction will continue limiting throughout the impact.

When P passes the straight line CC', compression ceases and restitution begins. But the passage is marked by no peculiarity except this. If R_1 be the abscissa of the point at which P crosses CC', the whole impact, for experimental reasons, is supposed to terminate when the abscissa of P is $R_2 = R_1(1+e)$, e being the measure of the elasticity of the two bodies.

It is obvious that a great variety of cases may occur according to the relative positions of the three straight lines AL, SS' and CC'. But in all cases the progress of the impact may be traced by the method just explained, which may be briefly stated thus. The representative point P travels along AL, until it meets SS'. It then proceeds either along SS', or along a straight line making the same angle with the axis of R as AL does, but on the opposite side. The one along which it proceeds is the steeper to the axis of F. It travels along this line in such a direction as to make the abscissa R increase. The complete value of R for the whole impact is found by multiplying the abscissa of the point at which P crosses CC' by $1+e$. The complete value of F is the corresponding ordinate of P. Substituting these in the dynamical equations (1) and (2), the motion just after impact may be easily found.

If the bodies be smooth, the straight line AL coincides with the axis of R. The representative point P must travel along the axis of R and the complete value of R for the whole impact is found by multiplying the abscissa of C by $1+e$.

167. It is not necessary that the friction should keep the same direction during the impact. The friction must keep one sign when P travels along AL. But when P reaches SS', its direction of motion changes, and the friction dF called into play in the time dt may have the same sign as before or the opposite. But it is clear that the friction can change sign only once during the impact.

It is possible that the friction may continue limiting throughout the impact, so that the bodies slide on each other throughout. The necessary conditions are that either the straight line SS' must be less steep to the axis of F than AL, or the point P must not reach the straight line SS' until its abscissa has become greater than R_2. The condition for the first case is, that b must be greater than μa. The abscissæ of the intersections of AL with SS' and CC' are respectively $R_0 = \dfrac{S_0}{a\mu + b}$ and

$R_1 = \dfrac{C_0}{b\mu + a'}$. The condition for the second case is necessary, that R_1 must be positive, and R_0 either negative or positively greater than $R_1 (1 + e)$.

168. Ex. 1. Show that the representative point P as it travels in the manner directed in the text must cross the line of greatest compression, and that the abscissa R of the point at which it crosses this straight line must be positive.

Ex. 2. Show that the conic whose equation referred to the axes of R and F is $aF^2 + 2bFR + a'R^2 = \epsilon$, where ϵ is some constant, is an ellipse, and that the straight lines of no sliding and greatest compression are parallel to the conjugates of the axes of F and R respectively. Show also that the intersection of the straight lines of no sliding and greatest compression must lie in that angle formed by the conjugate diameters which contains or is contained by the first quadrant.

Ex. 3. Two bodies, each turning about a fixed point, impinge on each other, find the motion just after impact.

Let G, G', in the figure of Art. 164, be taken as the fixed points. Taking moments about the fixed points, the results will be nearly the same as those given in the case considered in the text.

Initial Motions.

169. Suppose a system of bodies to be in equilibrium and that one of the supports suddenly gives way. It is required to find the initial motion of the bodies and the initial values of the reactions which exist between the several bodies.

The problem of finding the initial motion of a dynamical system is the same as that of expanding the co-ordinates of the moving particles in powers of the time t. Let (x, y, θ) be the co-ordinates of any body of the system. For the sake of brevity let us denote by accents differential coefficients with regard to the time, and let the suffix zero denote initial values. Thus x_0'' denotes the initial value of $\dfrac{d^2x}{dt^2}$. By Taylor's theorem we have

$$x = a + x_0'' \frac{t^2}{\underline{|2}} + x_0''' \frac{t^3}{\underline{|3}} + \dots\dots\dots\dots\dots\dots\dots (1):$$

the term x_0' is omitted because we shall suppose the system to start from rest.

First, let only the initial values of the reactions be required. The dynamical equations will contain the co-ordinates, their second differential coefficients with regard to t, and the unknown reactions. There will be as many geometrical equations as reactions. From these we have to eliminate the second differential

coefficients and find the reactions. The process will be as follows, which is really the same as the first method of solution described in Art. 125.

Write down the geometrical equations, differentiate each twice and then simplify the results by substituting for the co-ordinates their initial values. Thus, if we use Cartesian co-ordinates, let $\phi(x, y, \theta) = 0$ be any geometrical relation, we have since $x_0' = 0$, $y_0' = 0$, $\theta_0' = 0$,

$$\frac{d\phi}{dx} x_0'' + \frac{d\phi}{dy} y_0'' + \frac{d\phi}{d\theta} \theta_0'' = 0.$$

The process of differentiating the equations may sometimes be much simplified when the origin has been so chosen that the initial values of some at least of the co-ordinates are zero. We may then simplify the equations by neglecting the squares and products of all such co-ordinates. For if we have a term x^2, its second differential coefficient is $2(xx'' + x'^2)$, and if the initial value of x is zero, this vanishes.

The geometrical equations must be obtained by supposing the bodies to have their *displaced* position, because we require to differentiate them. But this is not the case with the dynamical equations. These we may write down on the supposition that each body is in its *initial* position. These equations may be obtained according to the rules given in Art. 125. The forms there given for the effective forces admit in this problem of some simplifications. Thus since $r_0' = 0$, $\phi_0' = 0$, the accelerations along and perpendicular to the radius vector take the simple forms r_0'' and $r\phi_0''$. So again the acceleration $\frac{v^2}{\rho}$ along the normal vanishes. If, for example, we know the initial direction of motion of the centre of gravity of any one of the bodies, we might conveniently resolve along the normal to the path. This will supply an equation which contains only the impressed forces and such tensions or reactions as may act on that body. If there be only one reaction, this equation will suffice to determine its initial value.

We may also deduce from the equations the values of x_0'', y_0'', θ_0'', and thus by substituting in equation (1) we have found the initial motion up to terms depending on t^2.

170. *Secondly, let the initial motion be required.* How many terms of the series (1) it may be necessary to retain will depend on the nature of the problem. Suppose the radius of curvature of the path described by the centre of gravity of one of the bodies to be required. We have

$$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''},$$

10—2

and by differentiating equation (1)

$$x' = x_0''t + x_0'''\frac{t^2}{\underline{|2}} + x_0^{iv}\frac{t^3}{\underline{|3}} + \ldots$$

$$x'' = x_0'' + x_0'''t + x_0^{iv}\frac{t^2}{\underline{|2}} + \ldots$$

$$\&c. = \&c.;$$

$$\therefore (x'^2 + y'^2)^{\frac{3}{2}} = (x_0''^2 + y_0''^2)^{\frac{3}{2}} t^3 + \ldots$$

$$x'y'' - y'x'' = (x_0''y_0''' - x_0'''y_0'')\frac{t^2}{2} + (x_0''y_0^{iv} - x_0^{iv}y_0'')\frac{t^3}{3} + \ldots$$

results which may also be obtained by a direct use of Taylor's theorem.

If then the body start from rest, the radius of curvature is zero. But if $x_0''y_0''' - x_0'''y_0'' = 0$, we have

$$\rho = 3\frac{(x_0''^2 + y_0''^2)^{\frac{3}{2}}}{x_0''y_0^{iv} - x_0^{iv}y_0''}.$$

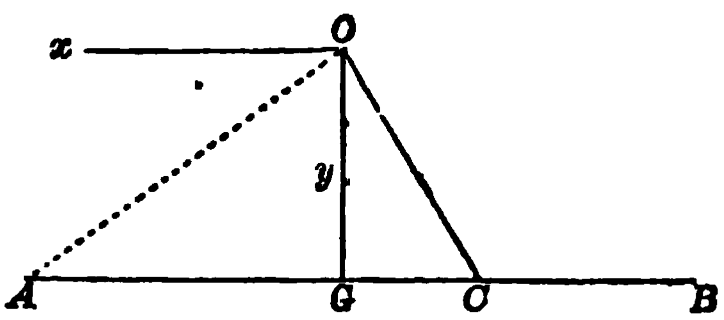
To find these differential coefficients we may proceed thus. Differentiate each dynamical equation twice and then reduce it to its initial form by writing for x, y, θ, &c. their initial values, and for x', y', θ' zero. Differentiate each geometrical equation four times and then reduce each to its initial form. We shall thus have sufficient equations to determine x_0'', x_0''', x_0^{iv}, &c., R_0, R_0', R_0'', &c., where R is any one of the unknown reactions. It will often be an advantage to eliminate the unknown reactions from the equations *before* differentiation. We shall then have only the unknown coefficients x_0'', x_0''', &c. entering into the equations.

If we know the direction of motion of one of the centres of gravity under consideration, we can take the axis of y a tangent to its path. Then we have $\rho = \frac{y^2}{2x}$, where x is of the second order, y of the first order, of small quantities. We may therefore neglect the squares of x and the cubes of y. This will greatly simplify the equations. If the body start from rest we have $x_0' = 0$, and if $x_0'' = 0$, we may then use the formula

$$\rho = 3\frac{y_0''^2}{x_0^{iv}}.$$

171. **Ex.** *A circular disc is hung up by three equal strings attached to three points at equal distances in its circumference, and fastened to a peg vertically over the centre of the disc. One of these strings is suddenly cut. Determine the initial circumstances of motion.*

Let O be the peg, AB the circle seen by an eye in its plane. Let OA be the string which is cut and C be the middle point of the chord joining the points of the



circle to which the two other strings are attached. Then the two tensions, each equal to T, are throughout the motion equivalent to a resultant tension R along CO. If 2α be the angle between the two strings, we have

$$R = 2T\cos\alpha.$$

Let l be the length of OC, β be the angle GOC, a be the radius of the disc. Let (x, y) be the co-ordinates of the *displaced* position of the centre of gravity with reference to the origin O, x being measured horizontally to the left and y vertically downwards. Let θ be the angle the displaced position of the disc makes with AB.

By drawing the disc in its displaced position it will be seen that the co-ordinates of the displaced position of C are $x - l\sin\beta\cos\theta$ and $y - l\sin\beta\sin\theta$. Hence since the length OC remains constant and equal to l we have

$$x^2 + y^2 - 2l\sin\beta\,(x\cos\theta + y\sin\theta) = l^2\cos^2\beta.$$

Suppose the initial tensions only to be required. It will be sufficient to differentiate this twice. Since we may neglect the squares of small quantities, we may omit x^2, put $\cos\theta = 1$, $\sin\theta = \theta$. The process of differentiation will not then be very long, for it is easy to see beforehand what terms will disappear when we equate the differential coefficients (x', y', θ') to zero, and put for (x, y, θ) their initial values $(0, l\cos\beta, 0)$. We get

$$y_0''\cos\beta = \sin\beta\,(x_0'' + l\cos\beta\theta_0'').$$

This equation may also be obtained by an artifice which is often useful. The motion of G is made up of the motion of C and the motion of G relatively to C. Since C begins to describe a circle from rest, its acceleration along CO is zero. Again, the acceleration of G relatively to C when resolved along CO is $GC\,\dfrac{d^2\theta}{dt^2}\cos\beta$. The resolved acceleration of G is the sum of these two, but it is also

$$y_0''\cos\beta - x_0''\sin\beta.$$

Hence the equation follows at once.

In this case we require the differential equations only in their initial form. These are

$$\left.\begin{aligned} mx_0'' &= R_0\sin\beta \\ my_0'' &= mg - R_0\cos\beta \\ mk^2\theta_0'' &= R_0\, l\sin\beta\cos\beta \end{aligned}\right\},$$

where m is the mass of the body. Substituting in the geometrical equation we find

$$R_0 = mg\,.\,\frac{\cos\beta}{1 + \dfrac{l^2}{k^2}\sin^2\beta\cos^2\beta}\,.$$

The tension of any string, *before* the string OA was cut, may be found by the rules of Statics, and is clearly $T_1 = \frac{mg}{3 \cos \gamma}$, where γ is the angle AOG. Hence the change of tension can be found.

172. Ex. 1. Two strings of equal length have each an extremity tied to a weight C and their other extremities tied to two points A, B in the same horizontal line. If one be cut the tension of the other is instantaneously altered in the ratio $1 : 2 \cos^2 \frac{C}{2}$. [St Pet. Coll.]

Ex. 2. An elliptic lamina is supported with its plane vertical and transverse axis horizontal by two weightless pegs passing through the foci. If one pin be released show that if the eccentricity of the ellipse be $\sqrt{\frac{2}{5}}$, the pressure on the other pin will be initially unaltered. [Coll. Exam.]

Ex. 3. Three equal particles A, B, C repelling each other with any forces, are tied together by three strings of unequal length, so as to form a triangle right-angled at A. If the string joining B and C be cut, prove that the instantaneous changes of tension of the strings joining BA, CA are $\frac{1}{3} T \cos B$ and $\frac{1}{3} T \cos C$ respectively, where B and C are the angles opposite the strings joining CA, AB respectively, and T is the repulsive force between B and C.

Ex. 4. Two uniform equal rods, each of mass m, are placed in the form of the letter X on a smooth horizontal plane, the upper and lower extremities being connected by equal strings; show that whichever string be cut, the tension of the other is the same function of the inclination of the rods, and initially is $\frac{3}{8} mg \sin \alpha$, where α is the initial inclination of the rods. [St Pet. Coll.]

Ex. 5. A horizontal rod of mass m and length $2a$ hangs by two parallel strings of length $2a$ attached to its ends: an angular velocity ω being suddenly communicated to it about a vertical axis through its centre, show that the initial increase of tension of either string equals $\frac{ma\omega^2}{4}$, and that the rod will rise through a space $\frac{a^2\omega^2}{6g}$. [Coll. Exam.]

Ex. 6. A particle is suspended by three equal strings of length a from three points forming an equilateral triangle of side $2b$ in a horizontal plane. If one string be cut the tension of each of the others is instantaneously changed in the ratio $\frac{3a^2 - 4b^2}{2(a^2 - b^2)}$. [Coll. Exam.]

Ex. 7. A sphere resting on a rough horizontal plane is divided into an infinite number of solid lines and tied together again with a string; the axis through which the plane faces of the lines pass being vertical. Show that if the string be cut the pressure on the plane is diminished instantaneously in the ratio $45\pi^2 : 2048$. [Emm. Coll.]

On Relative Motion or Moving Axes.

173. In many dynamical problems the relative motion of the different bodies of the system is frequently all that is required. In these cases it will be an advantage if we can determine this without finding the absolute motion of each body in space. Let us suppose that the motion relative to some one body (A) is required. There are then two cases to be considered, (1) when the body (A) has a motion of translation only, and (2) when it has a motion of rotation only. The case in which the body (A) has a motion both of translation and rotation may be regarded as a combination of these two cases. Let us consider these in order.

174. Let it be required to find the motion of any dynamical system relative to some moving point C. We may clearly reduce C to rest by applying to every element of the system an acceleration equal and opposite to that of C. It will also be necessary to suppose that an *initial* velocity equal and opposite to that of C has been applied to each element.

Let f be the acceleration of C at any time t. If every particle m of a body be acted on by the same accelerating force f parallel to any given direction, it is clear that these are together equivalent to a force $f\Sigma m$ acting at the centre of gravity. Hence to reduce any point C of a system to rest, it will be sufficient to apply to the centre of gravity of each body in a direction opposite to that of the acceleration of C a force measured by Mf, where M is the mass of the body and f the acceleration of C.

The point C may now be taken as the origin of co-ordinates. We may also take moments about it as if it were a point fixed in space.

Let us consider the equation of moments a little more minutely. Let (r, θ) be the polar co-ordinates of any element of a body whose mass is m referred to C as origin. The accelerations of the particle are $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$, along and perpendicular to the radius vector r. Taking moments about C, we get

$$\Sigma m \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \left\{\begin{array}{l}\text{moment round } C \text{ of the impressed forces}\\ \text{plus the moment round } C \text{ of the reversed}\\ \text{effective forces of } C \text{ supposed to act at the}\\ \text{centre of gravity.}\end{array}\right.$$

If the point C be fixed in the body and move with it, $\frac{d\theta}{dt}$ will be the same for every element of the body, and, as in Art. 88, we have $\Sigma m \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Mk^2\frac{d^2\theta}{dt^2}$.

175. From the general equation of moments about a moving point C we learn that we may use the equation

$$\frac{d\omega}{dt} = \frac{\text{moment of forces about } C}{\text{moment of inertia about } C}$$

in the following cases.

First. If the point C be fixed both in the body and in space; or, if the point C being fixed in the body move in space with uniform velocity; for the acceleration of C is zero.

Secondly. If the point C be the centre of gravity; for in that case, though the acceleration of C is not zero, yet the moment vanishes.

Thirdly. If the point C be the instantaneous centre of rotation*, and the motion be a small oscillation or an initial motion which starts from rest. At the time t the body is turning about C, and the velocity of C is therefore zero. At the time $t + dt$, the body is turning about some point C' very near to C. Let $CC' = d\sigma$, then the velocity of C is $\omega d\sigma$. Hence in the time dt the velocity of C has increased from zero to $\omega d\sigma$, therefore its acceleration is $\omega \frac{d\sigma}{dt}$. To obtain the accurate equation of moments about C we must apply the effective force $\Sigma m . \omega \frac{d\sigma}{dt}$ in the reversed direction at the centre of gravity. But in small oscillations ω and $\frac{d\sigma}{dt}$ are both small quantities whose squares and products are to be neglected, and in an initial motion ω is zero. Hence the moment of this force must be neglected, and the equation of motion will be the same as if C had been a fixed point.

It is to be observed that we may take moments about any point very near to the instantaneous centre of rotation, but it will usually be most convenient to take moments about the centre in its disturbed position. If there be any unknown reactions at the centre of rotation, their moments will then be zero.

176. If the accurate equation of moments about the instantaneous centre be required, we may proceed thus. Let L be the moment of the impressed forces about the instantaneous centre,

* If a body be in motion in one plane it is known that the actual displacement of every particle in the time dt is the same as if the body had been turned through some angle ωdt about some fixed point C. This may be proved in the same way as the corresponding proposition in Three Dimensions is proved in the next Chapter. See Art. 183. The point C is called the instantaneous centre of rotation, and ω is called the instantaneous angular velocity. See also Salmon's *Higher Plane Curves*, 1852, Arts. 246 and 264.

G the centre of gravity, r the distance between the centre of gravity and the instantaneous centre C, M the mass of the body; then the moment of the impressed forces and the reversed effective forces about C is

$$L - M\omega \frac{d\sigma}{dt} . r \cos GC'C.$$

If k be the radius of gyration about the centre of gravity, the equation of motion becomes

$$M(k^2 + r^2)\frac{d\omega}{dt} = L - M\omega r \frac{dr}{dt},$$

writing for $\cos GC'C$ its value $\frac{dr}{d\sigma}$.

177. Ex. 1. *Two heavy particles whose masses are* m *and* m' *are connected by an inextensible string, which is laid over the vertex of a double inclined plane whose mass is* M, *and which is capable of moving freely on a smooth horizontal plane. Find the force which must act on the wedge that the system may be in a state of relative equilibrium.*

Here it will be convenient to reduce the wedge to rest by applying to every particle an acceleration f equal and opposite to that of the wedge. Supposing this done the whole system is in equilibrium. If F be the required force, we have by resolving horizontally $(M + m + m')f = F$.

Let α, α' be the inclinations of the sides of the wedge to the horizontal. The particle m is acted on by mg vertically and mf horizontally. Hence the tension of the string is $m(g\sin\alpha + f\cos\alpha)$. By considering the particle m', we find the tension to be also $m'(g\sin\alpha' - f\cos\alpha')$. Equating these two we have

$$f = \frac{m'\sin\alpha' - m\sin\alpha}{m'\cos\alpha' + m\cos\alpha}g.$$

Hence F is found.

178. Ex. 2. *A cylindrical cavity whose section is any oval curve and whose generating lines are horizontal is made in a cubical mass which can slide freely on a smooth horizontal plane. The surface of the cavity is perfectly rough and a sphere is placed in it at rest so that the vertical plane through the centres of gravity of the mass and the sphere is perpendicular to the generating lines of the cylinder. A momentum* B *is communicated to the cube by a blow in this vertical plane. Find the motion of the sphere relatively to the cube and the least value of the blow that the sphere may not leave the surface of the cavity.*

Simultaneously with the blow B there will be an impulsive friction between the cube and the sphere. Let M, m be the masses of the cube and sphere, a the radius of the sphere, k its radius of gyration about a diameter. Let V_0 be the initial velocity of the cube, v_0 that of the centre of the sphere *relatively* to the cube, ω_0 the initial angular velocity. Then by resolving horizontally for the whole system, and taking moments for the sphere alone about the point of contact, we have

$$\left.\begin{aligned} m(v_0 + V_0) + MV_0 &= B \\ a(v_0 + V_0) + k^2\omega_0 &= 0 \end{aligned}\right\} \quad\quad\quad (1),$$

and since there is no sliding

$$v_0 - a\omega_0 = 0 \qquad (2).$$

To find the subsequent motion, let (x, y) be the co-ordinates of the centre of the sphere referred to rectangular axes attached to the cubical mass, x being horizontal and y vertical, then the equation to the cylindrical cavity being given, y is a known function of x. Let ψ be the angle the tangent to the cavity at the point of contact of the sphere makes with the horizon, then $\tan\psi = \frac{dy}{dx}$. Let V be the velocity of the cubical mass, then, by Art. 131,

$$m\left(\frac{dx}{dt} + V\right) + MV = B \qquad (3).$$

If T_0 be the initial vis viva and y_0 the initial value of y, we have by the equation of vis viva

$$m\left\{\left(\frac{dx}{dt} + V\right)^2 + \left(\frac{dy}{dt}\right)^2 + k^2\omega^2\right\} + MV^2 = T_0 - 2mg(y - y_0) \qquad (4),$$

where ω is the angular velocity of the sphere at the time t. If v be the velocity of the centre of the sphere relatively to the cube, we have since there is no sliding $v = a\omega$. Eliminating V and ω from these equations, we have

$$\left(\frac{dx}{dt}\right)^2 . \left\{(1 + \tan^2\psi)\left(1 + \frac{k^2}{a^2}\right) - \frac{m}{M+m}\right\} = Cg - 2gy \qquad (5),$$

where
$$Cg = \frac{B^2}{(M+m)\left\{M + (M+m)\frac{k^2}{a^2}\right\}} + 2gy_0.$$

This equation gives the motion of the sphere relatively to the cube.

To find the pressure on the cube, let us reduce the cube to rest. Let R be the pressure of the sphere on the cube, then the whole effective force on the cube is $R\sin\psi$ parallel to the axis of x. By Art. 174 we must therefore apply to every particle an acceleration $\frac{R\sin\psi}{M}$ opposite to this effective force. The sphere will then be acted on by $\frac{m}{M}R\sin\psi$ in a horizontal direction in addition to the reaction R, the friction and its own weight. Resolving the forces on its centre along a normal to its path we have

$$\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}\frac{m}{\rho} = R + \frac{m}{M}R\sin^2\psi - mg\cos\psi \qquad (6),$$

where ρ is the radius of curvature of the path of the centre of the sphere. Eliminating $\frac{dx}{dt}$ by the help of the equation of vis viva, we have

$$C - 2y + \rho\cos\psi\left(1 + \frac{k^2}{a^2} - \frac{m\cos^2\psi}{M+m}\right) = RF \qquad (7),$$

where $mgF = \rho\left(1 + \frac{k^2}{a^2} - \frac{m\cos^2\psi}{M+m}\right)\left(1 + \frac{m}{M}\sin^2\psi\right)$, and if ρ does not change sign, is essentially a positive quantity.

At the point where the sphere leaves the surface of the cavity R vanishes. Putting $R = 0$, we have an equation to determine ψ at this point, C being a known function of the initial conditions. If the sphere is to go all round the cylindrical cavity, the values of $\cos\psi$ given by this equation must be all imaginary or numeri-

cally greater than unity. If the sphere is *just* to go all round, then R must be positive throughout and must vanish at the point where it is least. In this case we have R and $\frac{dR}{d\psi}$ simultaneously zero. Differentiating we have

$$\frac{d\log\rho}{d\psi}\cos\psi\left(1+\frac{k^2}{a^2}-\frac{m}{M+m}\cos^2\psi\right)=\left(\frac{k^2}{a^2}+3-\frac{3m}{M+m}\cos^2\psi\right)\sin\psi\ldots\ldots(8).$$

This equation, since ρ is given as a function of ψ from the equation to the cylinder, determines ψ; C is then known from (7) when R is put equal to zero, and thence the required value of B.

We may notice that the position of the point at which R is to be put zero is independent of the initial conditions, and depends on the form of the cavity and the ratio of the masses of the cube and sphere. This point cannot be at the highest point of the cavity unless the radius of curvature of the cavity is at that point a maximum or minimum. If the section of the cavity be a circle or an ellipse having its major axis horizontal, then the equation to find ψ is satisfied only when $\psi=\pi$. In this case we find as the least value of the blow B to be given to the cube that the sphere may go all round

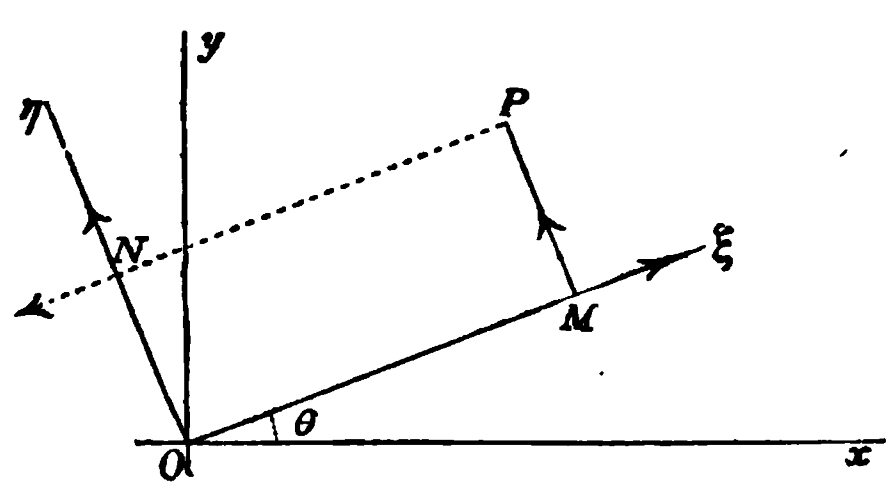
$$\frac{B^2}{g}=\left\{M+(M+m)\frac{k^2}{a^2}\right\}\cdot\left\{4(M+m)\beta+\left(M+(M+m)\frac{k^2}{a^2}\right)\frac{\alpha^2}{\beta}\right\},$$

where α and β are the semi-axes of the ellipse.

179. Next, let us consider the case in which we wish to refer the motion to two straight lines $O\xi$, $O\eta$, turning round a fixed origin O with angular velocity ω.

Let Ox, Oy be any fixed axes and let the angle $xO\xi=\theta$. Let $\xi=OM$, $\eta=PM$ be the co-ordinates of any point P.

It is evident that the motion of P is made up of the motions of the two points M, N by simple addition. The resolved parts of the velocity of M are $\frac{d\xi}{dt}$ and $\xi\omega$ along and perpendicular to OM.



The resolved parts of the velocity of N are in the same way $\frac{d\eta}{dt}$ and $\eta\omega$ along and perpendicular to ON. By adding these with their proper signs we have

$$\left.\begin{array}{l}\text{velocity of } P\\ \text{parallel to } O\xi\end{array}\right\} = \frac{d\xi}{dt} - \eta\omega,$$

$$\left.\begin{array}{l}\text{velocity of } P\\ \text{parallel to } O\eta\end{array}\right\} = \frac{d\eta}{dt} + \xi\omega.$$

In the same way by adding the accelerations of M and N we have

$$\left.\begin{array}{r}\text{acceleration of } P\\ \text{parallel to } O\xi\end{array}\right\} = \frac{d^2\xi}{dt^2} - \xi\omega^2 - \frac{1}{\eta}\frac{d}{dt}(\eta^2\omega),$$

$$\left.\begin{array}{r}\text{acceleration of } P\\ \text{parallel to } O\eta\end{array}\right\} = \frac{d^2\eta}{dt^2} - \eta\omega^2 + \frac{1}{\xi}\frac{d}{dt}(\xi^2\omega).$$

By using these formulæ instead of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ we may refer the motion to the moving axes $O\xi$, $O\eta$.

In a similar manner we may use polar co-ordinates. In this case if (r, ϕ) be the polar co-ordinates of P, we have

$$\left.\begin{array}{l}\text{acceleration of } P\\ \text{along rad. vect.}\end{array}\right\} = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt} + \omega\right)^2,$$

$$\left.\begin{array}{l}\text{acceleration of } P\\ \text{perp. to rad. vect.}\end{array}\right\} = \frac{1}{r}\frac{d}{dt}\left\{r^2\left(\frac{d\phi}{dt} + \omega\right)\right\}.$$

180. Ex. 1. Let the axes $O\xi$, $O\eta$ be oblique and make an angle α with each other, prove that if the velocity be represented by the two components u, v parallel to the axes,

$$u = \frac{d\xi}{dt} - \omega\xi\cot\alpha - \omega\eta\operatorname{cosec}\alpha,$$

$$v = \frac{d\eta}{dt} + \omega\eta\cot\alpha + \omega\xi\operatorname{cosec}\alpha.$$

In this case PM is parallel to $O\eta$. The velocities of M and N are the same as before. Their resultant is, by the question, the same as the resultant of u and v. By resolving in any two directions and equating the components we get two equations to find u and v. The best directions to resolve along are those perpendicular to $O\xi$ and $O\eta$, for then u is absent from one of the equations and v from the other. Thus u or v may be found separately when the other is not wanted.

Ex. 2. If the acceleration be represented by the components X and Y, prove

$$X = \frac{du}{dt} - \omega u\cot\alpha - \omega v\operatorname{cosec}\alpha,$$

$$Y = \frac{dv}{dt} + \omega v\cot\alpha + \omega u\operatorname{cosec}\alpha.$$

These may be obtained in the same way by resolving velocities and accelerations perpendicular to $O\xi$ and $O\eta$.

181. Ex. *A particle under the action of any forces moves on a smooth curve which is constrained to turn with angular velocity ω about a fixed axis. Find the motion relative to the curve.*

Let us suppose the motion to be in three dimensions. Take the axis of Z as the fixed axis, and let the axes of ξ, η be fixed relatively to the curve. Then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \xi\omega^2 - \frac{1}{\eta}\frac{d}{dt}(\eta^2\omega) &= X + Rl \\ \frac{d^2\eta}{dt^2} - \eta\omega^2 + \frac{1}{\xi}\frac{d}{dt}(\xi^2\omega) &= Y + Rm \\ \frac{d^2z}{dt} &= Z + Rn \end{aligned}\right\} \qquad (1),$$

where X, Y, Z are the resolved parts of the impressed accelerating forces resolved parallel to the axes, R is the pressure on the curve, and (l, m, n) the direction-cosines of the direction of R. Then since R acts perpendicularly to the curve

$$l\frac{d\xi}{ds} + m\frac{d\eta}{ds} + n\frac{d\zeta}{ds} = 0.$$

Suppose the moving curve to be projected orthogonally on the plane of ξ, η, let σ be the arc of the projection, and $v' = \frac{d\sigma}{dt}$ be the resolved part of the velocity parallel to the plane of projection. Then the equations may be written in the form

$$\frac{d^2\xi}{dt^2} = X + \omega^2\xi + \frac{d\omega}{dt}\eta + 2\omega v'\frac{d\eta}{d\sigma} + Rl,$$

$$\frac{d^2\eta}{dt^2} = Y + \omega^2\eta - \frac{d\omega}{dt}\xi - 2\omega v'\frac{d\xi}{d\sigma} + Rm,$$

$$\frac{d^2z}{dt^2} = Z + Rn.$$

The two terms $2\omega v'\frac{d\eta}{d\sigma}$ and $-2\omega v'\frac{d\xi}{d\sigma}$ may be regarded as the resolved parts of a force $2\omega v'$ acting in a direction whose direction-cosines are

$$l' = \frac{d\eta}{d\sigma}, \quad m' = \frac{-d\xi}{d\sigma}, \quad n' = 0.$$

These satisfy the equation $l'\frac{d\xi}{ds} + m'\frac{d\eta}{ds} + n'\frac{d\zeta}{ds} = 0.$

Hence the force is perpendicular to the tangent to the curve, and also perpendicular to the axis of rotation. Let R' be the resultant of the reaction R and of the force $2\omega v'$. Then R' also acts perpendicularly to the tangent, let (l'', m'', n'') be the direction-cosines of its direction.

The equations of motion therefore become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= X + \omega^2\xi + \frac{d\omega}{dt}\eta + R'l'' \\ \frac{d^2\eta}{dt^2} &= Y + \omega^2\eta - \frac{d\omega}{dt}\xi + R'm'' \\ \frac{d^2z}{dt^2} &= Z + R'n''. \end{aligned}\right\} \qquad (2).$$

These are the equations of motion of a particle moving on a *fixed curve*, and acted on in addition to the impressed forces by two extra forces, viz. (1) a force $\omega^2 r$ tending directly from the axis, where r is the distance of the particle from the axis, and (2) a force $\frac{d\omega}{dt} r$ perpendicular to the plane containing the particle and the axis, and tending opposite to the direction of rotation of the curve.

In any particular problem we may therefore treat the curve as fixed. Thus suppose the curve to be turning round the axis with uniform angular velocity.

Then resolving along the tangent we have

$$v\frac{dv}{ds} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} + \omega^2 r\frac{dr}{ds},$$

where r is the distance of the particle from the axis. Let V be the initial value of v, r_0 that of r. Then

$$v^2 - V^2 = 2\int(X\,dx + Y\,dy + Z\,dz) + \omega^2(r^2 - r_0^2).$$

Let v_0 be the velocity the particle would have had under the action of the same forces if the curve had been fixed. Then

$$v_0^2 - V^2 = 2\int(X\,dx + Y\,dy + Z\,dz).$$

Hence
$$v^2 - v_0^2 = \omega^2(r^2 - r_0^2).$$

The pressure on the moving curve is not equal to the pressure on the fixed curve. The pressure R on the moving curve is clearly the resultant of the pressure R' on the fixed curve, and a pressure $2\omega v'$ acting perpendicular both to the curve and to the axis in the direction of motion of the curve.

Thus suppose the curve to be plane and revolving uniformly about an axis perpendicular to its plane, and that there are no impressed forces. We have, resolving along the normal,

$$\frac{v^2}{\rho} = -\omega^2 r\sin\phi + R',$$

where ϕ is the angle r makes with the tangent.

If p be the perpendicular drawn from the axis on the tangent, we have, therefore,

$$R = \frac{v^2}{\rho} + \omega^2 p + 2\omega v.$$

This example might also have been advantageously solved by cylindrical co-ordinates. The fixed axis might be taken as axis of z and the projection on the plane of xy referred to polar co-ordinates. This method of treating the question is left to the student as an exercise.

Ex. If ω be variable, we have in a similar manner

$$R = \frac{v^2}{\rho} + \omega^2 p + 2\omega v + \frac{dv}{dt}\sqrt{r^2 - p^2}.$$

EXAMPLES*.

1. A circular hoop, which is free to move on a smooth horizontal plane, carries on it a small ring $\frac{1}{n}$th of its weight, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω about the centre of the hoop. Show that the ring will be at rest on the hoop after a time $\frac{1+n}{\mu\omega}$.

2. A heavy circular wire has its plane vertical and its lowest point at a height h above a horizontal plane. A small ring is projected along the wire from its highest point with an angular velocity about its centre equal to $\pi n\sqrt{\frac{2g}{h}}$ at the instant that the wire is let go. Show that when the wire reaches the horizontal plane, the particle will just have described n revolutions.

3. A heavy uniform sphere rolls on a rough plane and is acted on by a fixed centre of force in the plane varying inversely as the square of the distance; if the sphere be projected along the plane from a given point in it, in a direction opposite to that of the centre of force, find the roughness of the plane at any point, supposing the whole of it to be required.

4. Two equal uniform rods of length $2a$, loosely jointed at one extremity, are placed symmetrically upon a fixed smooth sphere of radius $\frac{a\sqrt{2}}{3}$, and raised into a horizontal position so that the hinge is in contact with the sphere. If they be allowed to descend under the action of gravity, show that, when they are first at rest, they are inclined at an angle $\cos^{-1}\frac{1}{3}$ to the horizon, that the points of contact with the sphere are the centres of oscillation of the rods relatively to the hinge, that the pressure on the sphere at each point of contact equals one-fourth the weight of either rod, and that there is no strain on the hinge.

5. Two circular discs are on a smooth horizontal plane; one, whose radius is n times that of the other, is fixed: an elastic string wraps round them so that those portions of it not in contact with the discs are common interior tangents, the natural length of the string being the sum of the circumferences. The moveable disc is drawn from the other till the tension of the string is T, prove that if it be now let go, the velocity acquired when it comes in contact with the fixed disc will be $\frac{T}{\lambda}\sqrt{\frac{2(n+1)\pi a \,.\, \lambda}{m}}$, where m is the mass of the moving disc, λ the modulus of elasticity, a the radius of the moving disc.

6. Two straight equal and uniform rods are connected at their ends by two strings of equal length a, so as to form a parallelogram. One rod is supported at its centre by a fixed axis about which it can turn freely, this axis being perpendicular to the plane of motion which is vertical. Show that the middle point of the lower rod will oscillate in the same way as a simple pendulum of length a, and that the *angular* motion of the rods is independent of this oscillation.

* These examples are taken from the Examination Papers which have been set in the University and in the Colleges.

7. A fine string is attached to two points A, B in the same horizontal plane, and carries a weight W at its middle point. A rod whose length is AB and weight W, has a ring at either end, through which the string passes, and is let fall from the position AB. Show that the string must be at least $\frac{5}{3} AB$, in order that the weight may ever reach the rod.

Also if the system be in equilibrium, and the weight be slightly and vertically displaced, the time of its small oscillations is $2\pi \sqrt{\dfrac{AB}{8g\sqrt{3}}}$.

8. A fine thread is enclosed in a smooth circular tube which rotates freely about a vertical diameter; prove that, in the position of relative equilibrium, the inclination (θ) to the vertical, of the diameter through the centre of gravity of the thread, will be given by the equation $\cos\theta = \dfrac{g}{a\omega^2 \cos\beta}$, where ω is the angular velocity of the tube, a its radius, and $2a\beta$ the length of the thread. Explain the case in which the value of $a\omega^2 \cos\beta$ lies between g and $-g$.

9. A smooth wire without inertia is bent into the form of a helix which is capable of revolving about a vertical axis coinciding with a generating line of the cylinder on which it is traced. A small heavy ring slides down the helix, starting from a point in which this vertical axis meets the helix: prove that the angular velocity of the helix will be a maximum when it has turned through an angle θ given by the equation $\cos^2\theta + \tan^2\alpha + \theta \sin 2\theta = 0$, α being the inclination of the helix to the horizon.

10. A spherical hollow of radius a is made in a cube of glass of mass M, and a particle of mass m is placed within. The cube is then set in motion on a smooth horizontal plane so that the particle just gets round the sphere, remaining in contact with it. If the velocity of projection be V, prove that $V^2 = 5ag + 4ag\dfrac{m}{M}$.

11. A perfectly rough ball is placed within a hollow cylindrical garden-roller at its lowest point, and the roller is then drawn along a level walk with a uniform velocity V. Show that the ball will roll quite round the interior of the roller, if V^2 be $> \frac{27}{7} g(b-a)$, a being the radius of the ball, and b of the roller.

12. AB, BC are two equal uniform rods loosely jointed at B, and moving with the same velocity in a direction perpendicular to their length; if the end A be suddenly fixed, show that the initial angular velocity of AB is three times that of BC. Also show that in the subsequent motion of the rods, the greatest angle between them equals $\cos^{-1}\frac{2}{3}$, and that when they are next in a straight line, the angular velocity of BC is nine times that of AB.

13. Three equal heavy uniform beams jointed together are laid in the same right line on a smooth table, and a given horizontal impulse is applied at the middle point of the centre beam in a direction perpendicular to its length; show that the instantaneous impulse on each of the other beams is one-sixth of the given impulse.

14. Three beams of like substance, joined together so as to form one beam, are laid on a smooth horizontal table. The two extreme beams are equal in length, and one of them receives a blow at its free extremity in a direction perpendicular to its length. Determine the length of the middle beam in order that the greatest possible angular velocity may be given to the third.

Result. If m be the mass of either of the outer rods, βm that of the inner rod, P the momentum of the blow, ω the angular velocity communicated to the third rod, then $ma\omega\left(\frac{1}{\beta}+\frac{4}{3}+\frac{4\beta}{3}\right)=P$. Hence when ω is a maximum $\beta=\frac{1}{2}\sqrt{3}$.

15. Two rough rods A, B are placed parallel to each other and in the same horizontal plane. Another rough rod C is laid across them at right angles, its centre of gravity being half way between them. If C be raised through any angle α and let fall, determine the conditions that it may oscillate, and show that if its length be equal to twice the distance between A and B, the angle θ through which it will rise in the n^{th} oscillation is given by the equation $\sin\theta=\left(\frac{1}{7}\right)^{2n}.\sin\alpha$.

16. A rod moveable in a vertical plane about a hinge at its upper end has a given uniform rod attached to its lower end by a hinge about which it can turn freely in the same vertical plane as the upper rod; at what point must the lower rod be struck horizontally in that same vertical plane that the upper rod may initially be unaffected by the blow?

17. A ball spinning about a vertical axis moves on a smooth table and impinges directly on a perfectly rough vertical cushion; show that the vis viva of the ball is diminished in the ratio $10+14\tan^2\theta:\frac{10}{e^2}+49\tan^2\theta$, where e is the elasticity of the ball and θ the angle of reflexion.

18. A rhombus is formed of four rigid uniform rods, each of length $2a$, freely jointed at their extremities. If the rhombus be laid on a smooth horizontal table and a blow be applied at right angles to any one of the rods, the rhombus will begin to move as a rigid body if the blow be applied at a point distant $a(1-\cos\alpha)$ from an acute angle, where α is the acute angle.

19. A rectangle is formed of four uniform rods of lengths $2a$ and $2b$ respectively, which are connected by hinges at their ends. The rectangle is revolving about its centre on a smooth horizontal plane with an angular velocity n, when a point in one of the sides of length $2a$ suddenly becomes fixed. Show that the angular velocity of the sides of length $2b$ immediately becomes $\frac{3a+b}{6a+4b}n$. Find also the change in the angular velocity of the other sides and the impulsive action at the point which becomes fixed.

20. Three equal uniform inelastic rods loosely jointed together are laid in a straight line on a smooth horizontal table, and the two outer ones are set in motion about the ends of the middle one with equal angular velocities (1) in the same direction and (2) in opposite directions. Prove that in the first case, when the outer rods make the greatest angle with the direction of the middle one produced on each side the common angular velocity of the three is $\frac{4\omega}{7}$, and in the second case after the impact of the two outer rods the triangle formed by them will move with uniform velocity $\frac{2a\omega}{3}$, $2a$ being the length of each rod.

21. An equilateral triangle formed of three equal heavy uniform rods of length a hinged at their extremities is held in a vertical plane with one side horizontal and the vertex downwards. If after falling through any height, the middle point of the

upper rod be suddenly stopped, the impulsive strains on the upper and lower hinges will be in the ratio of $\sqrt{13}$ to 1. If the lower hinge would just break if the system fell through a height $\dfrac{8a}{\sqrt{3}}$, prove that if the system fell through a height $\dfrac{32a}{\sqrt{3}}$ the lower rods would just swing through two right angles.

22. A perfectly rough and rigid hoop rolling down an inclined plane comes in contact with an obstacle in the shape of a spike. Show that if the radius of the hoop $=r$, height of spike above the plane $\dfrac{r}{2}$ and $V=$ velocity just before impact, then the condition that the hoop will surmount the spike is $V^2 > \frac{16}{9} gr \left\{1 - \sin\left(\alpha + \dfrac{\pi}{6}\right)\right\}$, α being the inclination of the plane to the horizon.

Show that unless $V^2 < \frac{16}{9} gr . \sin\left(\alpha + \dfrac{\pi}{6}\right)$, the hoop will not remain in contact with the spike at all.

If this inequality be satisfied the hoop will leave the spike when the diameter through the point of contact makes an angle with the horizon

$$=\sin^{-1}\left\{\frac{9}{32}\frac{V^2}{gr} + \tfrac{1}{2}\sin\left(\alpha + \frac{\pi}{6}\right)\right\}.$$

23. A flat circular disc of radius a is projected on a rough horizontal table, which is such that the friction upon an element α is $cV^3 m\alpha$ where V is the velocity of the element, m the mass of a unit of area: find the path of the centre of the disc.

If the initial velocity of the centre of gravity and the angular velocity of the disc be $u_0 \omega_0$, prove that the velocity u and angular velocity ω at any subsequent time satisfy the relation $\left(\dfrac{3u^2 - a^2\omega^2}{3{u_0}^2 - a^2{\omega_0}^2}\right)^2 = \dfrac{u^2\omega}{{u_0}^2\omega_0}$.

24. A heavy circular lamina of radius a and mass M rolls on the inside of a rough circular arc of twice its radius fixed in a vertical plane. Find the motion. If the lamina be placed at rest in contact with the lowest point, the impulse which must be applied horizontally that it may rise as high as possible (not going all round), without falling off, is $M\sqrt{3ag}$.

25. A string without weight is coiled round a rough horizontal cylinder, of which the mass is M and radius a, and which is capable of turning round its axis. To the free extremity of the string is attached a chain of which the mass is m and the length l; if the chain be gathered close up and then let go, prove that if θ be the angle through which the cylinder has turned after a time t before the chain is fully stretched, $Ma\theta = \dfrac{m}{l}\left(\dfrac{gt^2}{2} - a\theta\right)^2$.

26. Two equal rods AC, BC, are freely connected at C, and hooked to A and B, two points in the same horizontal line, each rod being then inclined at an angle α to the horizon. The hook B suddenly giving way, prove that the direction of the strain at C is instantaneously shifted through an angle $\tan^{-1}\left(\dfrac{1+6\sin^2\alpha}{1+6\cos^2\alpha}\cdot\dfrac{2-3\cos^2\alpha}{3\sin\alpha\cos\alpha}\right)$.

27. Two particles A, B are connected by a fine string; A rests on a rough horizontal table and B hangs vertically at a distance l below the edge of the table. If A be on the point of motion and B be projected horizontally with a velocity u, show that A will begin to move with acceleration $\frac{\mu}{\mu+1}\frac{u^2}{l}$, and that the initial radius of curvature of B's path will be $(\mu+1)\,l$, where μ is the coefficient of friction.

28. Two particles (m, m') are connected by a string passing through a small fixed ring and are held so that the string is horizontal; their distances from the ring being a and a', they are let go. If ρ, ρ' be the initial radii of curvature of their paths, prove that $\frac{m}{\rho}=\frac{m'}{\rho'}$, and $\frac{1}{\rho}+\frac{1}{\rho'}=\frac{1}{a}+\frac{1}{a'}$.

29. A sphere whose centre of gravity is not in its centre is placed on a rough table; the coefficient of friction being μ, determine whether it will begin to slide or to roll.

30. A circular ring is fixed in a vertical position upon a smooth horizontal plane, and a small ring is placed on the circle, and attached to the highest point by a string, which subtends an angle α at the centre; prove that if the string be cut and the circle left free, the pressures on the ring before and after the string is cut are in the ratio $M+m\sin^2\alpha : M\cos\alpha$, m and M being the masses of the ring and circle.

31. One extremity C of a rod is made to revolve with uniform angular velocity n in the circumference of a circle of radius a, while the rod itself is made to revolve in the opposite direction with the same angular velocity about that extremity. The rod initially coincides with a diameter, and a smooth ring capable of sliding freely along the rod is placed at the centre of the circle. If r be the distance of the ring from C at the time t, prove $r=\frac{2a}{5}(e^{nt}+e^{-nt})+\frac{a}{5}\cos 2nt$.

32. Two equal uniform rods of length $2a$ are joined together by a hinge at one extremity, their other extremities being connected by an inextensible string of length $2l$. The system rests upon two smooth pegs in the same horizontal line, distant $2c$ from each other. If the string be cut prove that the initial angular acceleration of either rod will be $g\dfrac{8a^2c-l^3}{\dfrac{8a^2l^2}{3}+\dfrac{32a^4c^2}{l^2}-8a^2cl}$.

33. A smooth horizontal disc revolves with angular velocity $\sqrt{\mu}$ about a vertical axis at which is placed a material particle attracted to a certain point of the disc by a force whose acceleration is $\mu\times$distance; prove that the path on the disc will be a cycloid.

CHAPTER V.

MOTION OF A RIGID BODY IN THREE DIMENSIONS.

Translation and Rotation.

182. If the particles of a body be rigidly connected, then whatever be the nature of the motion generated by the forces, there must be some general relations between the motions of the particles of the body. These must be such that if the motion of three points not in the same straight line be known, that of every other point may be deduced. It will then in the first place be our object to consider the general character of the motion of a rigid body apart from the forces that produce it, and to reduce the determination of the motion of every particle to as few independent quantities as possible: and in the second place we shall consider how when the forces are given these independent quantities may be found.

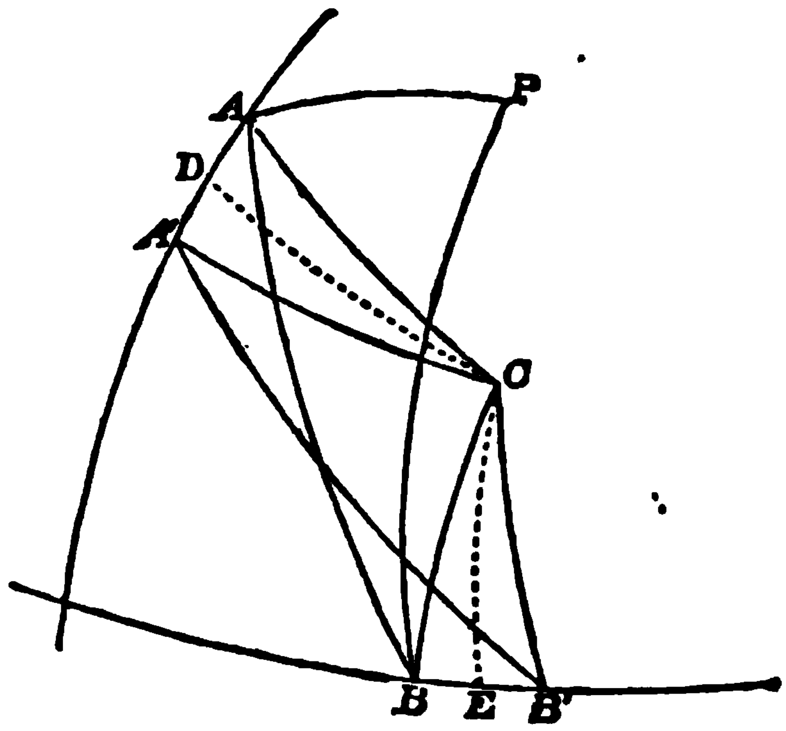
183. *One point of a moving rigid body being fixed, it is required to deduce the general relations between the motions of the other points of the body.*

Let O be the fixed point and let it be taken as the centre of a moveable sphere which we shall suppose fixed in the body. Let the radius vector to any point Q of the body cut the sphere in P, then the motion of every point Q of the body will be represented by that of P.

If the displacements of two points A, B, on the sphere in any time be given as AA', BB', then clearly the displacement of any other point P on the sphere may be found by constructing on $A'B'$ as base a triangle $A'P'B'$ similar and equal to APB. Then PP' will represent the displacement of P. It may be assumed as evident, or it may be proved as in Euclid, that on the same base and on the same side of it there cannot be two triangles on the same sphere, which have their sides terminated in one extremity of the base equal to one another, and likewise those terminated in the other extremity.

Let D and E be the middle points of the arcs AA', BB', and let DC, EC be arcs of great circles drawn perpendicular to AA',

BB' respectively. Then clearly $CA = CA'$ and $CB = CB'$, and therefore since the bases AB, $A'B'$ are equal, the two triangles



ACB, $A'CB'$ are equal and similar. Hence the displacement of C is zero. Also it is evident since the displacements of O and C are zero, that the displacement of every point in the straight line OC is also zero.

Hence a body may be brought from any position, which we may call AB, *into another* A'B' *by a rotation about* OC *as an axis through an angle* PCP' *such that any one point* P *is brought into coincidence with its new position* P'. Then every point of the body will be brought from its first to its final position.

184. A body is referred to rectangular axes x, y, z, and the origin remaining the same the axes are changed to x', y', z', according to the scheme in the margin. Show that this is equivalent to turning the body round an axis whose equations are any two of the following three:

	x',	y',	z'
x	a_1,	a_2,	a_3
y	b_1,	b_2,	b_3
z	c_1,	c_2,	c_3

$$(a_1 - 1)\,x + a_2 y + a_3 z = 0,$$
$$b_1 x + (b_2 - 1)\,y + b_3 z = 0,$$
$$c_1 x + c_2 y + (c_3 - 1)\,z = 0,$$

through an angle θ, where

$$3 - 4\sin^2\frac{\theta}{2} = a_1 + b_2 + c_3.$$

What is the condition that these three equations are consistent?

Take two points one on each of the axes of z and z' at a distance h from the origin. Their co-ordinates are $(0, 0, h)$ $(a_3 h, b_3 h, c_3 h)$ therefore their distance is $\sqrt{2(1-c_3)}\,h$. But it is also $2h\sin\gamma\sin\frac{\theta}{2}$; $\therefore 2\sin^2\frac{\theta}{2}\sin^2\gamma = 1 - c_3$. We have by similar reasoning $2\sin^2\frac{\theta}{2}\sin^2\alpha = 1 - a_1$ and $2\sin^2\frac{\theta}{2}\sin^2\beta = 1 - b_2$, whence the equation to find θ follows at once.

185. When a body is in motion we have to consider not merely its first and last positions, but also the intermediate posi-

tions. Let us then suppose AB, $A'B'$ to be two positions at any indefinitely small interval of time dt. We see that when a body moves about a fixed point O, there is, at every instant of the motion, a straight line OC, such that the displacement of every point in it during an indefinitely short time dt is zero. This straight line is called the instantaneous axis.

Let $d\theta$ be the angle through which the body must be turned round the instantaneous axis to bring any point P from its position at the time t to its position at the time $t + dt$, then the ultimate ratio of $d\theta$ to dt is called the angular velocity of the body about the instantaneous axis. The angular velocity may also be defined as the angle through which the body would turn in a unit of time if it continued to turn uniformly about the same axis throughout that unit with the angular velocity it had at the proposed instant.

186. Let us now remove the restriction that the body is moving with some one point fixed. We may establish the following proposition.

Every displacement of a rigid body may be represented by a combination of the two following motions, (1) *a motion of translation whereby every particle is moved parallel to the direction of motion of any assumed point* P *rigidly connected with the body and through the same space.* (2) *A motion of rotation of the whole body about some axis through this assumed point* P.

It is evident that the change of position may be effected by moving P from its old to its new position P' by a motion of translation and then retaining P' as a fixed point by moving any two points of the body not in one straight line with P into their final positions. This last motion has been proved to be equivalent to a rotation about some axis through P'.

Since these motions are quite independent, it is evident that their order may be reversed, *i.e.* we may rotate the body first and then translate it. We may even suppose them to take place simultaneously.

It is clear that any point P of the body may be chosen as the base point of the double operation. Hence the given displacement may be constructed in an infinite variety of ways.

187. *To find the relations between the axes and angles of rotation when different points* P, Q *are chosen as bases.*

Let the displacement of the body be represented by a rotation θ about an axis PR and a translation PP'. Let the same displacement be also represented by a rotation θ' about an axis QS and a translation QQ'. It is clear that any point has two dis-

placements, (1) a translation equal and parallel to PP', and (2) a rotation through an arc in a plane perpendicular to the axis of rotation PR. This second displacement is zero only when the point is on the axis PR. Hence the only points whose displacements are the same as the base point lie on the axis of rotation corresponding to that base point. Through the second base point Q draw a parallel to PR. Then for all points in this parallel, the displacements due to the translation PP', and the rotation θ round PR, are the same as the corresponding displacements for the point Q. Hence this parallel must be the axis of rotation corresponding to the base point Q. We infer *that the axes of rotation corresponding to all base points are parallel.*

188. The axes of rotation at P and Q having been proved parallel, let a be the distance between them. The rotation θ about PR will cause Q to describe an arc of a circle of radius a and angle θ, the chord Qq of this arc is $2a \sin \frac{\theta}{2}$ and is the displacement due to rotation. The whole displacement of Q is the resultant of Qq and the displacement of P. In the same way the rotation θ' about QS will cause P to describe an arc, whose chord Pp is equal to $2a \sin \frac{\theta'}{2}$. The whole displacement of P is the resultant of Pp and the displacement of Q. But if the displacement of Q is equal to that of P together with Qq, and the displacement of P is equal to that of Q together with Pp, we must have Pp and Qq equal and opposite. This requires that the two rotations θ, θ' about PR and QS should be equal and in the same direction. We infer that *the angles of rotation corresponding to all base points are equal.*

189. Since the translation QQ' is the resultant of PP' and Qq, we may by this theorem find both the translation and rotation corresponding to any proposed base point Q when those for P are given.

Since Qq, the displacement due to rotation round PR, is perpendicular to PR, the projection of QQ' on the axis of rotation is the same as that of PP'. *Hence the projections on the axis of rotation of the displacements of all points of the body are equal.*

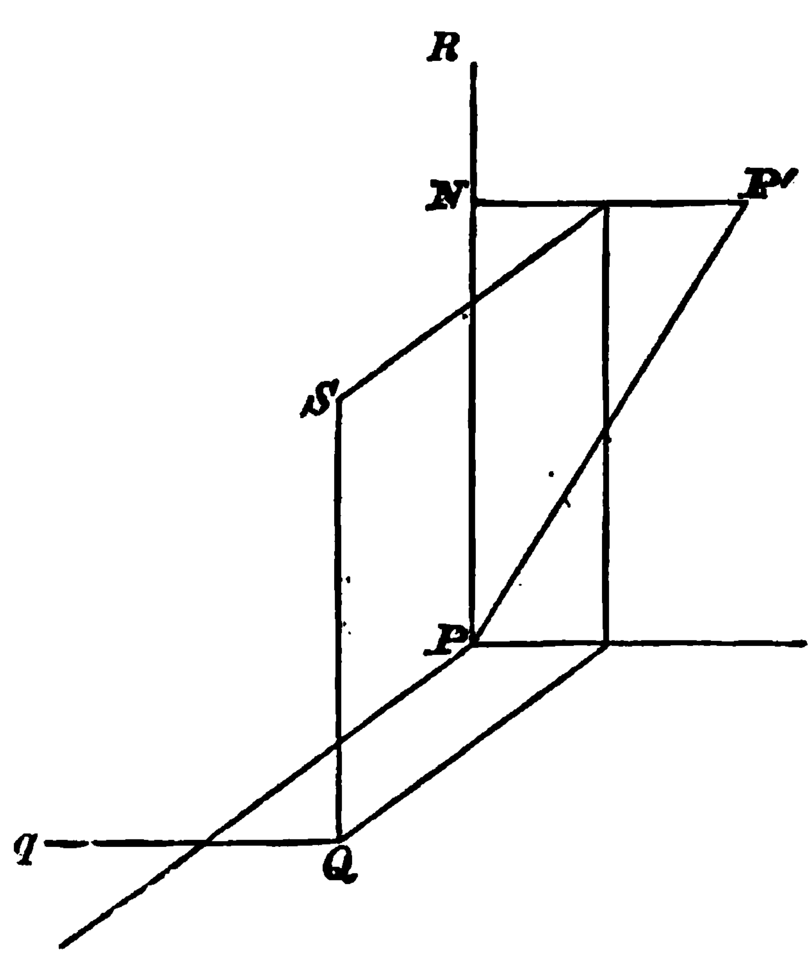
190. An important case is that in which the displacement is a simple rotation θ about an axis PR without any translation. If any point Q distant a from PR be chosen as the base, the same displacement is represented by a translation of Q through a chord $Qq = 2a \sin \frac{\theta}{2}$ in a direction making an angle $\frac{\pi - \theta}{2}$ with the plane QPR and a rotation which must be equal to θ about an axis which

must be parallel to PR. Hence *a rotation about any axis may be replaced by an equal rotation about any parallel axis together with a motion of translation.*

191. When the rotation is indefinitely small, the proposition can be enunciated thus, a motion of rotation ωdt about an axis PR is equivalent to an equal motion of rotation about any parallel axis QS, distant a from PR, together with a motion of translation $a\omega dt$ perpendicular to the plane containing the axes and in the direction in which QS moves.

192. It is often important to choose the base point so that the direction of translation may coincide with the axis of rotation. Let us consider how this may be done.

Let the given displacement of the body be represented by a rotation θ about PR, and a translation PP'. Draw $P'N$ perpendicular to PR. If possible let this same displacement be represented by a rotation about an axis QS, and a translation QQ' *along* QS. By Arts. 187 and 188 QS must be parallel to PR and the rotation about it must be θ. This translation will move P a length QQ' along PR, and the rotation about QS will move P along an arc perpendicular to PR. Hence QQ' must equal PN and NP' must be the chord of the arc. It follows that QS must lie on a plane bisecting NP' at right angles and at a distance a from PR where $NP' = 2a \sin \dfrac{\theta}{2}$, or, which is more convenient, at a distance y from the plane NPP' where $NP' = 2y \tan \dfrac{\theta}{2}$. The rotation θ round QS is to bring N to P' and is in the same direction as the rotation θ round PR. Hence the distance y must be measured from the



middle point of NP' in the direction in which that middle point is moved by its rotation round PR.

Having found the only possible position of QS, it remains to show that the displacement of Q is really along QS. The rotation θ round PR will cause Q to describe an arc whose chord Qq is parallel to $P'N$ and equal to $2a \sin \frac{\theta}{2}$. The chord Qq is therefore equal to NP', and the translation NP' brings q back to its position at Q. Hence Q is only moved by the translation PN, i.e. Q is moved along QS.

193. It follows from this reasoning that any displacement of a body can be represented by a rotation about some straight line and a translation *parallel* to that straight line. This mode of constructing the displacement is called a *screw*. The straight line is sometimes called the *central axis* and sometimes the *axis of the screw*. The ratio of the translation to the angle of rotation is called the *pitch* of the screw.

194. The same displacement of a body cannot be constructed by two different screws. For if possible let there be two central axes AB, CD. Then AB and CD by Art. 187 are parallel. The displacement of any point Q on CD is found by turning the body round AB and moving it parallel to AB, hence Q has a displacement perpendicular to the plane ABQ and therefore cannot move only along CD.

195. When the rotations are indefinitely small, the construction to find the central axis may be simply stated thus. Let the displacement be represented by a rotation ωdt about an axis PR and a translation Vdt in the *direction* PP'. Measure a distance $y = \dfrac{V \sin P'PR}{\omega}$ from P perpendicular to the plane $P'PR$ on that side of the plane towards which P' is moving. A parallel to PR through the extremity of y is the central axis.

196. Ex. 1. Given the displacements AA', BB', CC' of three points of a body in direction and magnitude, but not necessarily in position, find the direction of the axis of rotation corresponding to any base point P.

Through any assumed point O draw $O\alpha$, $O\beta$, $O\gamma$ parallel and equal to AA', BB', CC'. If $O\rho$ be the direction of the axis of rotation, the projections of $O\alpha$, $O\beta$, $O\gamma$ on $O\rho$ are all equal. Hence $O\rho$ is the perpendicular drawn from O on the plane $\alpha\beta\gamma$. This also shows that the direction of the axis of rotation is the same for all base points.

Ex. 2. If in the last example the motion be referred to the central axis, find the translation along it.

It is clearly equal to $O\rho$.

Ex. 3. Given the displacements AA', BB' of two points A, B of the body and the direction of the central axis, find the position of the central axis. Draw planes through AA', BB' parallel to the central axis. Bisect AA', BB' by planes perpendicular to these planes respectively and parallel to the direction of the central axis. These two last planes intersect in the central axis.

Composition of Rotations.

197. It is often necessary to compound rotations about axes OA, OB which meet at a point O. But as the only case which occurs in Rigid Dynamics is that in which these rotations are indefinitely small we shall first consider this case with some particularity, and then indicate generally the mode of proceeding when the rotations are of finite magnitude.

198. *To explain what is meant by a body having angular velocities about more than one axis at the same time.*

A body in motion is said to have an angular velocity ω about a straight line, when, the body being turned round this straight line through an angle ωdt, every point of the body is brought from its position at the time t to its position at the time $t + dt$.

Suppose that during three successive intervals each of time dt, the body is turned successively round three different straight lines OA, OB, OC meeting at a point O through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. Then we shall first prove that the final position is the same in whatever order these rotations are effected. Let P be any point in the body, and let its distances from OA, OB, OC, respectively be r_1, r_2, r_3. First let the body be turned round OA, then P receives a displacement $\omega_1 r_1 dt$. By this motion let r_2 be increased to $r_2 + dr_2$, then the displacement caused by the rotation about OB will be in magnitude $\omega_2 (r_2 + dr_2) dt$. But according to the principles of the Differential Calculus we may in the limit neglect the quantities of the second order, and the displacement becomes $\omega_2 r_2 dt$. So also the displacement due to the remaining rotation will be $\omega_3 r_3 dt$. And these three results will be the same in whatever order the rotations take place. In a similar manner we can prove that the *directions* of these displacements will be independent of the order. The final displacement is the diagonal of the parallelopiped described on these three lines as sides, and is therefore independent of the order of the rotations. Since then the three rotations are quite independent, they may be said to take place simultaneously.

When a body is said to have angular velocities about three different axes it is only meant that the motion may be determined as follows. Divide the whole time into a number of small intervals each equal to dt. During each of these, turn the body

round the three axes successively, through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. Then when dt diminishes without limit the motion during the whole time will be accurately represented.

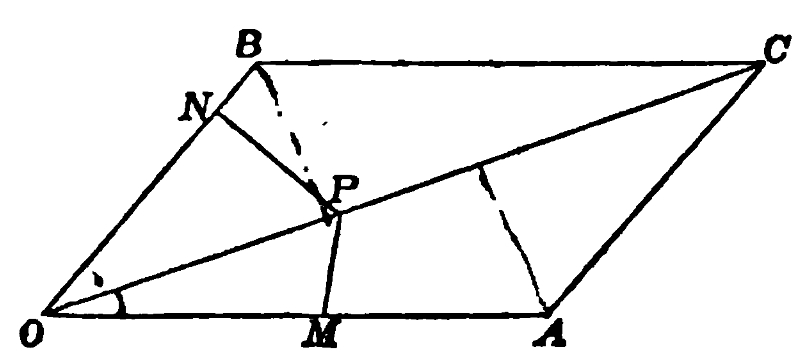
199. It is clear that a rotation about an axis OA may be represented in magnitude by a length measured along the axis. This length will also represent its direction if we follow the same rule as in Statics, viz. the rotation shall appear to be in some standard direction to a spectator placed along the axis so that OA is measured from his feet at O towards his head. This direction of OA is called the positive direction of the axis.

200. *If two angular velocities about two axes* OA, OB *be represented in magnitude and direction by the two lengths* OA, OB; *then the diagonal* OC *of the parallelogram constructed on* OA, OB *as sides will be the resultant axis of rotation, and its length will represent the magnitude of the resultant angular velocity.* This Prop. is usually called "The parallelogram of angular velocities."

Let P be any point in OC, and let PM, PN be drawn perpendicular to OA, OB. Since OA represents the angular velocity about OA and PM is the perpendicular distance of P from OA, the product $OA \,.\, PM$ will represent the velocity of P due to the angular velocity about OA. Similarly $OB \,.\, PN$ will represent the velocity of P due to the angular velocity about OB. Since P is on the left-hand side of OA and on the right-hand side of OB, as we respectively look along these directions, it is evident that these velocities are in opposite directions.

Hence the velocity of any point P is represented by

$$\begin{aligned} & OA \,.\, PM - OB \,.\, PN \\ & = OP\{OA \,.\, \sin COA - OB \,.\, \sin COB\} \\ & = 0. \end{aligned}$$



Therefore the point P is at rest and OC is the resultant axis of rotation.

Let ω be the angular velocity about OC, then the velocity of any point A in OA is perpendicular to the plane AOB and is represented by the product of ω into the perpendicular distance of A from $OC = \omega \,.\, OA \sin COA$. But since the motion is also

determined by the two given angular velocities about OA, OB, the motion of the point A is also represented by the product of OB into the perpendicular distance of A from $OB = OB \,.\, OA \sin BOA$;

$$\therefore \omega = OB \,.\, \frac{\sin BOA}{\sin COA} = OC.$$

Hence the angular velocity about OC is represented in magnitude by OC.

From this proposition we may deduce as a corollary "the parallelogram of angular accelerations." For if OA, OB represent the additional angular velocities impressed on a body at any instant, it follows that the diagonal OC will represent the resultant additional angular velocity in direction and magnitude.

201. This proposition shows that angular velocities and angular accelerations may be compounded and resolved by the same rules and in the same way as if they were forces. Thus an angular velocity ω about any given axis may be resolved into two, $\omega \cos \alpha$ and $\omega \sin \alpha$, about axes at right angles to each other and making angles α and $\frac{\pi}{2} - \alpha$ with the given axis.

If a body have angular velocities ω_1, ω_2, ω_3 about three axes Ox, Oy, Oz at right angles, they are together equivalent to a single angular velocity ω, where $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$, about an axis making angles with the given axes whose cosines are respectively $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$. This may be proved, as in the corresponding proposition in Statics, by compounding the three angular velocities, taking them two at a time.

It will however be needless to recapitulate the several propositions proved for forces in Statics with special reference to angular velocities. We may use "the triangle of angular velocities" or the other rules for compounding several angular velocities together, without any further demonstration.

202. *A body has angular velocities ω, ω' about two parallel axes* OA, O′B *distant* a *from each other, to find the resulting motion.*

Since parallel straight lines may be regarded as the limit of two straight lines which intersect at a very great distance, it follows from the parallelogram of angular velocities that the two given angular velocities are equivalent to an angular velocity about some parallel axis $O''C$ lying in the plane containing OA, $O'B$.

Let x be the distance of this axis from OA, and suppose it to be on the same side of OA as $O'B$. Let Ω be the angular velocity about it.

Consider any point P, distant y from OA and lying in the plane of the three axes. The velocity of P due to the rotation about OA is ωy, the velocity due to the rotation about $O'B$ is $\omega'(y-a)$. But these two together must be equivalent to the velocity due to the resultant angular velocity Ω about $O''C$, and this is $\Omega\ (y-x)$,

$$\therefore\ \omega y + \omega'(y-a) = \Omega\ (y-x).$$

This equation is true for all values of y, $\therefore \Omega = \omega + \omega'$, $x = \dfrac{a\omega'}{\Omega}$.

This is the same result we should have obtained if we had been seeking the resultant of two *forces* ω, ω' acting along OA, OB.

If $\omega = -\omega'$, the resultant angular velocity vanishes, but x is infinite. The velocity of any point P is in this case $\omega y + \omega'(y-a) = a\omega$, which is independent of the position of P.

The result is that two angular velocities, each equal to ω but tending to turn the body in opposite directions about two parallel axes at a distance a from each other, are equivalent to a *linear velocity* represented by $a\omega$. This corresponds to the proposition in Statics that "a couple" is properly measured by its moment.

We may deduce as a corollary, that a motion of rotation ω about an axis OA is equivalent to an equal motion of rotation about a parallel axis $O'B$ plus a motion of translation $a\omega$ perpendicular to the plane containing OA, $O'B$, and in the direction in which $O'B$ moves.

203. *To explain a certain analogy which exists between Statics and Dynamics.*

All propositions in Statics relating to the composition and resolution of forces and couples are founded on these theorems:

1. The parallelogram of forces and the parallelogram of couples.

2. A force F is equivalent to any equal and parallel force together with a couple Fp, where p is the distance between the forces.

Corresponding to these we have in Dynamics the following theorems on the instantaneous motion of a rigid body:

1. The parallelogram of angular velocities and the parallelogram of linear velocities.

2. An angular velocity ω is equivalent to an equal angular velocity about a parallel axis together with a linear velocity equal to ωp, where p is the distance between the parallel axes.

It follows that every proposition in Statics relating to forces has a corresponding proposition in Dynamics relating to the motion of a rigid body, and these two may be proved in the same way.

To complete the analogy it may be stated (i) that an angular velocity like a force in Statics requires, for its complete determination, five constants, and (ii) that a velocity like a couple in Statics requires but three. Four constants are required to determine the line of action of the force or of the axis of rotation, and one to determine the magnitude of either. There will also be a convention in either case to determine the positive direction of the line. Two constants and a convention are required to determine the positive direction of the axis of the couple or of the velocity and one the magnitude of either.

It is proved in Statics that a system of forces and couples is generally equivalent to a single force and a single couple, and that these may be reduced to a resultant R acting along a line called the central axis, and a couple G about that axis. Or they may also be reduced to a resultant R of the same magnitude as before, acting along any line parallel to the central axis at any chosen distance c from it, together with a couple G' about an axis perpendicular to the line whose length is c, and inclined to the resultant R at an angle θ. Then we know that $G' = \sqrt{G^2 + R^2c^2}$, and is a minimum when $c = 0$, and also that $\tan\theta = \dfrac{Rc}{G}$.

The same train of reasoning by which these results were established, will establish the following proposition. The instantaneous motion of a body having been reduced to a motion of translation and one of rotation, these are equivalent to a motion of rotation ω about a line called the central axis, and a translation V along that axis. Or they may also be reduced to a rotation ω of the same magnitude as before about any line parallel to the central axis, and at any chosen distance c from it, together with a translation V' along a line perpendicular to the line c, and inclined to the axis of ω at an angle θ. Then we know that $V' = \sqrt{V^2 + c^2\omega^2}$, and is a minimum when $c = 0$, and also that $\tan\theta = \dfrac{c\omega}{V}$. In a similar manner many other propositions may be established.

204. Ex. 1. The locus of points in a body moving about a fixed point which at any proposed instant have the same actual velocity is a circular cylinder.

Ex. 2. The geometrical motion of a body is represented by angular velocities inversely proportional to $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ about three lines forming three edges of a cube which do not meet nor are parallel. Prove that the body rotates about the line

$$(\beta-\gamma)x-a\alpha=(\gamma-\alpha)y-a\beta=(\alpha-\beta)z-a\gamma,$$

$2a$ being an edge of the cube, the centre being the origin, and the axes parallel to the edges.

Ex. 3. A body has an angular velocity ω about the axis

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

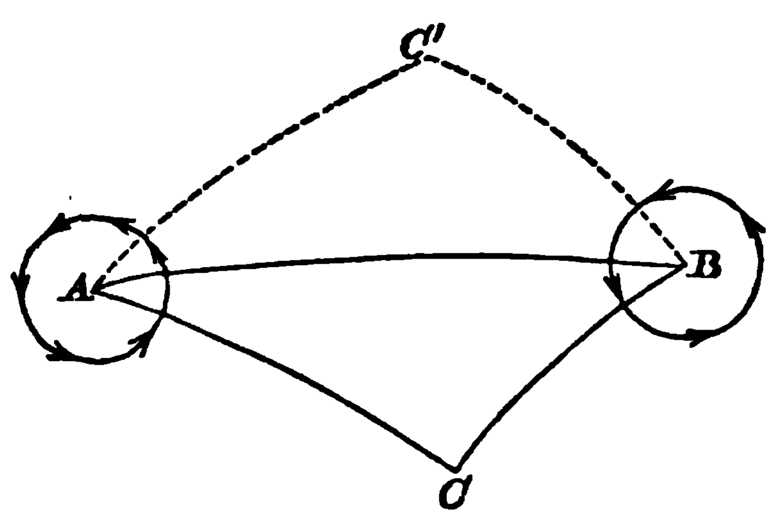
where $l^2+m^2+n^2=1$. The motion is equivalent to rotations $l\omega$, $m\omega$, $n\omega$ about the co-ordinate axes, and translations $(m\gamma-n\beta)\omega$, $(n\alpha-l\gamma)\omega$, $(l\beta-m\alpha)\omega$ in the directions of the axes.

This follows from the analogy of forces in Statics to angular velocities in Dynamics. See Art. 203.

Ex. 4. A body has equal angular velocities about two axes which neither meet nor are parallel. Prove that the central axis of the motion is equally inclined to each of the axes.

205. When the rotations to be compounded are finite in magnitude, the rule to find the resultant is somewhat more complicated. Let the given rotations be (1) a rotation about an axis OA through an angle θ; (2) a subsequent rotation about an axis OB through an angle θ', and let both these axes be fixed in space. Let lengths measured along OA, OB represent these rotations in the manner explained in Art. 199.

Let the directions of the axes OA, OB cut a sphere whose centre is at O in A and B. On this sphere measure the angle BAC equal to $\frac{\theta}{2}$ in a direction opposite



to the rotation round OA and also the angle ABC equal to $\frac{\theta'}{2}$ in the same direction as the rotation round OB and let the arcs intersect in C. Lastly, measure the angles BAC', ABC' respectively equal to BAC, ABC, but on the other side of AB.

The rotation θ round OA will then carry any point P in OC into the straight line OC' and the subsequent rotation θ' about OB will carry the point P back into OC. Thus the points in OC are unmoved by the double rotation and OC is therefore the axis of the single rotation by which the given displacement of the body may be constructed. The straight line OC is called the resultant axis of rotation. If the order of the rotations were reversed, so that the body is rotated first about OB and then about OA, the resultant axis would be OC'.

If the axes OA, OB were fixed in the body, the rotation θ about OA would bring OB into a position OB'. Then the body may be brought from its first into its last position by rotations θ, θ' about the axes OA, OB' *fixed in space*. Hence the same construction will again give the position of the resultant axis and the rotation about it.

To find the magnitude θ'' of the rotation about the resultant axis OC we notice that if a point P be taken in OA, it is unmoved by the rotation θ about OA, and the subsequent rotation θ' about OB will bring it into the position P', where PP' is bisected at right angles by the plane OBC. But the rotation θ'' about OC must give P the same displacement, hence in the standard case θ'' is twice the external angle between the planes OCA, OCB. If the order of the rotations be reversed, the rotation about the resultant axis OC' would be twice the external angle at C', which is the same as that at C. So that though the position of the resultant axis of rotation depends on the order of rotation the resultant angle of rotation is independent of that order.

206. A rotation represented by twice any internal angle of the spherical triangle ABC is equal and opposite to that represented by twice the corresponding external angle. For since the sum of the internal and external angles is π, these two rotations only differ by 2π; and it is evident that a rotation through an angle 2π cannot alter the position of any point of the body. This is merely another way of saying that when a body turns about a fixed axis it may be brought from one given position to another by turning the body either way round the axis.

207. The rule for compounding finite rotations may be stated thus:

If ABC *be a spherical triangle, a rotation round* OA *from* C *to* B *through twice the internal angle at* A, *followed by a rotation round* OB *from* A *to* C *through twice the internal angle at* B *is equal and opposite to a rotation round* OC *from* B *to* A *through twice the internal angle at* C.

It will be noticed that the order in which the axes are to be taken as we travel round the triangle is opposite to that of the rotations.

As the demonstrations in Art. 205 are only modifications of those of Rodrigues, we may call this theorem after his name.

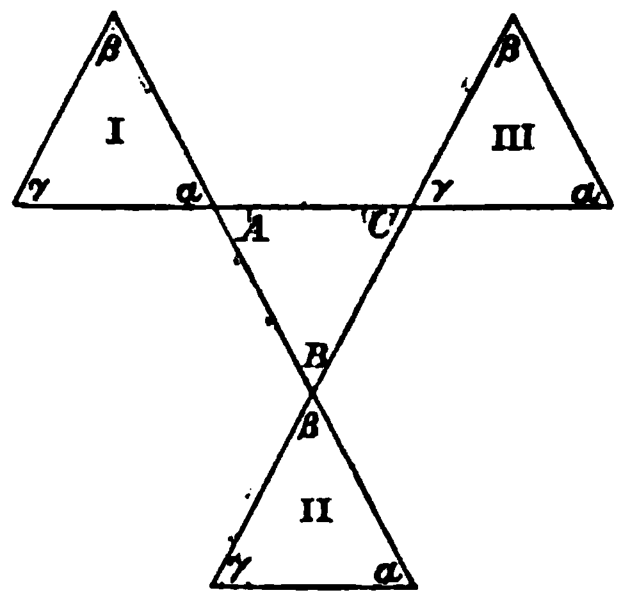
208. Ex. 1. If two rotations θ, θ' about two axes OA, OB at right angles be compounded into a single rotation ϕ about an axis OC, then

$$\tan COA = \tan\frac{\theta'}{2}\operatorname{cosec}\frac{\theta}{2},\ \tan COB = \tan\frac{\theta}{2}\operatorname{cosec}\frac{\theta'}{2}\ \text{and}\ \cos\frac{\phi}{2} = \cos\frac{\theta}{2}\cos\frac{\theta'}{2}.$$

209. From Rodrigues' theorem we may deduce Sylvester's theorem by drawing the polar triangle $A'B'C'$. Since a side $B'C'$ is the supplement of the angle A, a rotation represented in direction and magnitude by $2B'C'$ differs from that represented by $2A$ in the opposite direction by a rotation through an angle 2π. But a rotation through 2π cannot alter the position of the body, hence the two rotations $2B'C'$ and $2A$ are equivalent in magnitude but opposite in direction. *If therefore $A'B'C'$ be any spherical triangle, a rotation represented by twice $B'C'$ followed by a rotation twice $C'A'$ produces the same displacement of the body as a rotation twice $B'A'$.* By a rotation $B'C'$ is meant a rotation about an axis perpendicular to the plane of $B'C'$ which will bring the point B' to C'.

210. The following proof of the preceding theorem was given by Prof. Donkin in the *Phil. Mag.* for 1851. Let ABC be any triangle on a sphere fixed in space,

$\alpha\beta\gamma$ a triangle on an equal and concentric sphere moveable about its centre. The sides and angles of $\alpha\beta\gamma$ are equal to those of ABC, but differently arranged, one triangle being the inverse or reflection of the other. If the triangle $\alpha\beta\gamma$ be placed in the position I, so that the sides containing the angle α may be in the same great circles with those containing A, it is obvious that it may slide along AB into the position II, and then along BC into the position III; into which last position it might also be brought by sliding along AC. To slide $\alpha\beta\gamma$ along AB is equivalent to moving β and α each through an arc twice the arc AB about an axis perpendicular to the plane of AB. A similar remark applies when the triangle slides



along BC or AC. Hence, twice the rotation AB followed by twice the rotation BC produces the same displacement as twice the rotation AC.

211. If it be required to compound the rotations about two parallel axes, the construction of Rodrigues requires only a slight modification. Instead of arcs drawn on a sphere, let planes be drawn through the axes making with the plane containing the axes the same angles as before; their intersection will be the resultant axis. One case deserves notice. If $\theta = -\theta'$, the resultant axis is at infinity. A rotation about an axis at infinity is evidently equivalent to a translation. Hence a rotation θ about any axis OA followed by an equal and opposite rotation about a parallel axis $O'B$ distant a from OA is equivalent to a translation $2a \sin \frac{\theta}{2}$ perpendicular to a plane through OA making an angle $\frac{\theta}{2}$ with the plane containing the axes and in the direction of the chord of the arc described by any point in OA. These results also follow easily from Art. 190.

212. *Any given displacement of a body may be represented by two finite rotations, one about any given straight line and the other about some other straight line which does not necessarily intersect the first.* When a displacement is thus represented, the axes are called *conjugate axes* and the rotations are called *conjugate rotations.*

Let OA be the given straight line and let the given displacement be represented by a rotation ϕ about a straight line OR and a translation OT. We wish to resolve this rotation about OR into two rotations, one about OA to be followed by a rotation about OB, where OB is some straight line perpendicular to OT. To do this we follow the rule in Art. 205, we describe a sphere whose centre is O and radius unity and let it intersect OA, OR, OT in A, R and T. Make the angle ARB

equal to the supplement of $\frac{\phi}{2}$ and produce RB to B so that $TB = \frac{\pi}{2}$ and join AB. By the triangle of rotations the rotation ϕ is now represented by a rotation about OA which we may call θ, followed by a rotation about OB which we may call θ'.

By Art. 211 the rotation θ' is equivalent to an equal rotation θ' about a parallel axis CD, together with a translation, which may be made to destroy the translation OT. This will be the case if the angle OT makes with the plane of OB, CD be $\frac{\pi - \theta'}{2}$ on the one side or the other of OT according to the direction of the rotation, and if the distance r between AB, CD be such that $2r \sin \frac{\theta'}{2} = OT$.

The whole displacement has thus been reduced to a rotation θ about OA followed by a rotation θ' about CD.

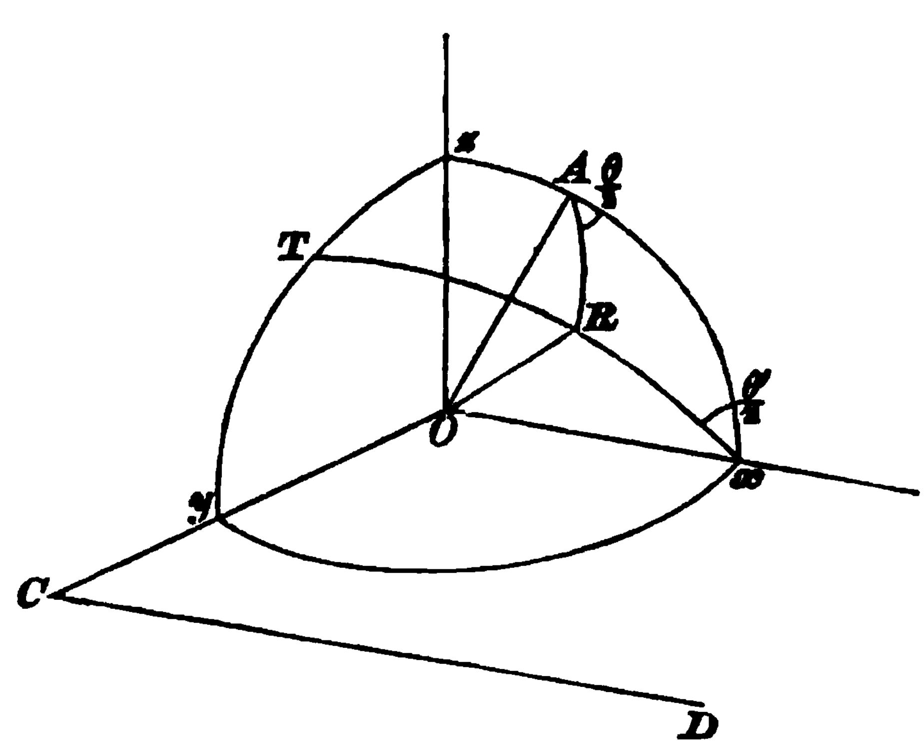
213. Analytically, we might reason thus—A screw motion is given when we know (1) its axis, (2) the rotation about it, (3) the translation along it. The axis is known when its inclination to two of the axes and the two co-ordinates of the point in which it cuts the plane of xy are given. Thus *six* constants are required to determine a screw.

Let a given screw be resolved into two screws. We have then twelve constants, but since they are together equivalent to the given screw there are six relations between the constants. We are therefore at liberty to choose any six relations we please between these twelve constants. We might, for example, resolve a given screw into two screws of any given pitches, the remaining four constants being chosen to make the axis of one screw coincide with any given straight line. If the given pitch of each screw be zero, the screws are reduced to simple rotations, and thus any displacement can be reduced to two conjugate rotations. It has been shown in the preceding article that the two rotations are real.

214. Ex. Show that any screw may be resolved into two real screws having the axis of one in a given direction and the axis of the other intersecting the first at a given angle.

215. *Any two successive displacements of a body may be represented by two successive screw motions. It is required to compound these.*

Let the body be screwed first along the axis OA with linear displacement a and



angle of rotation θ, and secondly along the axis CD with displacement a' and angle θ'. Let OC be the shortest distance between OA and CD, and for the sake of the perspective let it be called the axis of y. Let O be the origin and let the axis of x be parallel to CD, so that OA lies in the plane of xz. Let $OC=r$, and the angle $AOx=\alpha$. Draw a plane xOT making with the plane of xz an angle $\frac{\theta'}{2}$, and let it cut yz in OT. Draw another plane AOR making with xz an angle $\frac{\theta}{2}$, and cutting the plane xOT in OR.

Produce AO to a point P, not marked in the figure, so that $PO=a$, and let us choose P as a base point to which the whole displacement of the body may be referred. The rotation θ' is equivalent to a rotation θ' about Ox together with a translation along $OT=2r\sin\frac{\theta'}{2}$ by Art. 190. By Art. 205 the rotation θ about OA followed by θ' about Ox is equivalent to a rotation Ω about OR where Ω is twice the angle ART, so that $\sin\frac{\Omega}{2}=\sin\frac{\theta}{2}\cdot\frac{\sin Ax}{\sin Rx}$. The whole displacement is now represented by (1) a translation of the base point P to O, (2) the rotation Ω, (3) a further linear translation which is the resultant of the translations $2r\sin\frac{\theta'}{2}$ along OT and a' along Ox. By Art. 186 these displacements may be made in any order, being all connected with the same base point. They may therefore be compounded into a single screw by the rule given in Art. 192. This is called the *resultant screw*. A screw equal and opposite to the resultant screw will bring the body back to its original position.

The angle of rotation of the resultant screw is Ω and its axis is parallel to OR by Art. 187. It follows by Art. 206 that the sine of half the angle of rotation of each screw is proportional to the sine of the angle between the axes of the other two screws.

To find the linear displacement along the axis of the resultant screw, we must by Art. 189 add together the projections on OR of the three displacements OT, a, a'. The projection of $OT=2r\sin\frac{\theta'}{2}\cos TR=2r\cos Ty\,.\cos TR$ which is twice the projection of the shortest distance r on the axis of rotation. If T be the linear displacement, we have $T=2r\cos Ry+a\cos RA+a'\cos Rx$.

216. If the component screws be simple rotations we have $a=0$, $a'=0$, and it may be shown without difficulty that $T\sin\frac{\Omega}{2}=2r\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\alpha$. It has been shown in Art. 212 that any displacement may be represented by two conjugate rotations in an infinite number of ways, but it now follows that in all these $r\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\alpha$ is the same. When the rotations are indefinitely small, and equal to ωdt, $\omega' dt$ respectively, this becomes $\frac{1}{4}r\omega\omega'\sin\alpha$; that is, the product of an angular velocity into the moment of its conjugate angular velocity about its axis is the same for all conjugates representing the same motion.

Ex. 1. If the component screws be simple finite rotations, show that the equations to the axis of the resultant screw are

$$-x\tan\phi'+y\sin\frac{\theta'}{2}+z\cos\frac{\theta'}{2}=r\sin\frac{\theta'}{2},\quad y\cos\frac{\theta'}{2}-z\sin\frac{\theta'}{2}=r\sin\frac{\theta'}{2}\cos\phi'\cot\frac{\Omega}{2},$$

where ϕ' is the angle xOR and Ω is the resultant rotation. The first equation expresses the fact that the central axis lies in a plane which bisects at right angles a straight line drawn from O perpendicular to OR in the plane xOR to represent the linear translation in that direction. The second expresses that the central axis lies in a plane parallel to TOR at a distance from it determined by Art. 192.

These equations may also be deduced from those of Rodrigues given in Art. 223. To effect this we must write for (a, b, c) the resolved parts of the translation along OT. Since however the positive direction of the rotation in Rodrigues' formulæ has been taken opposite to that chosen in the preceding article, we must write for (l, m, n) the direction cosines of OR with their signs changed.

The equations to the central axis of any two screws may be found by either of these methods.

Ex. 2. Let the motion be constructed by two finite rotations θ, θ' taken in order round axes OA, CD at right angles to each other and let CO be the shortest distance between the axes. Let the two straight lines OP, CP be drawn in the plane DCO such that the angle $POC=\frac{\theta}{2}$ and $\tan PCO=\sin^2\frac{\theta'}{2}\cot\frac{\theta}{2}$. Then if P be moved backwards by the rotation θ or forwards by the rotation θ', in either case its new position is a point on the central axis.

Ex. 3. If OA, OB be the axes of two screws at right angles, with linear displacements a and b, the point P is the intersection of two parallels to the straight lines described in the last example; these parallels being drawn respectively at distances $\frac{a}{2}\tan\phi$ and $\frac{b}{2}\left(1+\cot^2\phi'\sin^2\frac{\theta'}{2}\right)^{-\frac{1}{2}}$, where ϕ, ϕ' are the angles the resultant axis of rotations makes with OA and CD. Then if P be screwed backwards by the first screw or forwards by the second, in either case its new position is a point on the central axis.

217. Ex. 1. If the instantaneous motion of a body be represented by two conjugate rotations ωdt and $\omega' dt$, the axis of the resultant screw intersects at right angles the shortest distance between the conjugate axes. Let γ, γ' be the angles the conjugate axes make with the axis of their resultant, a the angle they make with each other; c, c' the shortest distances between the conjugate axes and the axis of the screw, V and Ω the linear and angular velocities of the screw, then prove that

$$\frac{\omega}{\sin\gamma'}=\frac{\omega'}{\sin\gamma}=\frac{\Omega}{\sin a},$$

$$\frac{c\omega}{\cos\gamma'}=\frac{c'\omega'}{\cos\gamma}=\frac{V}{\sin a},$$

$$c\tan\gamma'=c'\tan\gamma=\frac{V}{\Omega}.$$

The first line follows from Art. 201. The second expresses the fact that the direction of the linear motion of the point where the axis cuts the shortest distance is along the axis of the screw.

Ex. 2. If one conjugate axis of an instantaneous motion is at right angles to the central axis, the other meets it, and conversely.

Ex. 3. If one conjugate axis of an instantaneous motion is parallel to the central axis, the other is at an infinite distance, and conversely.

Ex. 4. The locus of tangents to the trajectories of different points of the same straight line in the instantaneous motion of a body is a hyperbolic paraboloid.

Let AB be the given straight line, CD its conjugate. The points on AB are turning round CD and therefore the tangents all pass through two straight lines, viz. AB and its consecutive position $A'B'$, and are also all parallel to a plane which is perpendicular to CD.

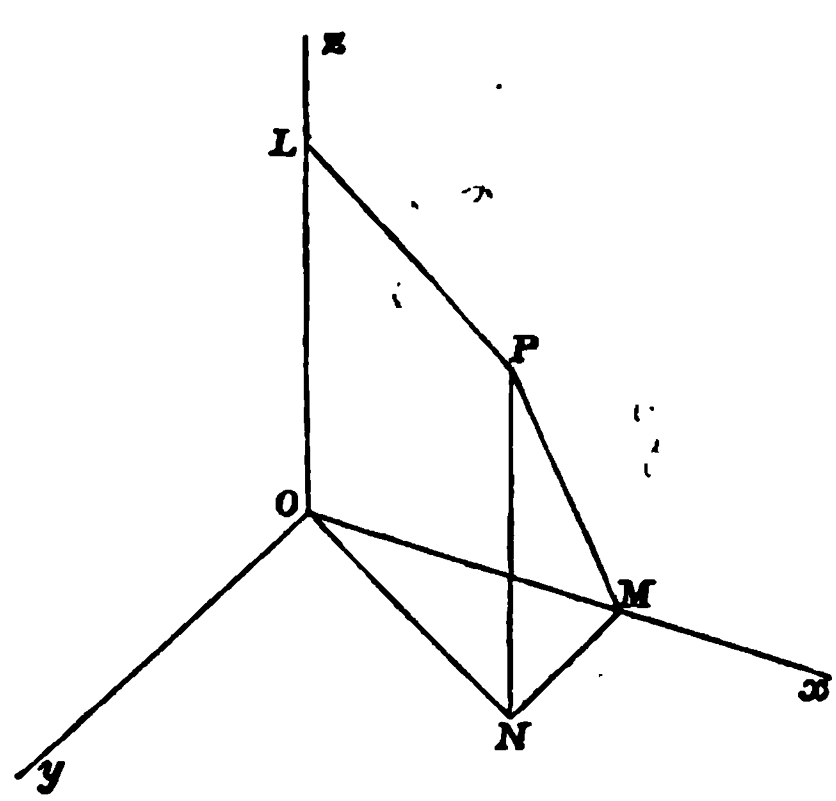
Ex. 5. If radii vectores be drawn from a fixed point to represent in direction and magnitude the velocities of all points of a rigid body in motion, prove that the extremities of these radii vectores at any one instant lie in a plane. [Coll. Exam.]

Motion referred to fixed axes.

218. The general equations of motion given in Art. 71 of Chapter II. involve the differential coefficients $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{d^2x}{dt^2}$, &c. It will now be necessary to express these in terms of the instantaneous angular velocities of the body.

219. Let us suppose in the first instance that one point in the body is fixed. Let us take this point as the origin of coordinates, and let the axes Ox, Oy, Oz be any directions fixed in space and at right angles to each other. The body at the time t is turning about some axis of instantaneous rotation. Let its angular velocity be Ω, and let this be resolved into the angular velocities ω_1, ω_2, ω_3 about the co-ordinate axes. We have to find the resolved velocities $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ of a particle whose coordinates are x, y, z.

These angular velocities are supposed positive when they tend the same way round the axes that positive couples tend in Statics. Thus the positive directions of ω_1, ω_2, ω_3 are respectively from y to z, from z to x, and from x to y.



Let us determine the velocity of P parallel to the axis of z. Let PN be the ordinate z, and let PM be drawn perpendicular to Ox. The velocity of P due to the rotation about Ox is clearly $\omega_1 PM$. Resolving this along NP we get $\omega_1 PM \sin NPM = \omega_1 y$. Similarly that due to the rotation about Oy is $-\omega_2 x$; and that due to the rotation about Oz is zero. Hence the whole velocity of P parallel to Oz is

$$\frac{dz}{dt} = \omega_1 y - \omega_2 x,$$

and the velocities parallel to the other axes are

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y,$$

$$\frac{dy}{dt} = \omega_3 x - \omega_1 z.$$

220. The quantities $\omega_1, \omega_2, \omega_3$ are called the angular velocities of the *body* about the axes of x, y, z respectively, but they must be carefully distinguished from the angular velocities of any particular *particle* of the body about the same axes. Let P be any particle of the body whose co-ordinates are x, y, z, and draw $PL = r$ perpendicular to the axis of z. Let θ be the angle xON, then the instantaneous angular velocity of P about Oz is $\frac{d\theta}{dt}$.

But $r^2 \frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = \omega_3 r^2 - xz\omega_1 - yz\omega_2$, by substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, their values just found;

$$\therefore \frac{d\theta}{dt} = \omega_3 - \omega_1 \frac{xz}{r^2} - \omega_2 \frac{yz}{r^2}.$$

Hence the angular velocity of a particle about Oz is the same as that of the body when the particle lies in the plane of xy, or when it lies in the plane given by $y = -x\frac{\omega_1}{\omega_2}$.

If the axes be themselves moving in any manner, these equations only give the linear velocities of the particle relatively to the axes. Thus suppose the directions of the axes to be fixed in space, but the origin O to be in motion with a velocity V whose resolved parts parallel to the axes are respectively u, v, w. Then the velocities in space resolved parallel to the axes will be

$$\left.\begin{aligned} u' &= u + \omega_2 z - \omega_3 y \\ v' &= v + \omega_3 x - \omega_1 z \\ w' &= w + \omega_1 y - \omega_2 x \end{aligned}\right\}.$$

221. *The motion being given, as before, by the linear velocities* (u, v, w) *of some point* O *and the angular velocities* $(\omega_1, \omega_2, \omega_3)$, *find the equations to the central axis.*

Let the same motion be also represented by the linear velocities u', v', w' parallel to the axes, of some other point O' and by angular velocities $\omega_1', \omega_2', \omega_3'$ about axes parallel to the coordinate axes and meeting in O'. Let (ξ, η, ζ) be the co-ordinates of O'. We have now two representations of the same motion, both these must give the same result for the linear velocities of any point. Hence

$$\left.\begin{aligned} u+\omega_2 z-\omega_3 y &= u'+\omega_2'(z-\zeta)-\omega_3'(y-\eta) \\ v+\omega_3 x-\omega_1 z &= v'+\omega_3'(x-\xi)-\omega_1'(z-\zeta) \\ w+\omega_1 y-\omega_2 x &= w'+\omega_1'(y-\eta)-\omega_2'(x-\xi) \end{aligned}\right\} \quad \ldots\ldots\ldots (1),$$

must be true for all values of x, y, z.

This gives $\omega_1'=\omega_1$, $\omega_2'=\omega_2$, $\omega_3'=\omega_3$, so that whatever origin is chosen, the angular velocity is always the same in direction and magnitude. See Art. 188.

Also (ξ, η, ζ) may be so chosen that the velocity of O' is along the axis of rotation; in this case we have (u', v', w') proportional to $(\omega_1, \omega_2, \omega_3)$. The equation to the locus of O' is therefore

$$\frac{u+\omega_2\zeta-\omega_3\eta}{\omega_1}=\frac{v+\omega_3\xi-\omega_1\zeta}{\omega_2}=\frac{w+\omega_1\eta-\omega_2\xi}{\omega_3} \quad \ldots\ldots\ldots (2).$$

By multiplying the numerator and denominator of each of these fractions by $\omega_1, \omega_2, \omega_3$ respectively, and adding them together, we see that each of them is

$$=\frac{u\omega_1+v\omega_2+w\omega_3}{\Omega^2}.$$

The motion of the body is thus represented by a motion of translation along the straight line whose equations are (2) and an angular velocity equal to Ω about it.

This straight line has been called the central axis, and the fraction just written down is equal to the ratio of the velocity of translation along the central axis to the angular velocity about it, *i.e.* the pitch of the screw.

If the motion be such that $u\omega_1+v\omega_2+w\omega_3=0$, and $\omega_1, \omega_2, \omega_3$ do not all vanish, each of the equalities in (2) is zero, and hence by equation (1) $u'=0$, $v'=0$, $w'=0$. The motion is therefore equivalent to a rotation about the central axis, without translation. This is also evident from the analogy explained in Art. 203.

222. When the rotations are finite the corresponding formulæ are somewhat more complicated. Let the given displacement of the body be a rotation through a

finite angle θ about an axis passing through the origin whose direction cosines are (l, m, n). It is required to find the changes produced in the co-ordinates (x, y, z) of any point P.

Let PP' be the chord of the arc described by P and let Q be the middle point of PP'. Let $x+\delta x$, $y+\delta y$, $z+\delta z$ be the co-ordinates of P' and ξ, η, ζ those of Q. Since the abscissæ of Q is the arithmetic mean of those of P and P' we have $\xi = x + \frac{\delta x}{2}$; similarly $\eta = y + \frac{\delta y}{2}$, $\zeta = z + \frac{\delta z}{2}$. Let QM be a perpendicular from Q on the axis, then $PP' = 2\,QM \tan\frac{\theta}{2}$.

Let (λ, μ, ν) be the direction cosines of PP', then since PP' is perpendicular to the axis, we have $l\lambda + m\mu + n\nu = 0$, and since it is also perpendicular to OQ we have $\xi\lambda + \eta\mu + \zeta\nu = 0$, hence

$$\frac{\lambda}{m\zeta - n\eta} = \frac{\mu}{n\xi - l\zeta} = \frac{\nu}{l\eta - m\xi}.$$

The sum of the squares of the denominators is

$$(\xi^2 + \eta^2 + \zeta^2)(l^2 + m^2 + n^2) - (l\xi + m\eta + n\zeta)^2,$$

which is $OQ^2 - OM^2 = QM^2$. Hence each of these ratios is $= \frac{1}{QM}$.

Now δx is the projection of PP' on the axis of x,

$$\therefore\ \delta x = 2QM \,.\, \tan\frac{\theta}{2}\lambda = 2\tan\frac{\theta}{2}(m\zeta - n\eta);$$

similarly $\delta y = 2\tan\frac{\theta}{2}(n\xi - l\zeta)$, $\delta z = 2\tan\frac{\theta}{2}(l\eta - m\xi)$, which are the required formulæ.

If the origin have a linear displacement whose resolved parts parallel to the axes are (a, b, c), we must add those displacements to the values of δx, δy, δz found by solving these equations. Let the co-ordinates of the middle point of the *whole* displacement of P be represented by ξ', η', ζ'. Then we have, as before, $\xi' = x + \frac{\delta x}{2}$ &c., but since δx, δy, δz, are increased by a, b, c we must write $\xi' - \frac{a}{2}$, $\eta' - \frac{b}{2}$, $\zeta' - \frac{c}{2}$ for ξ, η, ζ. We thus obtain

$$\delta x = a + 2\tan\frac{\theta}{2}\left\{m\left(\zeta' - \frac{c}{2}\right) - n\left(\eta' - \frac{b}{2}\right)\right\},$$

with similar expressions for δy and δz.

223. The equations to the central axis follow from these expressions without difficulty. The whole displacement of any point in the central axis is along the axis, so that (ξ', η', ζ') the co-ordinates of the middle point of the displacement are co-ordinates of a point in the axis, and δx, δy, δz are proportional to (l, m, n) the direction cosines of the axis. Hence

$$\frac{a + 2\tan\frac{\theta}{2}\left\{m\left(\zeta' - \frac{c}{2}\right) - n\left(\eta' - \frac{b}{2}\right)\right\}}{l} = \frac{b + 2\tan\frac{\theta}{2}\left\{n\left(\xi' - \frac{a}{2}\right) - l\left(\zeta' - \frac{c}{2}\right)\right\}}{m}$$
$$= \frac{c + 2\tan\frac{\theta}{2}\left\{l\left(\eta' - \frac{b}{2}\right) - m\left(\xi' - \frac{a}{2}\right)\right\}}{n}.$$

Each of these is evidently equal to $la + mb + nc$, which is the linear displacement along the central axis. The results of this and the preceding Article are due to Rodrigues.

224. Ex. Let the restraints on a body be such that it admits of two motions A and B each of which may be represented by a screw motion, and let m, m' be the pitches of these screws. Then the body must admit of a screw motion compounded of any indefinitely small rotations ωdt, $\omega' dt$ about the axes of these screws accompanied of course by the translations $m\omega dt$, $m'\omega' dt$. Prove that (1) the locus of the axes of all these screws is the surface $z(x^2+y^2)=2axy$. (2) If the body be screwed along any generator of this surface the pitch is $c+a\cos 2\theta$, where c is a constant which is the same for all generators and θ is the angle the generator makes with the axis of x. (3) The size and position of the surface being chosen so that the two given screws A and B lie on the surface with their appropriate pitch, show that only one surface can be drawn to contain two given screws. (4) If any three screws of the surface be taken and a body be displaced by being screwed along each of these through a small angle proportional to the sine of the angle between the other two, the body after the last displacement will occupy the same position that it did before the first.

This surface has been called the *cylindroid* by Prof. Ball, to whom these four theorems are due.

225. Ex. 1. If an instantaneous motion be given by the linear velocities (u, v, w) along and the angular velocities $(\omega_1, \omega_2, \omega_3)$ about the co-ordinate axes, show that the equations to the conjugate of $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ are

$$\begin{vmatrix} x & y & z \\ \omega_1 & \omega_2 & \omega_3 \\ l & m & n \end{vmatrix} = lu+mv+nw,$$

$$\begin{vmatrix} x & y & z \\ \omega_1 & \omega_2 & \omega_3 \\ f & g & h \end{vmatrix} = (f-x)u+(g-y)v+(h-z)w.$$

The first equation follows from the fact that the direction of motion of any point on the conjugate is perpendicular to the given axis, and the second from the fact that the direction of motion is also perpendicular to the straight line joining the point to (f, g, h).

Ex. 2. If an instantaneous motion be represented by a screw along the axis of z, the linear and angular velocities being V and Ω, prove that the equations to the conjugate of $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ are $mx-ly+n\frac{V}{\Omega}=0$ and $gx-fy-\frac{V}{\Omega}(z-h)=0$.

Ex. 3. The locus of the conjugates of all axes of instantaneous rotation which are parallel to a fixed straight line is a plane parallel to the central axis and to the fixed straight line.

Ex. 4. The locus of the conjugates of all axes of instantaneous rotation which pass through a given point is a plane. If two axes intersect, their conjugates also intersect.

226. If the instantaneous motion of a body be represented by two conjugate rotations about two axes *at right angles*, a plane can be drawn through either axis perpendicular to the other. The axis in the plane has been called the *characteristic* of that plane, and the axis perpendicular to the plane is said to cut the plane in its *focus*. These names were given by M. Chasles in the *Comptes Rendus* for 1843. Some of the following examples were also given by him, though without demonstrations.

Ex. 1. Show that every plane has a characteristic and a focus.

Let the central axis cut the plane in O. Resolve the linear and angular velocities in two directions Ox, Oz, the first in the plane and the second perpendicular to it. The translations along Ox, Oz may be removed if we move the axes of rotation Ox, Oz parallel to themselves, by Art. 202. Thus the motion is represented by a rotation about an axis in the plane and a rotation about an axis perpendicular to it. It also follows that the characteristic of a plane is parallel to the projection of the central axis.

Ex. 2. If a plane be fixed in the body and move with the body, it intersects its consecutive position in its characteristic. The velocity of any point P in the plane when resolved perpendicular to the plane is proportional to its distance from the characteristic, and when resolved in the plane is proportional to its distance from the focus and is perpendicular to that distance.

Ex. 3. If two conjugate axes cut a plane in F and G, then FG passes through the focus.

If two conjugate axes be projected on a plane, they meet in the characteristic of that plane.

Ex. 4. If two axes CM, CN meet in a point C, their conjugates lie in a plane whose focus is C and intersect in the focus of the plane CMN.

This follows from the fact that if a straight line cut an axis the direction of motion of every point on it is perpendicular to the straight line only when it also cuts the conjugate.

Ex. 5. Any two axes being given and their conjugates, the four straight lines lie on the same hyperboloid.

Ex. 6. If the instantaneous motion of a body be given by the linear and angular velocities (u, v, w) $(\omega_1, \omega_2, \omega_3)$, prove that the characteristic of the plane

$$Ax + By + Cz + D = 0$$

is its intersection with

$$A(u + \omega_2 z - \omega_3 y) + B(v + \omega_3 x - \omega_1 z) + C(w + \omega_1 y - \omega_2 x) = 0,$$

and its focus may be found from

$$\frac{u + \omega_2 z - \omega_3 y}{A} = \frac{v + \omega_3 x - \omega_1 z}{B} = \frac{w + \omega_1 y - \omega_2 x}{C}.$$

For the characteristic is the locus of the points whose directions of motion are perpendicular to the normal to the plane, and the focus is the point whose direction of motion is perpendicular to the plane.

What do these equations become when the central axis is the axis of z?

Ex. 7. The locus of the characteristics of planes which pass through a given straight line is a hyperboloid of one sheet; the shortest distance between the given straight line and the central axis being the direction of one principal diameter, and the other two being the internal and external bisectors of the angle between the given straight line and the central axis. Prove also that the locus of the foci of the planes is the conjugate of the given straight line.

Ex. 8. Let any surface A be fixed in a body and move with it, the normal planes to the trajectories of all its points envelope a second surface B. Prove that if the surface B be fixed in the body and move with it, the normal planes to the

trajectories of its points will envelope the surface A: so that the surfaces A and B have conjugate properties, each surface being the locus of the foci of the tangent planes to the other.

Prove that if one surface is a quadric the other is also a quadric.

Ex. 9. A body is moved from any position in space to any other, and every point of the body in the first position is joined to the same point in the second position. If all the straight lines thus found be taken which pass through a given point, they will form a cone of the second order. Also if the middle points of all these lines be taken, they will together form a body capable of an infinitesimal motion, each point of it along the line on which the same is situate. Cayley's *Report to the Brit. Assoc.*, 1862.

Euler's Equations.

227. *To determine the general equations of motion of a body about a fixed point.*

Let the fixed point O be taken as origin, and let x, y, z be the co-ordinates at time t of any particle m referred to any rectangular axes fixed in space. Let Xm, Ym, Zm be the impressed forces acting on this element parallel to the axes of co-ordinates, and let L, M, N be the moments of all these forces about the axes.

Then by D'Alembert's Principle, if the effective forces $m\dfrac{d^2x}{dt^2}$, $m\dfrac{d^2y}{dt^2}$, $m\dfrac{d^2z}{dt^2}$ be applied to every particle m in a reversed direction, there will be equilibrium between these forces and the impressed forces. Taking moments therefore about the axes, we have

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=N, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and two similar equations.

To simplify these equations, let ω_x, ω_y, ω_z be the angular velocities about the axes. Then $\dfrac{dx}{dt}=\omega_y z-\omega_z y$, $\dfrac{dy}{dt}=\omega_z x-\omega_x z$, $\dfrac{dz}{dt}=\omega_x y-\omega_y x$;

$$\therefore\ \frac{d^2x}{dt^2}=z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}+\omega_y(\omega_x y-\omega_y x)-\omega_z(\omega_z x-\omega_x z),$$

$$\frac{d^2y}{dt^2}=x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}+\omega_z(\omega_y z-\omega_z y)-\omega_x(\omega_x y-\omega_y x).$$

Substituting in equation (1) we get

$$\left.\begin{array}{l}\Sigma m(x^2+y^2)\dfrac{d\omega_z}{dt}-\Sigma myz\,.\dfrac{d\omega_y}{dt}-\Sigma mxz\,.\dfrac{d\omega_x}{dt}\\ -\Sigma mxy\,.(\omega_x^2-\omega_y^2)+\Sigma m(x^2-y^2)\,\omega_x\omega_y-\Sigma myz\,.\,\omega_x\omega_z\\ \qquad\qquad\qquad\qquad\qquad\qquad+\Sigma mxz\,.\,\omega_y\omega_z\end{array}\right\}=N.$$

The other two equations may be treated in the same manner.

The coefficients in this equation are the moments and products of inertia of the body with regard to axes fixed in space and are therefore variable as the body moves about. Let us then take a second set of rectangular axes OA, OB, OC fixed in the body, and let ω_1, ω_2, ω_3 be the angular velocities about these axes. Since the axes Ox, Oy, Oz are perfectly arbitrary, let them be so chosen that the axes OA, OB, OC are passing through them at the moment under consideration. Then $\omega_x=\omega_1$, $\omega_y=\omega_2$, $\omega_z=\omega_3$. If the principal axes at the fixed point have been chosen as the set of axes fixed in the body, and A, B, C be the moments of inertia about them, the equation takes the form

$$C\frac{d\omega_z}{dt}-(A-B)\,\omega_1\omega_2=N,$$

in which all the coefficients are constants.

228. We shall now show that $\dfrac{d\omega_3}{dt}=\dfrac{d\omega_z}{dt}$. This may appear at first sight to follow at once from the equation $\omega_3=\omega_z$. But it is not so; ω_3 denotes the angular velocity of the body about OC *fixed in the body*, while ω_z denotes the angular velocity about a line Oz *fixed in space* and determined by the condition that *at the time t* OC coincides with it. At the time $t+dt$ OC will have separated from Oz and we cannot therefore assert *a priori* that the angular velocity about OC will continue to be the same as that about Oz. We have to prove that this is the case as far as the first order of small quantities. Let OR, OR' be the resultant axes of rotation at the times t and $t+dt$, *i.e.* let a rotation Ωdt about OR bring OC into coincidence with Oz at the time t, and let a further rotation $\Omega' dt$ about OR' bring OC into the position OC' in space at the time $t+dt$. Then according to the definition of a differential coefficient

$$\frac{d\omega_3}{dt}=L^t\text{ of }\frac{\Omega'\cos R'C'-\Omega\cos RC}{dt},$$

$$\frac{d\omega_z}{dt}=L^t\text{ of }\frac{\Omega'\cos R'z-\Omega\cos Rz}{dt}.$$

Since a rotation about OR' brings OC from the position Oz to OC', $R'C'$ and $R'z$ differ by quantities of the second order, and therefore these two differential coefficients are ultimately equal.

229. The following demonstration of this equality has been given by the late Professor Slesser of Queen's College, Belfast, and is instructive as founded on a different principle. Let A, B, C be the points in which the principal axes cut a sphere whose centre is at the fixed point. Let OL be any other axis, and let Ω be the angular velocity about it. Let the angles LOA, LOB, LOC be called respectively α, β, γ. Then by Art. 201

$$\Omega = \omega_1 \cos\alpha + \omega_2 \cos\beta + \omega_3 \cos\gamma;$$

$$\therefore \frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \frac{d\omega_2}{dt}\cos\beta + \frac{d\omega_3}{dt}\cos\gamma$$

$$- \omega_1 \sin\alpha \frac{d\alpha}{dt} - \omega_2 \sin\beta \frac{d\beta}{dt} - \omega_3 \sin\gamma \frac{d\gamma}{dt}.$$

Now let the line OL be fixed in space and coincide with OC at the moment under consideration. Then $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$;

therefore $$\frac{d\Omega}{dt} = \frac{d\omega_3}{dt} - \omega_1 \frac{d\alpha}{dt} - \omega_2 \frac{d\beta}{dt}.$$

Also $\frac{d\alpha}{dt}$ is the angular rate at which A separates from a *fixed* point at C, this is clearly ω_2. Similarly $\frac{d\beta}{dt} = -\omega_1$. Hence $\frac{d\Omega}{dt} = \frac{d\omega_3}{dt}$. Thus $\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}$, $\frac{d\omega_y}{dt} = \frac{d\omega_2}{dt}$, $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}$.

230. The three equations of motion of the body referred to the principal axes at the fixed point are therefore

$$A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = L,$$

$$B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 = M,$$

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = N.$$

These are called Euler's equations.

231. We know by D'Alembert's principle that the moment of the effective forces about any straight line is equal to that of the impressed forces. The equations of Euler therefore indicate that the moment of the effective force about the principal axes at the fixed point are expressed by the left-hand sides of the above equations. If there is no point of the body which is fixed in space, the motion of the body about its centre of gravity is the same as if that point were fixed. In this case, if A, B, C be the principal moments at the centre of gravity, the left-hand sides of Euler's equations give the moments of the effective forces about

the principal axes at the centre of gravity. If we want the moment about any other straight line passing through the fixed point, we may find it by simply resolving these moments by the rules of Statics.

232. Ex. 1. If $2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2$ and G be the moment of the impressed forces about the instantaneous axis, Ω the resultant angular velocity, prove that $\frac{dT}{dt} = G\Omega$.

Ex. 2. A body turning about a fixed point is acted on by forces which tend to *produce* rotation about an axis at right angles to the instantaneous axis, show that the angular velocity cannot be uniform unless two of the principal moments at the fixed point are equal. The axis about which the forces tend to produce rotation is that axis about which it would begin to turn if the body were placed at rest.

233. *To determine the pressure on the fixed point.*

Let x, y, z be the co-ordinates of the centre of gravity referred to rectangular axes fixed in space meeting at the fixed point, and let P, Q, R be the resolved parts of the pressures on the body in these directions. Let μ be the mass of the body. Then we have

$$\mu \frac{d^2x}{dt^2} = P + \Sigma mX$$

and two similar equations. Substituting for $\frac{d^2x}{dt^2}$ its value in terms $\omega_x, \omega_y, \omega_z$ we have

$$\mu \left\{ z \frac{d\omega_y}{dt} - y \frac{d\omega_z}{dt} + \omega_y (\omega_x y - \omega_y x) - \omega_z (\omega_z x - \omega_x z) \right\} = P + \Sigma mX$$

and two similar equations.

If we now take the axes fixed in space to coincide with the principal axes at the fixed point at the moment under consideration we may substitute for $\frac{d\omega_y}{dt}$ and $\frac{d\omega_z}{dt}$ from Euler's equations. We then have

$$\mu \left\{ \omega_1 (B + C - A) \left(\frac{y\omega_2}{C} + \frac{z\omega_3}{B} \right) - (\omega_2^2 + \omega_3^2) x \right\} = P + \Sigma mX - \mu \left(\frac{M}{B} z - \frac{N}{C} y \right),$$

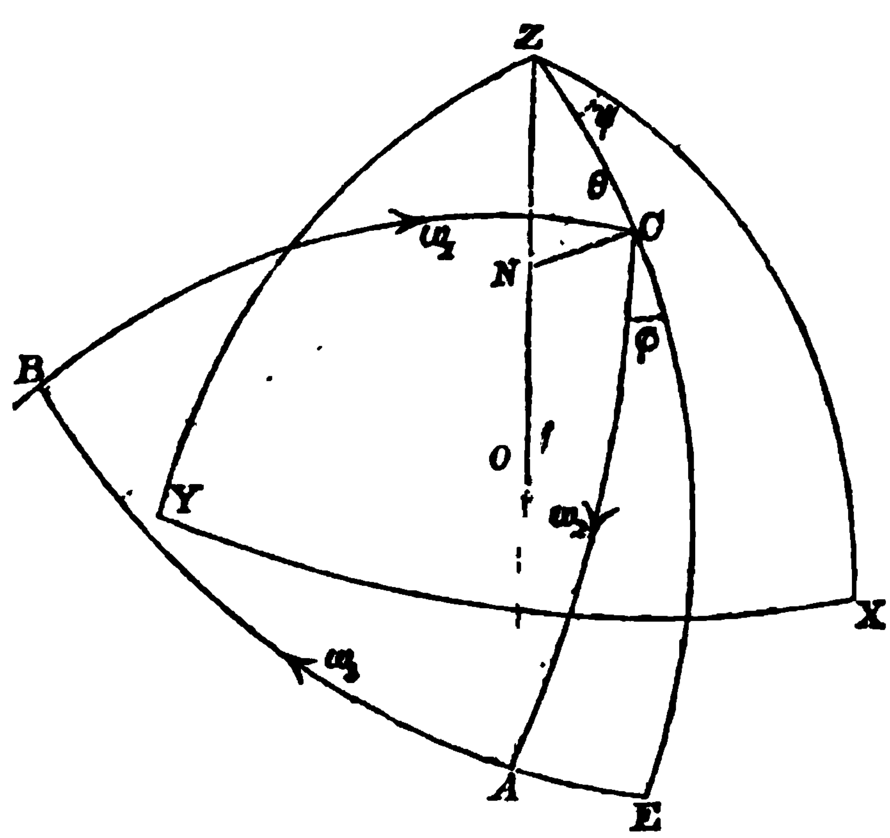
with similar expressions for Q and R.

234. Ex. If G be the centre of gravity of the body, show that the terms on the left-hand sides of the equations which give the pressures on the fixed point are the components of two forces, one $\Omega^2 . GH$ along GH which is a perpendicular on the instantaneous axis OI, Ω being the resultant angular velocity, and the other $\Omega'^2 . GK$ perpendicular to the plane OGK, where GK is a perpendicular on a straight line OJ whose direction cosines are proportional to $\frac{B-C}{A}\omega_2\omega_3$, $\frac{C-A}{B}\omega_3\omega_1$, $\frac{A-B}{C}\omega_1\omega_2$, and Ω'^4 is the sum of the squares of these quantities.

235. *To determine the geometrical equations connecting the motion of the body in space with the angular velocities of the body about the three moving axes,* OA, OB, OC.

Let the fixed point O be taken as the centre of a sphere of radius unity; let X, Y, Z and A, B, C be the points in which the sphere is cut by the fixed and moving axes respectively. Let ZC, BA produced if necessary, meet in E. Let the angle $XZC = \psi$, $ZC = \theta$, $ECA = \phi$. It is required to determine the geometrical relations between θ, ϕ, ψ, and $\omega_1, \omega_2, \omega_3$.

Draw CN perpendicular to OZ. Then since ψ is the angle the plane COZ makes with a plane XOZ fixed in space, the velocity of C perpendicular to the plane ZOC is $CN\dfrac{d\psi}{dt}$, which is the same as $\sin\theta\dfrac{d\psi}{dt}$, the radius OC of the sphere being unity. Also the velocity of C along ZC is $\dfrac{d\theta}{dt}$. Thus the motion of C is re-



presented by $\dfrac{d\theta}{dt}$ and $\sin\theta\dfrac{d\psi}{dt}$ respectively along and perpendicular to ZC. But the motion of C is also expressed by the angular velocities ω_1 and ω_2 respectively along BC and CA. These two representations of the same motion must therefore be equivalent. Hence resolving along and perpendicular to ZC we have

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_1 \sin\phi + \omega_2 \cos\phi \\ \sin\theta\frac{d\psi}{dt} &= -\omega_1 \cos\phi + \omega_2 \sin\phi \end{aligned}\right\}.$$

Similarly by resolving along CB and CA we have

$$\left.\begin{aligned}\omega_1 &= \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi \\ \omega_2 &= \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi\end{aligned}\right\}.$$

These two sets of equations are precisely equivalent to each other and one may be deduced from the other by an algebraic transformation.

In the same way by drawing a perpendicular from E on OZ we may show that the velocity of E perpendicular to ZE is $\frac{d\psi}{dt}\sin ZE$, and this is the same as $\frac{d\psi}{dt}\cos\theta$. Also the velocity of A relative to E along EA is in the same way $\frac{d\phi}{dt}\sin CA$, and this is the same as $\frac{d\phi}{dt}$. Hence the whole velocity of A in space along AB is represented by $\frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}$. But this motion is also expressed by ω_3. As before these two representations of the same motion must be equivalent. Hence we have

$$\omega_3 = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}.$$

If in a similar manner we had expressed the motion of any other point of the body as B, both in terms of ω_1, ω_2, ω_3 and θ, ϕ, ψ, we should have obtained other equations. But as we cannot have more than three independent relations, we should only arrive at equations which are algebraic transformations of those already obtained.

236. **Ex.** If p, q, r be the direction cosines of OZ with regard to the axes OA, OB, OC, show that these equations may be put into the symmetrical form

$$\frac{dp}{dt} - q\omega_3 + r\omega_2 = 0, \qquad \frac{dq}{dt} - r\omega_1 + p\omega_3 = 0, \qquad \frac{dr}{dt} - p\omega_2 + q\omega_1 = 0.$$

Any one of these may be obtained by differentiating one of the expressions $p = -\sin\theta\cos\phi$, $q = \sin\theta\sin\phi$, $r = \cos\theta$. The others may be inferred by the rule of symmetry.

237. It is clear that instead of referring the motion of the body to the principal axes at the fixed point, as Euler has done, we may use any axes fixed in the body. But these are in general so complicated as to be nearly useless. When, however, a body is making small oscillations about a fixed point, so that some three rectangular axes fixed in the body never deviate far from three axes fixed in space, it is often convenient to refer the motion to

these even though they are not principal axes. In this case ω_1, ω_2, ω_3 are all small quantities, and we may neglect their products and squares. The general equation of Art. 227 reduces in this case to

$$C\frac{d\omega_3}{dt} - D\frac{d\omega_2}{dt} - E\frac{d\omega_1}{dt} = N,$$

where the coefficients have the usual meanings given to them in Chap. I. We have thus three linear equations which may be written thus:

$$A\frac{d\omega_1}{dt} - F\frac{d\omega_2}{dt} - E\frac{d\omega_3}{dt} = L,$$

$$-F\frac{d\omega_1}{dt} + B\frac{d\omega}{dt} - D\frac{d\omega_3}{dt} = M,$$

$$-E\frac{d\omega_1}{dt} - D\frac{d\omega_2}{dt} + C\frac{d\omega_3}{dt} = N.$$

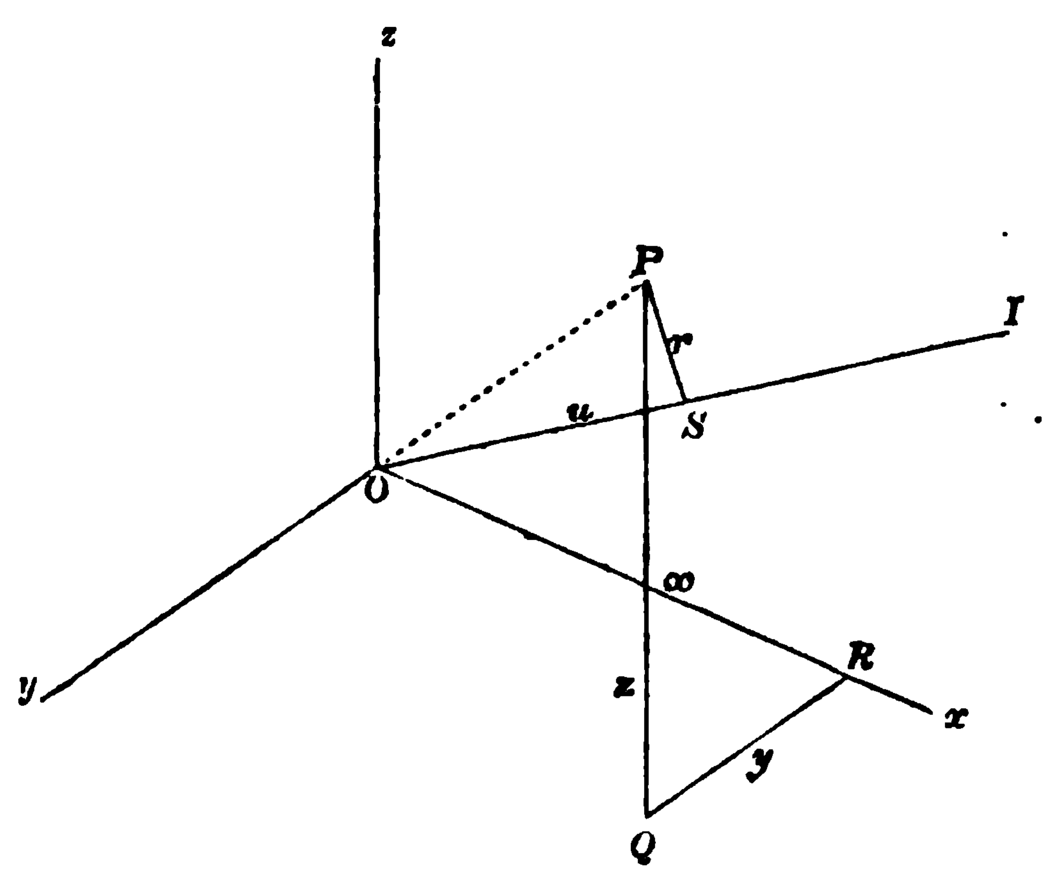
238. It appears from Euler's Equations that the whole changes of ω_1, ω_2, ω_3 are not due merely to the direct action of the forces, but are in part due to the centrifugal force of the particles tending to carry them away from the axis about which they are revolving. For consider the equation

$$\frac{d\omega_3}{dt} = \frac{N}{C} + \frac{A-B}{C}\omega_1\omega_2.$$

Of the increase $d\omega_3$ in the time dt, the part $\frac{N}{C}dt$ is due to the direct action of the forces whose moment is N, and the part $\frac{A-B}{C}\omega_1\omega_2\,dt$ is due to the centrifugal force. This may be proved as follows.

If a body be rotating about an axis OI with an angular velocity ω, then the moment of the centrifugal forces of the whole body about the axis Oz is $(A-B)\,\omega_1\omega_2$.

Let P be the position of any particle m and let x, y, z be its co-ordinates. Then $x=OR$, $y=RQ$, $z=QP$. Let PS be a perpendicular on OI, let $OS=u$, and $PS=r$. Then the centrifugal force of the particle m is $\omega^2 rm$ tending from OI.



The force $\omega^2 rm$ is evidently equivalent to the *four* forces $\omega^2 xm$, $\omega^2 ym$, $\omega^2 zm$, and $-\omega^2 um$ acting at P parallel to x, y, z, and u respectively.

$$\left.\begin{array}{l} \text{The moment of } \omega^2 xm \text{ round } Oz = -\omega^2 xym \\ \text{.................... } \omega^2 ym \text{ .... } = \omega^2 xym \\ \text{.................... } \omega^2 zm \text{ .... } = 0 \end{array}\right\};$$

these three therefore produce no effect.

The force $-\omega^2 um$ parallel to OI is equivalent to the three, $-\omega\omega_1 um$, $-\omega\omega_2 um$, $-\omega\omega_3 um$, acting at P parallel to the axes, and their moment round Oz is evidently $\omega um(\omega_1 y - \omega_2 x)$. Now the direction cosines of OI being $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, we get by projecting the broken line x, y, z on OI, $u = \frac{\omega_1}{\omega}x + \frac{\omega_2}{\omega}y + \frac{\omega_3}{\omega}z$; therefore substituting for u, the moment of centrifugal forces about Oz is

$$= (\omega_1 y - \omega_2 x)(\omega_1 x + \omega_2 y + \omega_3 z)\, m,$$

$$= (\omega_1^2 xy + \omega_1\omega_2 y^2 + \omega_1\omega_3 yz - \omega_1\omega_2 x^2 - \omega_2^2 xy - \omega_2\omega_3 xz)\, m.$$

Writing Σ before every term, and supposing the axes of x, y, z, to be principal axes, then the moment of the centrifugal forces about the principal axis Oz

$$= \omega_1\omega_2 \Sigma m(y^2 - x^2) = \omega_1\omega_2 (A - B).$$

Let the moments of the centrifugal forces about the principal axes of the body be represented by L', M', N', so that

$$L' = (B - C)\,\omega_2\omega_3, \quad M' = (C - A)\,\omega_3\omega_1, \quad N' = (A - B)\,\omega_1\omega_2,$$

and let G be their resultant couple. The couple G is usually called the *centrifugal couple.*

Since $L'\omega_1 + M'\omega_2 + N'\omega_3 = 0$, it follows that the axis of the centrifugal couple is at right angles to the instantaneous axis.

Describe the momental ellipsoid at the fixed point O and let the instantaneous axis cut its surface in I. Let OH be a perpendicular from O on the tangent plane at I. The direction cosines of OH are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$. Since $A\omega_1 L' + B\omega_2 M' + C\omega_3 N' = 0$, it follows that the axis of the centrifugal couple is at right angles to the perpendicular OH.

The plane of the centrifugal couple is therefore the plane IOH.

If μk^2 be the moment of inertia of the body about the instantaneous axis of rotation we have $k^2 = \frac{\epsilon^4}{OI^2}$, and $T = \mu k^2 \omega^2$ is the Vis Viva of the body. We may then easily show that the magnitude G of the centrifugal couple is $G = T\tan\phi$, where ϕ is the angle IOH.

This couple will generate an angular velocity of known magnitude about the diametral line of its plane. By compounding this with the existing angular velocity, the change in the position of the instantaneous axis might be found.

Expressions for Angular Momentum.

239. We may now investigate convenient formulæ for the angular momentum of a body about any axis. The importance of these has been already pointed out in Art. 77. In fact, the general equations of motion of a rigid body as given in Art. 71,

cannot be completely expressed until these formulæ have been found.

When the body is moving in space of two dimensions about either a fixed point, or its centre of gravity regarded as a fixed point, the angular momentum about that point has been proved in Art. 88 to be $Mk^2\omega$ where Mk^2 is the moment of inertia, and ω the angular velocity about that point. Our object is to find corresponding formulæ when the body is moving in space of three dimensions. Following the same order as in Euler's Equations, we shall first find the angular momentum about any fixed straight line in space, taken as the axis of z and passing through the fixed point; secondly, the momentum about any fixed straight line in the body and also passing through the fixed point, and lastly, we shall show how the angular momenta about other axes may be found.

240. *A body is turning about a fixed point in any manner, to determine the moments of the momentum about the axes,* i.e. *to find the areas conserved round those axes.* See Chap. II. Art. 78.

Let (x, y, z) be the co-ordinates of any particle m of the body referred to axes fixed in space meeting at the fixed point. Let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the body about the fixed axes. Then the moment of the momentum about the axis of z is

$$h_3 = \Sigma m \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

Substituting for $\frac{dx}{dt}, \frac{dy}{dt}$ their values

$$\left.\begin{aligned}\frac{dx}{dt} &= \omega_y z - \omega_z y\\ \frac{dy}{dt} &= \omega_z x - \omega_x z\end{aligned}\right\},$$

we have $\quad h_3 = \Sigma m(x^2 + y^2)\,\omega_z - (\Sigma mxz)\,\omega_x - (\Sigma myz)\,\omega_y.$

241. The coefficients of $\omega_x, \omega_y, \omega_z$ are the moments and products of inertia of the body about the axes, and if the axes be fixed in space, these will generally be variable. In some cases it will be found more convenient to take as axes of reference three straight lines fixed in the body.

Let $\omega_1, \omega_2, \omega_3$ be the angular velocities of the body about rectangular axes Ox', Oy', Oz' fixed in the body and meeting at the fixed point O. Since in the expression given above for h_3 the fixed axes may be any whatever, let them be chosen so that the moving axes coincide with them at the time t. Then, $\omega_x = \omega_1$,

$\omega_y = \omega_2$, $\omega_z = \omega_3$, and the moment of the momentum about the moving axis of z will be expressed by the form

$$h_3' = C\omega_3 - E\omega_1 - D\omega_2,$$

where $\quad C = \Sigma m (x'^2 + y'^2), \quad E = \Sigma m x'z', \quad D = \Sigma m y'z'.$

These will be constant throughout the motion, and their values may be found by the rules given in Chapter I.

If the axes fixed in the body be principal axes, then the products of inertia will vanish. The expressions for the moments of the momentum will then take the simple forms

$$\left.\begin{aligned} h_1' &= A\omega_1 \\ h_2' &= B\omega_2 \\ h_3' &= C\omega_3 \end{aligned}\right\},$$

where A, B, C are the principal moments of the body.

Let the direction-cosines of the axes fixed in space but moving with reference to axes fixed in the body be given by the following diagram; where, for example, b_3 is the cosine of the angle between the axes of z and y'. It has just been proved that the resultant of the momenta of all the particles of the body is equivalent to the three "couples" h_1', h_2', h_3' about the axes Ox', Oy', Oz'. Hence the moment of the momentum about the axis of z which is fixed in space may be written in the form

	x,	y,	z
x'	a_1,	a_2,	a_3
y'	b_1,	b_2,	b_3
z'	c_1,	c_2,	c_3

$$h_3 = h_1'a_3 + h_2'b_3 + h_3'c_3,$$

which will be frequently found useful.

242. It may be required to find the moment of the momentum about axes neither fixed in space nor in the body, but moving in any arbitrary manner. This will be expressed by the same form as if the axes were fixed. If ω_x, ω_y, ω_z be the angular velocities about these axes, the moment required will be

$$= \Sigma m (x^2 + y^2)\, \omega_z - (\Sigma mxz)\, \omega_x - (\Sigma myz)\, \omega_y.$$

If the axis of z coincide with the instantaneous axis of rotation, $\omega_x = 0$, $\omega_y = 0$, and ω_z is the resultant angular velocity. The expressions for the moments of the momentum or areas conserved about the axes of x, y, z become respectively

$$-(\Sigma mxz)\, \omega_z, \quad -(\Sigma myz)\, \omega_z, \quad \Sigma m (x^2 + y^2)\, \omega_z.$$

The axis of the couple which is the resultant of the moments of the momentum about the axis is sometimes called the resultant axis of angular momentum and sometimes the resultant axis of areas. It is to be remarked that this axis does not in general

coincide with the instantaneous axis of rotation. The two are coincident only when the axis of rotation is a principal axis. If a body be turning about a straight line, which we may call the axis of z, as instantaneous axis, it is a common mistake to suppose that the angular momentum about a perpendicular axis is zero. We see from the last remark that this is not generally true.

If it be required to find the moment of the momentum about the axis of z of a rigid body moving in any manner in space, we may use the principle proved in Chapter II. Art. 76.

In the case of a system of rigid bodies, the moment of their momenta may be found by adding up the separate moments of the several bodies.

Ex. 1. A triangular area ACB whose mass is M is turning round the side CA with an angular velocity ω. Show that the angular momentum about the side CB is $\frac{1}{12} Mab \sin^2 C\omega$, where a and b are the sides containing the angle C.

Ex. 2. Two rods OA, AB, are hinged together at A and suspended from a fixed point O. The system turns with angular velocity ω about a vertical straight line through O so that the two rods are in a vertical plane. If θ, ϕ be the inclinations of the rods to the vertical, a, b their lengths, M, M' their masses, show that the angular momentum about the vertical axis is

$$\omega\left[(\tfrac{1}{3} M + M') a^2 \sin^2 \theta + M'ab \sin\theta \sin\phi + \tfrac{1}{3} M' b^2 \sin^2 \phi.\right]$$

Ex. 3. A right cone, whose vertex O is fixed, has an angular velocity ω communicated to it about its axis OC, while at the same time its axis is set moving in space. The semi-angle of the cone is $\frac{\pi}{4}$ and its altitude is h. If θ be the inclination of the axis to a fixed straight line Oz and ψ the angle the plane zOC makes with a fixed plane through Oz, prove that the angular momentum about Oz is $\frac{3}{5} Mh^2\omega \left(\sin^2\theta \frac{d\psi}{dt} + \frac{2}{3}\cos\theta\right)$, where M is the mass of the cone.

Ex. 4. A rod AB is suspended by a string from a fixed point O and is moving in any manner. If (l, m, n) (p, q, r) be the direction cosines of the string and rod referred to any rectangular axes Ox, Oy, Oz, show that the angular momentum about the axis of z is

$$Mb^2\left(l\frac{dm}{dt} - m\frac{dl}{dt}\right) + M\frac{a^2}{3}\left(p\frac{dq}{dt} - q\frac{dp}{dt}\right) + M\frac{ab}{2}\left(p\frac{dm}{dt} - m\frac{dp}{dt} + l\frac{dq}{dt} - q\frac{dl}{dt}\right),$$

where M is the mass of the rod, and a, b the lengths of the rod and string.

On Moving Axes and Relative Motion.

243. In many cases it will be found convenient to refer the motion of the body under consideration to axes moving in space in some manner about a fixed origin. If we refer the motion of these axes to other axes fixed in space we shall have the inconvenience of two sets of axes. For this reason their motion at any instant is sometimes defined by angular velocities $(\theta_1, \theta_2, \theta_3)$ about them-

selves. In this case we are to regard the axes as if they were a material system of three straight lines at right angles whose motion at any instant is given by three coexistent angular velocities about axes instantaneously coincident with them.

When the axes are moving we may suppose the motion of the body to be determined by the three angular velocities ω_1, ω_2, ω_3 about the axes, in the same manner as if the axes were fixed for an instant in space. The position of the body at the time $t + dt$ may be constructed from that at the time t by turning the body through the angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$ successively round the instantaneous position of the axes. But it must be remembered that $\omega_3 dt$ does not now give the angle the body has been turned through relatively to the plane xz, but relatively to some plane fixed in space passing through the instantaneous position of the axis of z. The angle turned through relatively to the plane of xz is $(\omega_3 - \theta_3)\, dt$.

244. *To find the resolved part of the velocity of any particle parallel to the moving axes.*

The resolved parts of the velocity of any point whose coordinates are (x, y, z) are not given by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. These are the resolved velocities of the particle *relatively* to the axes. To find the motion in space we must add to these the resolved velocities due to the motion of the axes themselves. If we supposed the particle to be rigidly connected with the axes, it is clear that its velocities would be expressed by the forms given in Art. 219 with θ_1, θ_2, θ_3 substituted for ω_1, ω_2, ω_3. So that the actual resolved velocities of the particle are

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1.$$

245. *To find the accelerations of any particle parallel to the axes we may proceed thus.*

The velocities of the particle at the time t resolved parallel to the axes Ox, Oy, Oz are respectively (u, v, w). At the time $t + dt$, the axes have been turned into the position Ox', Oy', Oz' by rotations equal to $\theta_1 dt$, $\theta_2 dt$, $\theta_3 dt$ round the axes Ox, Oy, Oz respectively, and the velocities of the particle parallel to the axes in their new position are

$$u + \frac{du}{dt}dt, \quad v + \frac{dv}{dt}dt, \quad w + \frac{dw}{dt}dt.$$

Describe a sphere of unit radius whose centre is at the fixed origin and let all these axes cut the sphere in the points x, y, z, x', y', z' respectively. Thus we have two spherical triangles xyz and $x'y'z'$, all whose sides are right angles. The resolved part of the velocity of the particle at the time $t+dt$ along the axis of z is

$$\left(u+\frac{du}{dt}dt\right)\cos zx' + \left(v+\frac{dv}{dt}dt\right)\cos zy' + \left(w+\frac{dw}{dt}dt\right)\cos zz'.$$

By the rotation round Oy, x' has receded from z by the arc $\theta_2 dt$, and by the rotation round Ox, y' has approached z by the arc $\theta_1 dt$. Therefore

$$zx' = zx + \theta_2\, dt,$$
$$zy' = zy - \theta_1\, dt.$$

Also the cosine of the arc zz' differs from unity by the squares of small quantities. Substituting these we find that the component velocity of the particle at the time $t+dt$ parallel to the axis of z is ultimately

$$w+\frac{dw}{dt}dt - u\theta_2\, dt + v\theta_1\, dt.$$

But the acceleration is by definition, the ratio of the velocity gained in any time dt to that time. Hence if Z be the acceleration resolved parallel to the axis of z, we have

$$Z=\frac{dw}{dt} - u\theta_2 + v\theta_1.$$

Similarly if X and Y be the accelerations parallel to the axes of x and y, we have

$$X=\frac{du}{dt} - v\theta_3 + w\theta_2,$$

$$Y=\frac{dv}{dt} - w\theta_1 + u\theta_3.$$

246. Ex. 1. Let the motion be referred to *oblique* moving axes so that the sides of the spherical triangle xyz are a, b, c and the angles A, B, C. Let the equal quantities $\sin a \sin b \sin C$, $\sin b \sin c \sin A$, $\sin c \sin a \sin B$ be called μ. Prove that if the velocity be represented by the three *components* u, v, w parallel to these axes, then the *resultant* acceleration parallel to the axis of z is

$$Z=\frac{dw}{dt}+\frac{du}{dt}\cos b+\frac{dv}{dt}\cos a - u\theta_2\mu + v\theta_1\mu,$$

with similar expressions for X and Y.

This may be done by the use of the spherical triangles xyz, $x'y'z'$, by first proving that $zx' = b + \theta_2 dt \sin c \sin A$, $zy' = a - \theta_1 dt \sin c \sin B$, and then substituting as before.

Ex. 2. Prove in the same way that if x, y, z be the co-ordinates referred to oblique axes, and u', v', w' the *resultant* velocities parallel to the axes,

$$w'=\frac{dz}{dt}+\frac{dx}{dt}\cos b+\frac{dy}{dt}\cos a - x\theta_2\mu + y\theta_1\mu,$$

with similar expressions for u' and v'.

Ex. 3. Prove also that the equations connecting u, v, w with the co-ordinates are

$$w = \frac{dz}{dt} + \begin{vmatrix} \frac{\sin^2 c}{\mu} & -\cot B & -\cot A \\ \theta_3 & \theta_1 & \theta_2 \\ z & x & y \end{vmatrix}$$

with two similar expressions for u and v.

Since w' is the resolved velocity parallel to z of $(u, v, w,)$ we have

$$u \cos b + v \cos a + w = w',$$

with similar expressions for u' and v'. By solving these we get the required values of u, v, w.

Ex. 4. If the whole acceleration be represented by the three components X, Y, Z parallel to the axes, prove that the expressions for these in terms of uvw, may be obtained from those given in the last example by changing x, y, z into u, v, w and u, v, w into X, Y, Z.

247. *To express the geometrical conditions that a straight line whose equations with reference to the moving axes are given is fixed in direction in space.*

Let the equation to the given straight line be

$$\frac{x-f}{p} = \frac{y-g}{q} = \frac{z-h}{r},$$

and let the equations be so prepared that (p, q, r) are the direction cosines of the line. Let a straight line be drawn through the origin parallel to this given straight line and let a point P be taken on this at any given distance L from the origin O. Then the co-ordinates of P are pL, qL, rL respectively. Since the straight line OP is fixed in direction in space, the resolved parts of the velocity of P parallel to the axes are zero. Hence we have

$$\frac{dLp}{dt} - Lq\theta_3 + Lr\theta_2 = 0,$$

and two similar equations. The required geometrical conditions are therefore

$$\frac{dp}{dt} - q\theta_3 + r\theta_2 = 0,$$

$$\frac{dq}{dt} - r\theta_1 + p\theta_3 = 0,$$

$$\frac{dr}{dt} - p\theta_2 + q\theta_1 = 0.$$

When it is necessary to refer the motion of these moving axes to other axes fixed in space, we may either use the equations of this article or those of Art. 235. Taking the notation of the

article referred to, it is obvious (the axes being treated as a body consisting simply of three straight lines) that we shall have the results

$$\left.\begin{aligned} \frac{d\psi}{dt}\sin\theta &= -\theta_1\cos\phi + \theta_2\sin\phi \\ \frac{d\theta}{dt} &= \theta_1\sin\phi + \theta_2\cos\phi \\ \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt} &= \theta_3 \end{aligned}\right\}.$$

These equations will determine θ, ϕ, ψ in terms of the angular velocities θ_1, θ_2, θ_3.

248. *To express the geometrical conditions that a point whose co-ordinates with reference to the moving axes are* (x, y, z) *is fixed in space.*

This may be done by equating to zero the resolved parts of the velocity of the point as given in Art. 244. If the origin of the moving axes be fixed, the conditions are

$$\frac{dx}{dt} - y\theta_3 + z\theta_2 = 0,$$

and two similar equations. If the origin be in motion, let u_0, v_0, w_0 be the resolved parts of its velocity parallel to the axes, then the required conditions are clearly

$$u_0 + \frac{dx}{dt} - y\theta_3 + z\theta_2 = 0,$$

and two similar equations.

249. **Ex.** Let the direction cosines of a straight line OM fixed relatively to the moving axes be (λ, μ, ν) and let it be required to refer the motion of OM to some straight line OL fixed in space whose direction cosines at the time t are (p, q, r). Let the angle LOM be θ and let ψ be the angle the plane LOM makes with any fixed plane in space passing through OL. Then show that

$$\left.\begin{aligned} &\cos\theta = p\lambda + q\mu + r\nu, \\ &\sin^2\theta\frac{d\psi}{dt} = \theta_1(p - \lambda\cos\theta) + \theta_2(q - \mu\cos\theta) + \theta_3(r - \nu\cos\theta) \end{aligned}\right\}.$$

If θ_l, θ_m be the resolved parts of the angular velocities about OL, OM respectively, the last equation may be written in the form

$$\sin^2\theta\frac{d\psi}{dt} = \theta_l - \theta_m\cos\theta.$$

If the straight line OM be not fixed relatively to the axes, then (λ, μ, ν) will be variable and we must add to the right-hand side of the second equation the determinant

$$\left(\lambda\frac{d\mu}{dt} - \mu\frac{d\lambda}{dt}\right)r + \left(\mu\frac{d\nu}{dt} - \nu\frac{d\mu}{dt}\right)p + \left(\nu\frac{d\lambda}{dt} - \lambda\frac{d\nu}{dt}\right)q.$$

The mode of proof may be indicated as follows. Let P be a point in OM at a distance unity from O and let P move about with OM. The moment of its velocity about OL is $\sin^2\theta\,\frac{d\psi}{dt}$. But if (x, y, z) be the co-ordinates of P, its velocities parallel to the axes are given by Art. 244, and the moments of these velocities about the axes will be $L = yw - zv$, $M = zu - xw$, $N = xv - yu$. Hence the moment about OL will be

$$\sin^2\theta\,\frac{d\psi}{dt} = Lp + Mq + Nr.$$

If we effect these substitutions, and since OM is unity, replace x, y, z by λ, μ, ν, we get the results in the example.

250. *To explain a method of changing from fixed to moving axes.*

If a body be moving about a fixed point and we have established any *general* proposition referring its motion to fixed axes meeting at the fixed point, then we may use the following method to infer the corresponding proposition referring the motion to axes moving in any proposed manner about the origin. Suppose the general equation established to be

$$\psi\left\{\omega_x, \frac{d\omega_x}{dt}, \text{\&c.} \ldots\right\} = 0,$$

where ω_x, ω_y, ω_z are the angular velocities about the fixed axes. Let ω_1, ω_2, ω_3 be the angular velocities of the body about the moving axes and let the motion of the axes be defined by the angular velocities θ_1, θ_2, θ_3 about themselves.

The fixed axes being arbitrary in position, let them be so chosen that, at the moment under consideration, the three moving axes are passing through them, so that the two sets are for an instant coincident. Then, by referring to Art. 243, we see that we may write $\omega_x = \omega_1$, $\omega_y = \omega_2$, $\omega_z = \omega_3$, but we cannot assert $\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}$, for the moving axes at the time $t + dt$ are not coincident with the fixed axes.

To determine the relation between $\frac{d\omega_x}{dt}$ and $\frac{d\omega_1}{dt}$ we may proceed thus. Let OL be any straight line *fixed in space* making with the moving axes the angles α, β, γ. Let Ω be the angular velocity of the body about this straight line. Then as in Art. 229

$$\Omega = \omega_1 \cos\alpha + \omega_2 \cos\beta + \omega_3 \cos\gamma,$$

$$\therefore\ \frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \frac{d\omega_2}{dt}\cos\beta + \frac{d\omega_3}{dt}\cos\gamma$$

$$- \omega_1 \sin\alpha\,\frac{d\alpha}{dt} - \omega_2 \sin\beta\,\frac{d\beta}{dt} - \omega_3 \sin\gamma\,\frac{d\gamma}{dt}.$$

Since OL is any fixed line in space, let it be so chosen that the moving axis of z coincides with it at the time t. Then $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, and $\gamma = 0$, also $\frac{d\Omega}{dt}$ will be $\frac{d\omega_z}{dt}$. Since α is the angle OL makes with the moving axis of x, $\frac{d\alpha}{dt}$ is the rate at which the axis of x is separating from a fixed straight line coincident with the axis of z and this is clearly θ_2. Similarly $\frac{d\beta}{dt} = -\theta_1$, hence $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1$.

Similarly

$$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2,$$

$$\frac{d\omega_y}{dt} = \frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3.$$

If we substitute these expressions in the given general equation

$$\psi\left\{\omega_x, \frac{d\omega_x}{dt}, \ldots\right\} = 0,$$

we shall have the corresponding equation referred to moving axes.

If the moving axes be fixed in the body, and move with it, we have $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, $\theta_3 = \omega_3$. In this case the relations will become $\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}$, $\frac{d\omega_y}{dt} = \frac{d\omega_2}{dt}$, $\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}$, as in Art. 229.

The preceding proof of the relation between $\frac{d\omega_3}{dt}$ and $\frac{d\omega_z}{dt}$ is a simple corollary from the parallelogram of angular velocities. The result will therefore be true for any other magnitude which obeys the "parallelogram law." In fact the demonstration is exactly the same. Now linear velocities and linear accelerations do obey this law. Hence the expressions obtained in Arts. 244, 245, for the velocities (u, v, w) and the accelerations (X, Y, Z) may be deduced from the one proved above.

If the general equation $\psi = 0$ should contain the velocity or acceleration of any particle of the body, then to obtain the corresponding equation referred to moving axes, we must substitute for these velocities or accelerations the expressions found in Arts. 244 and 245.

251. If the general equation should contain $\frac{d^2\omega_z}{dt^2}$ or any other second differential coefficients, the expressions to be substituted for them become more complicated.

Since $\frac{d\omega_x}{dt}$, $\frac{d\omega_y}{dt}$, $\frac{d\omega_z}{dt}$, being angular accelerations, follow the parallelogram law, we have

$$\frac{d\Omega}{dt} = \left(\frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2\right)\cos\alpha + \left(\frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3\right)\cos\beta + \left(\frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1\right)\cos\gamma.$$

We may repeat the same reasoning and we shall finally obtain

$$\frac{d^2\omega_z}{dt^2} = \frac{d}{dt}\left\{\frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1\right\} - \theta_2\left\{\frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2\right\} + \theta_1\left\{\frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3\right\}.$$

So we may proceed to treat third and higher differential coefficients.

252. *A body is turning about a fixed point in any manner, to determine the moments of the effective forces about the axes.*

Let (x, y, z) be the co-ordinates of any particle m of the body referred to axes fixed in space and meeting at the fixed point, and let h_1, h_2, h_3 be the moments of the momentum about the axes.

The moment of the effective forces about the axis of z is

$$\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right),$$

and this may be written in the form $\frac{dh_3}{dt}$. Thus the moments of the effective forces about axes Ox, Oy, Oz fixed in space are respectively $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$, $\frac{dh_3}{dt}$, where h_1, h_2, h_3 have the values found in Art. 240.

Let h_1', h_2', h_3' be the moments of the momentum, found by Art. 242, about axes Ox', Oy', Oz' moving in space about the fixed origin. Let θ_1, θ_2, θ_3 be the angular velocities of these axes about their instantaneous directions. Then since moments or couples follow the parallelogram law, we see by the proposition of Art. 250 that the moments of the effective forces about the moving axes are respectively

$$\frac{dh_1'}{dt} - h_2'\theta_3 + h_3'\theta_2,$$

$$\frac{dh_2'}{dt} - h_3'\theta_1 + h_1'\theta_3,$$

$$\frac{dh_3'}{dt} - h_1'\theta_2 + h_2'\theta_1.$$

If the moving axes be *fixed in the body*, we have $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, $\theta_3 = \omega_3$, and the equations admit of some simplification. If the axes be the *principal axes* we have $h_1' = A\omega_1$, $h_2' = B\omega_2$,

$h_3' = C\omega_3$, and the moments of the effective forces take the simple forms

$$A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3,$$

$$B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1,$$

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2,$$

where A, B, C are the principal moments. See Art. 230.

If it be required to find the moment about the axis of z of the effective forces on a rigid body moving in any manner in space, we may use the principle proved in Chap. II. Art. 72.

In the case of a system of rigid bodies, the moment of their effective forces may be found by adding up the separate moments of the several bodies.

253. *To obtain the general equations of motion of a system of rigid bodies.*

These equations have been already obtained in Chap. II. Art. 83, when the system is referred to axes fixed in space. If the axes be moveable we must replace the accelerations $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{dw}{dt}$ by the corresponding forms in Art. 245 and the couples $\frac{dh_1}{dt}$, $\frac{dh_2}{dt}$, $\frac{dh_3}{dt}$ by the expressions in Art. 252.

Thus, suppose we refer the motion to three axes moving in space about a fixed origin O. Let X, Y, Z be the impressed forces on any rigid body of the system, including the unknown reactions of the other bodies of the system. Let L, M, N be the moments of these forces about axes drawn through the centre of gravity of the body parallel to the co-ordinate axes. Let m be the mass of the body. Then if we adopt the notation of Arts. 245 and 252, the equations of motion for the rigid body under consideration will be

$$\left.\begin{aligned} \frac{du}{dt} - v\theta_3 + w\theta_2 &= \frac{X}{m}, \\ \frac{dv}{dt} - w\theta_1 + u\theta_3 &= \frac{Y}{m}, \\ \frac{dw}{dt} - u\theta_2 + v\theta_1 &= \frac{Z}{m}, \end{aligned}\right\}$$

and
$$\left.\begin{aligned} \frac{dh_1'}{dt} - h_2'\theta_3 + h_3'\theta_2 &= L, \\ \frac{dh_2'}{dt} - h_3'\theta_1 + h_1'\theta_3 &= M, \\ \frac{dh_3'}{dt} - h_1'\theta_2 + h_2'\theta_1 &= N, \end{aligned}\right\}$$

where h_1', h_2', h_3' have the values given in Art. 240*.

Similar equations will apply for each body of the system.

Besides these dynamical equations there will be the geometrical equations expressing the connections of the system. As every such forced connection is accompanied by some reaction, the number of geometrical equations will be the same as the number of unknown reactions in the system.

Thus we have sufficient equations to determine the motion.

254. If two of the principal moments at the fixed origin O are equal, it is often convenient to choose as axis of z' the axis OC of unequal moment, and as axes of x', y' two other axes OA, OB moving in any manner round OC. Let χ be the angle the plane of AOC makes with some plane fixed in the body and passing through OC. Then we have $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, and $\theta_3 = \omega_3 + \dfrac{d\chi}{dt}$. Also by Art. 241, we have $h_1' = A\omega_1$, $h_2' = A\omega_2$, $h_3' = C\omega_3$. The equations of motion of Art. 253 now become

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt}\right) - (A - C)\,\omega_2\omega_3 &= L, \\ A\left(\frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt}\right) + (A - C)\,\omega_3\omega_1 &= M, \\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad\qquad &= N. \end{aligned}\right\}$$

In this case the most convenient geometrical equations to express the relations of these moving axes to straight lines fixed in space will be those given in Art. 235.

Since $\dfrac{d\chi}{dt}$ is arbitrary, it may be chosen to simplify either the dynamical or the geometrical equations.

* The equations of Art. 253 were first given in *this* form by Prof. Slesser to whom the equations of Art. 254 had been shown by the author. It appears however that similar results had been previously published in *Liouville's Journal* in 1858.

First, we may put $\frac{d\chi}{dt} = -\omega_3$. The dynamical equations then become

$$A\frac{d\omega_1}{dt} + C\omega_2\omega_3 = L,$$

$$A\frac{d\omega_2}{dt} - C\omega_1\omega_3 = M,$$

$$C\frac{d\omega_3}{dt} \qquad = N.$$

Secondly, we may so choose $\frac{d\chi}{dt}$ that $\phi = 0$. In this case the plane COA always passes through a straight line OZ fixed in space. The geometrical equations then become,

$$\frac{d\theta}{dt} = \omega_2, \qquad \frac{d\psi}{dt}\sin\theta = -\omega_1, \qquad -\frac{d\chi}{dt} + \frac{d\psi}{dt}\cos\theta = \omega_3.$$

255. If three principal moments at the fixed origin O be equal, there are three sets of axes such that when the motion is referred to them, the equations take a simple form.

First. We may choose axes fixed in space. Since every axis is a principal axis in the body, the general equations of motion become

$$\frac{d\omega_1}{dt} = \frac{L}{A}, \quad \frac{d\omega_2}{dt} = \frac{M}{A}, \quad \frac{d\omega_3}{dt} = \frac{N}{A}.$$

The geometrical equations of Art. 235 are not required.

Secondly. We may choose one axis as that of OC fixed in space and let the other two move round it in any manner, then as in Art. 254, the equations of motion become

$$\left.\begin{aligned} \frac{d\omega_1}{dt} - \omega_2\frac{d\chi}{dt} &= \frac{L}{A} \\ \frac{d\omega_2}{dt} + \omega_1\frac{d\chi}{dt} &= \frac{M}{A} \\ \frac{d\omega_3}{dt} \qquad &= \frac{N}{A} \end{aligned}\right\}.$$

Thirdly. We can take as axes any three straight lines at right angles moving in space in any proposed manner. The equations of motion are then by Art. 253

$$\frac{d\omega_1}{dt} - \omega_2\theta_3 + \omega_3\theta_2 = \frac{L}{A},$$

$$\frac{d\omega_2}{dt} - \omega_3\theta_1 + \omega_1\theta_3 = \frac{M}{A},$$

$$\frac{d\omega_3}{dt} - \omega_1\theta_2 + \omega_2\theta_1 = \frac{N}{A}.$$

The geometrical equations will then be the same as those given in Art. 235 or Art. 247.

256. Ex. An ellipsoid, whose centre O is fixed, contracts by cooling and being set in motion in any manner is under the action of no forces. Find the motion.

The principal diameters are principal axes at O throughout the motion. Let us take them as axes of reference. The expressions for the angular momenta about the axes are by Art. 241 $h_1' = A\omega_1$, $h_2' = B\omega_2$, $h_3' = C\omega_3$. The equations of Art. 253 then become

$$\left.\begin{aligned} \frac{d}{dt}(A\omega_1) - (B-C)\,\omega_2\omega_3 = 0 \\ \frac{d}{dt}(B\omega_2) - (C-A)\,\omega_3\omega_1 = 0 \\ \frac{d}{dt}(C\omega_3) - (A-B)\,\omega_1\omega_2 = 0 \end{aligned}\right\}.$$

Multiplying these equations by $A\omega_1$, $B\omega_2$, $C\omega_3$, adding and integrating we see that $A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$ is constant throughout the motion. To obtain another integral, let $A = A_0 f(t)$, $B = B_0 f(t)$, $C = C_0 f(t)$ where $f(t)$ expresses the law of cooling which has been supposed such that the body changes its form very slowly. Let $\omega_1 f(t) = \Omega_1$, $\omega_2 f(t) = \Omega_2$, $\omega_3 f(t) = \Omega_3$, and put $\dfrac{dt'}{dt} = \dfrac{1}{f(t)}$, then the equations become

$$A_0 \frac{d\Omega_1}{dt'} - (B_0 - C_0)\,\Omega_2\Omega_3 = 0,$$

and two similar equations. These may be treated as in the chapter on the motion of a body under no forces. *Liouville's Journal.*

257. The theory of relative motion is best understood by viewing it in as many aspects as possible. We shall, therefore, now consider a method of determining the motion which is more elementary, and does not, in the result, make an exclusive use of Cartesian co-ordinates.

Let it be required to refer the motion of a particle P to any given system of moving axes. The motion of these axes during any interval of time dt may be constructed by a screw-motion along and round some straight line OI. Let Udt be the translation along and Ωdt the rotation round OI. Let P_0 be the position of P at the time t, and let P_0 be attached to the given axes and move with them during the interval dt. Let f represent the acceleration of P_0 in direction and magnitude. The particle P will, of course, separate from P_0; but, as is explained in Dynamics of a Particle, the actual acceleration of P in space is the resultant of its acceleration relative to P_0 treated as a fixed point, and the acceleration f of P_0.

To find the acceleration relative to P_0, we must treat P_0 as a fixed point. Draw P_0z' parallel to OI and let P_0y' be the projection of the direction of the relative motion of P on a plane perpendicular to P_0z', and let P_0x' be perpendicular to P_0y' and P_0z'. These axes are taken for the purposes of description, and but little

use will be made of co-ordinates. Let these axes move during the time dt, so as to preserve unchanged the angles they make with the given axes of reference. Let P_0P_1 be the displacement of P relative to P_0, and let P_0P_1 make an angle θ with P_0z', so that $P_0P_1 \sin\theta$ is the projection of the relative displacement on the plane of $x'y'$. Since these axes, in the interval of time dt, have turned round P_0z' through an angle Ωdt, the x co-ordinate of P, after that interval, is greater than what it would have been if referred to axes fixed in space by $P_0P_1 \sin\theta\Omega dt$, while the y and z co-ordinates are unaltered. We have here, according to the rules of the Differential Calculus, retained only the lowest powers of the small quantities which occur. Hence, if the acceleration of P relative to these axes be compounded with an acceleration equal and *opposite* to that which would produce a displacement $P_0P_1 \sin\theta\Omega dt$, we shall have the acceleration of P relative to axes whose directions are fixed in space, but having the moving point P_0 as origin. Let V be the velocity of the particle relative to the moving axes, then $P_0P_1 = Vdt$ in the limit, and therefore the change $\delta x'$ in the x co-ordinate of P is $\delta x' = V\Omega \sin\theta\,(dt)^2$. If f' be the acceleration corresponding to this displacement, we have $\delta x' = \frac{1}{2} f'(dt)^2$. Comparing these two expressions we see that $f' = 2V\Omega \sin\theta$. This acceleration must be supposed to act along the positive direction of the axis of x'.

The general conclusion is that the acceleration of P in space is the resultant of the accelerations f, $-f'$, and the acceleration relative to the given moving axes.

The equations of motion of a particle being comprised in the formula, "acceleration in any fixed direction equals the impressed force divided by the mass," it is more convenient to transpose the terms f and $-f'$ to the other side of the equation with opposite signs, we then have the following theorem:

In finding the motion of a particle of mass m *with reference to any moving axes, we may treat the axes as if they were fixed in space, provided we regard the particle as acted on, in addition to the impressed forces, by two forces:*

(1) *a force equal and opposite to that which would constrain the particle to remain fixed to the moving axes, and which is measured by* mf *where* f *is the reversed acceleration of the point of moving space occupied by the particle,*

(2) *a force perpendicular to both the direction of relative motion of the particle and to the central axis or axis of rotation of the moving axes, and which is measured by* $2\mathrm{m}\mathrm{V}\Omega\sin\theta$, *where* V *is the relative velocity of the particle,* Ω *the resultant angular velocity of the moving axes, and* θ *the angle between the direction of the velocity and the axis of rotation.*

To find the direction of this last force, we notice that in the investigation, the rotation Ω has been supposed to be, as usual, from the positive direction of x' to the positive direction of y', and that the positive direction of y' is a tangent to the projection of the relative velocity of P. Since the force acts along the positive direction of x', we have this rule: *Stand with the back along the axis of rotation, so that the rotation appears to be in the direction of the hands of the watch; then viewing the particle receding from the axis of rotation, the force acts on the left hand.* We may call these forces respectively the *force of moving space*, and the *compound centrifugal force of the particle.*

258. This method of determining the relative motion of a particle was first given by Clairaut in 1742, and afterwards the same rule was demonstrated in a different manner by Coriolis. The arguments of the former were criticized and improved by M. Bertrand in a paper published in the nineteenth volume of the *Journal Polytechnique.* We have here followed, with but slight variations, M. Bertrand's mode of proof, as being the most different of any from the analytical methods given in this chapter. But it will be important to perceive the connection between the two methods of expressing the relative motion, and this will be explained in the next article.

259. Let us refer the motion of P to any moving axes having a fixed origin, and let X, Y, Z be the impressed forces on the particle resolved parallel to the axes. If we eliminate u, v, w from the equations of Art. 244 and Art. 245 we get

$$\frac{X}{m}=\frac{d^2x}{dt^2}-2\frac{dy}{dt}\dot{\theta}_3+2\frac{dz}{dt}\dot{\theta}_2+Ax+By+Cz,$$

with similar expressions for Y and Z. Here A, B, C are functions of θ_1, θ_2, θ_3 and their differential coefficients with regard to t, which it is unnecessary to write down. If x, y, z were constants, all the terms of X would disappear except the three last. These then with the corresponding terms in Y and Z express the acceleration of a point P_0 rigidly attached to the axes, but occupying the instantaneous position of P. The second and third terms of X taken together, with the corresponding terms of Y and Z, express the resolved parts of an acceleration perpendicular both to the resultant axis of the rotations θ_1, θ_2, θ_3, and to the direction of the velocity which is the resultant of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. By adding up the squares we easily find the magnitude of the resultant acceleration to be $2\Omega V \sin\theta$, where Ω, V and θ have the meaning given in Art. 257*.

* Another demonstration by the use of polar co-ordinates is given in Vol. XII. of the *Quarterly Journal of Mathematics*, by the Rev. H. W. Watson.

To determine the manner in which these forces should be applied, we must transpose the terms which represent them to the other sides of the equations. The first equation will then become

$$m\frac{d^2x}{dt^2} = X + 2m\left(\frac{dy}{dt}\theta_3 - \frac{dz}{dt}\theta_2\right) - m(Ax + By + Cz),$$

and the other two will take similar forms. These are the equations of motion of a particle referred to fixed axes, moving under the same impressed forces as before, but with two additional forces. These are, first, a force equal and opposite to that represented by mf, where f is the acceleration of the point of moving space occupied by the particle; and secondly, a force whose magnitude has been shown to be $2mV\Omega \sin\theta$. To determine the direction of this force, let the axis of z be taken along the instantaneous axis of rotation of the moving space, and let the plane of yz be parallel to the direction of motion of the particle, then $\theta_1 = 0$, $\theta_2 = 0$ and $\frac{dx}{dt} = 0$. We then easily see that this force disappears from the equations giving $m\frac{d^2y}{dt^2}$ and $m\frac{d^2z}{dt^2}$; while in that giving $m\frac{d^2x}{dt^2}$, we have the single term $2m\frac{dy}{dt}\theta_3$. The magnitude of this force is obviously $2mV\Omega\sin\theta$, and it acts along the positive direction of the axis of x. This is the left-hand side when the receding particle is viewed from the axis of rotation and the rule given at the end of Art. 257 is therefore established.

When these equations have been integrated, the arbitrary constants are to be determined from the initial values of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. These differential coefficients are clearly the components of the initial velocity of the particle, taken relatively to the moving axes.

260. Ex. If the particle be constrained to move along a curve which is itself moving in any manner, the compound centrifugal force, being perpendicular to the direction of the relative velocity of the particle, may be included in the reaction of the curve. The only force which it is necessary to impress on the particle is the force of the moving space. If the curve be turning about a fixed axis with an angular velocity Ω in the manner described in Art. 181, the components of the accelerating force of moving space are clearly $\Omega^2 r$ tending directly from the axis of rotation, and $\frac{d\Omega}{dt}r$ perpendicular to the plane containing the particle and the axis. Here r, as in the article referred to, is the distance of the particle from the axis.

261. In finding the compound centrifugal force it will be found useful to remember, that we may resolve the angular velo-

city Ω or the linear velocity V in any manner that we please, and find the forces due to each of the components separately. Though we have thus more than two forces which must be applied to the particle, yet, by making a proper resolution, some of these may produce either no effect, and may therefore be omitted, or may produce an effect which it may be easy to take account of.

262. When we wish to determine the motion of a rigid body by this method, we must consider each particle to be acted on by the two forces corresponding to the position and velocity of that particle. This will generally require an integration to be performed; which, though not difficult, is not always convenient.

The forces of moving space for any body are the same as the effective forces of an imaginary body occupying the instantaneous position of the real body, and moving with the space occupied by it. The resultant of these forces may, therefore, be found by the method indicated in Art. 83.

The components of the compound centrifugal forces on any particle are, by Art. 259, algebraic functions of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. We may, therefore, use Art. 14 to help us in finding the resultants of the compound centrifugal forces of the whole body. If M be the mass of the body, V the velocity of its centre of gravity, Ω the angular velocity of the moving space, θ the angle between the direction of V and the axis of Ω, then the compound centrifugal forces of all the particles of the body are equivalent to a force $2MV\Omega \sin\theta$ acting at the centre of gravity perpendicular both to its direction of motion and the axis of Ω, together with the compound centrifugal forces of the body after the centre of gravity has been reduced to rest.

To find these latter forces, let us refer the body to the principal axes at the centre of gravity as axes of co-ordinates. Let $\omega_1, \omega_2, \omega_3$ be the resolved angular velocities of the body, $\Omega_1, \Omega_2, \Omega_3$ the resolved parts of Ω about these axes; A, B, C the principal moments of inertia at the centre of gravity. Then, by Art. 259, the compound centrifugal forces on any particle of the body whose co-ordinates are (x, y, z) and mass m, are

$$X = m\left\{2\frac{dy}{dt}\Omega_3 - 2\frac{dz}{dt}\Omega_2\right\},$$

with similar expressions for Y and Z. The centre of gravity being at the origin, the *resultant forces* of these are easily seen by integration to be all zero, while the resultant couples about the axes are

$$L = \omega_2\Omega_3(A + B - C) - \omega_3\Omega_2(A + C - B),$$

with similar expressions for M and N.

263. Ex. 1. A disc of mass M is constrained to move in a plane under any forces while the plane turns about a straight line parallel to the plane and distant a from it with angular velocity Ω. Show that in finding the motion of the disc, we may regard the plane as fixed, provided we impress on the disc in addition to the given forces, (1) a force $M\Omega^2 r - Ma\frac{d\Omega}{dt}$ acting through the centre of gravity tending directly from the projection of the axis of rotation on the plane, where r is the distance of the centre of gravity from the projection, (2) a couple $F\Omega^2$ where F is the product of inertia about two rectangular axes in the plane intersecting at the centre of gravity, and respectively parallel to the axis and perpendicular to it. The constants of integration are to be determined from the initial conditions taken relatively to the moving plane.

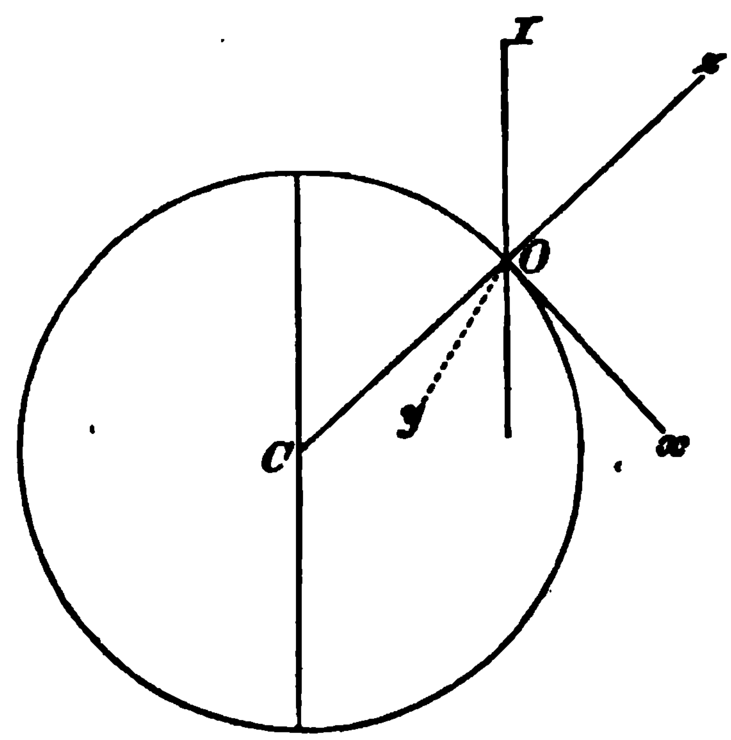
Ex. 2. A disc of mass M is constrained to move in a plane under any forces while the plane turns with angular velocity Ω about a straight line perpendicular to its plane and cutting the plane in the point O. Show that we may regard the plane as fixed provided we impress on the disc (1) a force $M\Omega^2 r$ acting at the centre of gravity and tending directly from the axis, where r is the distance of the centre of gravity from the axis, (2) a force $Mr\frac{d\Omega}{dt}$ acting at the centre of gravity perpendicular to r in the direction opposite to the rotation, (3) a couple $Mk^2\frac{d\Omega}{dt}$, where Mk^2 is the moment of inertia of the disc about an axis through its centre of gravity perpendicular to its plane, (4) a force $2MV\Omega$ acting at the centre of gravity perpendicular to its direction of motion, where V is the velocity of the centre of gravity.

Ex. 3. A sphere of mass M moves in space, show that the compound centrifugal forces of all its elements are equal to (1) a resultant force $2MV\Omega\sin\theta$ acting at the centre of gravity, where V is the velocity of the centre of gravity and Ω the angular velocity of the moving space and θ the angle the direction of V makes with the axis of Ω, (2) a couple $Mk^2\Omega\omega\sin\phi$, where ω is the angular velocity of the sphere, ϕ the angle its instantaneous axis makes with the axis of Ω, and the plane of the couple is parallel to the axes of Ω and ω.

On Motion relative to the Earth.

264. The motion of a body on the surface of the earth is not exactly the same as if the earth were at rest. As an illustration of the use of the equations of this chapter, we shall proceed to determine the equations of motion of a particle referred to axes of co-ordinates fixed in the earth and moving with it.

Let O be any point on the surface of the earth whose latitude is λ. Thus λ is the angle the normal to the surface of still water at O makes with the plane of the equator. Let the axis of z be vertical at O and measured positively in the direction opposite to gravity. Let the axes of x and y be respectively a tangent to the meridian and a perpendicular to it, their positive directions being respectively south and west. In the figure the axis of y is dotted



to indicate that it is perpendicular to the plane of the paper. Let ω be the angular velocity of the earth, b the distance of the point O from the axis of rotation.

We may reduce the point O to rest by applying to every point under consideration an acceleration equal and opposite to that of O, and therefore equal to $\omega^2 b$ and tending from the axis of rotation. We must also apply a velocity equal and opposite to the initial velocity of O. This velocity is ωb. The whole figure will then be turning about an axis OI, parallel to the axis of rotation of the earth with an angular velocity ω.

When the particle has been projected from the earth it is acted on by the attraction of the earth and the applied acceleration $\omega^2 b$. The attraction of the earth is not what we call gravity. Gravity is the resultant of the attraction of the earth and the centrifugal force, and the earth is of such a form that this resultant acts perpendicular to the surface of still water. If it were not so, particles resting on the earth would tend to slide along the surface. It appears, therefore, that the force on the particle, *after* O *has been reduced to rest*, is equal to gravity. Let this be represented by g. Besides this there may be other forces on the particle, let their resolved parts parallel to the axes be X, Y, Z.

Since the earth is turning round OI with angular velocity ω, the resolved part about Oz is $\omega \sin \lambda$, since the angle IOz is the complement of ω; since the rotation is from west to east, the resolved angular velocity is from y to x, which is the negative direction, hence $\theta_3 = -\omega \sin \lambda$. The resolved angular velocity round Ox is $\omega \cos \lambda$ and is from y to z, which is the positive direction, hence $\theta_1 = \omega \cos \lambda$. Also since OI is perpendicular to Oy, $\theta_2 = 0$. Hence, by Art. 244, the actual velocities of any particle whose co-ordinates are (x, y, z), are

$$\left.\begin{aligned} u &= \frac{dx}{dt} + \omega \sin\lambda\, y \\ v &= \frac{dy}{dt} - \omega \cos\lambda\, z - \omega \sin\lambda\, x \\ w &= \frac{dz}{dt} + \omega \cos\lambda\, y \end{aligned}\right\}.$$

To find the equations of motion it is only necessary to substitute these in the equations of Art. 245.

The resulting equations may be simplified if we neglect such small quantities as the difference between the force of gravity at different heights. If a be the equatorial radius of the earth and g' the force of gravity at a height z, we have $g' = g\left(1 - \frac{2z}{a}\right)$ nearly. Now $\omega^2 a$ is the centrifugal force at the equator, which is known to be $\frac{1}{289} g$. Hence if we neglect the small term $g\frac{z}{a}$ we must also neglect $\omega^2 z$. The equations will therefore become

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + 2\omega \sin\lambda \frac{dy}{dt} = X \\ &\frac{d^2y}{dt^2} - 2\omega \cos\lambda \frac{dz}{dt} - 2\omega \sin\lambda \frac{dx}{dt} = Y \\ &\frac{d^2z}{dt^2} + 2\omega \cos\lambda \frac{dy}{dt} = -g + Z \end{aligned}\right\},$$

where the terms (X, Y, Z) include all the accelerating forces, except gravity, which act on the particle. These equations agree with those given by Poisson, *Journal Polytechnique*, 1838.

265. If we do not neglect the term containing ω, the equations of motion are

$$\frac{d^2x}{dt^2} + 2\omega \sin\lambda \frac{dy}{dt} - \omega^2 \sin^2\lambda\, x - \omega^2 \sin\lambda \cos\lambda\, z = X,$$

$$\frac{d^2y}{dt^2} - 2\omega \cos\lambda \frac{dz}{dt} - 2\omega \sin\lambda \frac{dx}{dt} - \omega^2 y = Y,$$

$$\frac{d^2z}{dt^2} + 2\omega \cos\lambda \frac{dy}{dt} - \omega^2 \cos^2\lambda\, z - \omega^2 \sin\lambda \cos\lambda\, x = -g + Z.$$

266. As an example, let us consider the case of a particle dropped from a height h. The initial conditions are therefore x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ all zero, and $z = h$. As a first approximation, neglect all the terms containing the small factor ω. Then we have $x = 0$, $y = 0$, $z = h - \frac{1}{2}gt^2$.

For a second approximation, we may substitute these values of (x, y, z) in the small terms. We have after integration

$$x=0, \; y=-\omega \cos \lambda g \frac{t^3}{3}, \; z=h-\frac{1}{2}gt^2.$$

Thus there will be a small deviation towards the east, proportional to the cube of the time of descent. There will be no southerly deviation, and the vertical motion will be the same as if the earth were at rest.

An elementary demonstration of this result will make the whole argument clearer. Let the particle be dropped from a height h vertically over O. Then O being reduced to rest, the particle is really projected eastwards with a velocity $\omega h \cos \lambda$. Hence, if the direction of gravity did not alter owing to the rotation of the earth about OI, the particle would describe a parabola and the easterly deviation would be $(\omega h \cos \lambda)\, t$ where t is the time of falling. Since $h=\frac{1}{2}gt^2$, this deviation is $\omega \cos \lambda g \dfrac{t^3}{2}$. The rotation ω about OI is equivalent to $\omega \sin \lambda$ about Oz and $\omega \cos \lambda$ about Ox. The former does not alter the position of OC the normal to the surface of the earth, which is the direction of gravity. The latter turns OC in any time t through an angle $\omega \cos \lambda t$. Thus gravity gradually changes its direction as the particle falls. The particle is therefore acted on by a westerly component $=g \sin (\omega \cos \lambda t)$, which, since ωt is small, is nearly equal to $g\omega \cos \lambda t$. Let y' be the distance of the particle from the position of the plane xz in space at the moment when the particle began to fall, and let y' be measured positively to the west. The equation of motion of the particle in space is therefore

$$\frac{d^2 y'}{dt^2}=g\omega t \cos \lambda.$$

Integrating this and remembering that as explained above $\dfrac{dy'}{dt}=-\omega h \cos \lambda$ when $t=0$, we get

$$y'=-\omega h t \cos \lambda+\tfrac{1}{6} g\omega t^3 \cos \lambda.$$

When the particle reaches the ground we have $y'=y$ very nearly and $h=\frac{1}{2}gt^2$, thus the deviation westwards is $-\omega g \dfrac{t^3}{3} \cos \lambda$, which is the same as before. If it be not evident that $y'=y$, it may be shown thus. In the time t Oy, Oz have turned through a very small angle $\theta=\omega \cos \lambda t$, hence, as in transformation of axes,

$$y'=y \cos \theta - z \sin \theta,$$

which gives $y'=y$ when we reject the squares of θ.

267. In many cases it will be found convenient to refer the motion to axes more generally placed. Let O be the origin, and let the axes be fixed relatively to the earth, but in any directions at right angles to each other. Let θ_1, θ_2, θ_3 be the resolved parts of ω about these axes, then θ_1, θ_2, θ_3 are known constants. After substituting from Art. 244 in the equations of motion given in Art. 245 we get

$$\frac{d^2x}{dt^2}-2\frac{dy}{dt}\theta_3+2\frac{dz}{dt}\theta_2=X,$$

$$\frac{d^2y}{dt^2}-2\frac{dz}{dt}\theta_1+2\frac{dx}{dt}\theta_3=Y,$$

$$\frac{d^2z}{dt^2} - 2\frac{dx}{dt}\theta_2 + 2\frac{dy}{dt}\theta_1 = -g + Z.$$

For example, if we wished to determine the motion of a projectile, it will be convenient to take the axis of z vertical and the plane of xz to be the plane of projection. Let the axis of x make an angle β with the meridian, the angle being measured from the south towards the west. Then

$$\theta_1 = \omega\cos\lambda\cos\beta,\quad \theta_2 = -\omega\cos\lambda\sin\beta,\quad \theta_3 = -\omega\sin\lambda.$$

These equations may be solved in any particular case by the method of continued approximation. If we neglect the small terms we get a first approximation to the values of (x, y, z). To find a second approximation we may substitute these values in the terms containing ω and integrate the resulting equations. As these equations are only true on the supposition that ω^2 may be neglected, we cannot proceed to a third approximation.

268. Ex. 1. A particle is projected with a velocity V in a direction making an angle α with the horizontal plane, and such that the vertical plane through the direction of projection makes an angle β with the plane of the meridian, the angle β being measured from the south towards the west. If x be measured horizontally in the plane of projection, y be measured horizontally in a direction making an angle $\beta + \frac{\pi}{2}$ with the meridian, and z vertically upwards from the point of projection, prove that

$$x = V\cos\alpha t + \left(V\sin\alpha t^2 - \frac{1}{3}gt^3\right)\omega\cos\lambda\sin\beta,$$

$$y = \left(V\sin\alpha t^2 - \frac{1}{3}gt^3\right)\omega\cos\lambda\cos\beta + V\cos\alpha t^2\omega\sin\lambda,$$

$$z = V\sin\alpha t - \frac{1}{2}gt^2 - V\cos\alpha t^2\omega\cos\lambda\sin\beta,$$

where λ is the latitude of the place, and ω the angular velocity of the earth about its axis of figure.

Show also that the increase of range on the horizontal plane through the point of projection is

$$4\omega\frac{V^3}{g^2}\sin\beta\cos\lambda\sin\alpha\left(\frac{1}{3}\sin^2\alpha - \cos^2\alpha\right),$$

and the deviation to the right of the plane of projection is

$$4\omega\frac{V^3}{g^2}\sin^2\alpha\,(\cos\lambda\cos\beta\,\frac{\sin\alpha}{3} + \sin\lambda\cos\alpha).$$

Ex. 2. A bullet is projected from a gun nearly horizontally with great velocity so that the trajectory is nearly flat, prove that the deviation is nearly equal to $Rt\omega\sin\lambda$, where R is the range, and the other letters have the same meaning as in the last question. The deviation is always to the right of the plane of firing in the Northern hemisphere, and to the left in the Southern hemisphere. It is asserted (*Comptes Rendus*, 1866) that the deviation due to the earth's rotation as calculated by this formula is as much as half the actual deviation in Whitworth's gun.

Ex. 3. A spherical bullet is projected with so great a velocity that the resistance of the air must be taken into account. The resistance of the air being assumed to be $\frac{(\text{vel.})^2}{k}$, and the trajectory to be flat, prove that, neglecting the effects of the rotation of the earth,

$$x = k \log\left(1 + \frac{Vt}{k}\right)$$

$$y = \frac{2\omega \sin\lambda}{V} k^2 \left(e^{\frac{x}{k}} - \frac{x}{k} - 1\right),$$

$$z = x \tan\alpha - \frac{gk^2}{4V^2}\left(e^{2\frac{x}{k}} - 2\frac{x}{k} - 1\right) - \frac{2\omega \sin\beta \cos\lambda}{V} k^2 \left(e^{\frac{x}{k}} - \frac{x}{k} - 1\right).$$

These are given by Poisson, *Journal Polytechnique*, 1838.

269. Let us apply these equations to determine the effect of the rotation of the earth on the motion of a pendulum. In this as in some other cases, it will be found advantageous to refer the motion to axes not fixed in the earth but moving in some known manner. Let the axis of z be vertical as before and let the axes of x and y move slowly round the vertical with angular velocity $\omega \sin\lambda$ in the direction from the south towards the west. In this case we have

$$\theta_1 = \omega \cos\lambda \cos\beta, \qquad \theta_2 = -\omega \cos\lambda \sin\beta,$$

and
$$\theta_3 = -\omega \sin\lambda + \omega \sin\lambda = 0,$$

where β is the angle the axis of x makes with the tangent to the meridian, so that $\frac{d\beta}{dt} = \omega \sin\lambda$. If, as before, we neglect quantities which contain the square of ω as a factor, the terms which contain $\frac{d\theta_1}{dt}$ and $\frac{d\theta_2}{dt}$ must be omitted. Hence the required equations may be obtained from those of Art. 267, by putting $\theta_3 = 0$.

If m be the mass of the particle, l the length of the string, and T the tension; these equations are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\omega \cos\lambda \sin\beta \frac{dz}{dt} &= -\frac{T}{m}\frac{x}{l} \\ \frac{d^2y}{dt^2} - 2\omega \cos\lambda \cos\beta \frac{dz}{dt} &= -\frac{T}{m}\frac{y}{l} \\ \frac{d^2z}{dt^2} + 2\omega \cos\lambda \sin\beta \frac{dx}{dt} + 2\omega \cos\lambda \cos\beta \frac{dy}{dt} &= -g - \frac{T}{m}\frac{z}{l} \end{aligned}\right\},$$

the origin being taken at the point of suspension.

If the oscillation be sufficiently small z will differ from l by small quantities of the order α^2 where α is the semi-angle of oscillation. The last equation then shows that T differs from mg by quantities of the order $\omega\alpha$ at least. If then we neglect terms of the

order $\omega\alpha^2$ and α^3, we may put mg for T in the two first equations and neglect the terms containing $\omega \dfrac{dz}{dt}$. The equations of motion thus become the same as for a pendulum attached to a fixed point. The solutions of the equations are clearly

$$x = A \cos\left(\sqrt{\frac{g}{l}}\, t + C\right), \quad y = B \sin\left(\sqrt{\frac{g}{l}}\, t + D\right).$$

The small oscillations of a pendulum on the earth referred to axes turning round the vertical with angular velocity $\omega \sin \lambda$ are therefore the same as those of an imaginary pendulum suspended from an absolutely fixed point.

Let us then suppose the pendulum to be drawn aside so as to make with the vertical a small angle α and then let go. Relatively therefore to the axes moving round the vertical with angular velocity $\omega \sin \lambda$ we must suppose the particle to be projected with a velocity $l \sin \alpha \omega \sin \lambda$ perpendicular to the initial plane of displacement. We have then when $t = 0$, $x = l\alpha$, $y = 0$, $\dfrac{dx}{dt} = 0$, $\dfrac{dy}{dt} = l\alpha\omega \sin \lambda$. It is then easy to see that in the above values of x and y, C and D are both zero and that the particle describes an ellipse, the ratio of the axes being $\omega \sin l \sqrt{\dfrac{l}{g}}$. The effect of the rotation of the earth is to make this ellipse turn round the vertical with uniform angular velocity $\omega \sin \lambda$ in a direction from south to west. If the angle α be not so small that its square may be neglected, it is known by Dynamics of a particle that, independently of all considerations of the rotation of the earth, there will be a progression of the apsides of the ellipse. It is therefore necessary for the success of the experiment that the length l of the pendulum should be very great. This motion of the apsides depending on the magnitude of α is in the opposite direction to that caused by the rotation of the earth and cannot therefore be mistaken for it.

It also appears that the time of oscillation is unaffected by the rotation of the earth, provided the arc of oscillation be so small that the effects of forces whose magnitude contains the factor $\omega\alpha^2$ may be neglected.

270. In Chapter IV. we have considered the motion of a system of bodies constrained to remain in a fixed plane. Since no plane can be found which does not move with the earth, it is important to determine what effect the rotation of the earth will have on the motion of these bodies. Let us treat this as an example of the method of Coriolis given in Art. 257.

Let the plane make an angle α with the axis of the earth. Let a point O in this plane be on the surface of the earth and let it be reduced to rest. Then, as

proved in Art. 264, the moving bodies while in the neighbourhood of O are acted on by their weights in a direction normal to the surface of the earth. The earth is now turning round an axis through O parallel to the axis of figure with a constant angular velocity ω. Let this angular velocity be resolved into two, viz., $\omega \sin \alpha$ about an axis perpendicular to the plane and $\omega \cos \alpha$ about an axis in the plane. Now the square of ω is to be rejected, hence by the principle of the superposition of small motions, we may determine the whole effect of these two rotations by adding together the effects produced by each separately.

It is a known theorem that if a particle be constrained to move in a plane which turns round any axis in that plane with a constant angular velocity $\omega \cos \alpha$, the motion may be found by regarding the plane as fixed and impressing an acceleration $\omega^2 r \cos^2 \alpha$ on the particle, where r is the distance of the particle from the axis. This may be deduced, as in Art. 260, from the theorem of Coriolis. This impressed acceleration is to be neglected because it depends on the square of ω. The angular velocity $\omega \cos \alpha$ has therefore no sensible effect.

If the bodies be free to move in the plane, the effect of the rotation $\omega \sin \alpha$ is to turn the axes of reference round the normal to the plane drawn through the point O. If then we calculate the motion without regard to the rotation of the earth, taking the initial conditions relative to fixed space, the effect of the rotation of the earth may be allowed for by referring this motion to axes turning round the normal with angular velocity $\omega \sin \alpha$. For example, if the body be a heavy particle suspended by a long string from a point O fixed relatively to the earth, it is really constrained to move in a horizontal plane, and the reasoning given above shows that the plane of oscillation will appear to a spectator on the earth to revolve with angular velocity $\omega \sin \alpha$ round the vertical.

If the bodies be constrained to revolve with the plane, it will be required to find the motion relatively to that plane. We must therefore apply to each particle the force of moving space and the compound centrifugal force. If r be the distance of any particle of mass m from O, the former is $mr\omega^2 \sin^2 \alpha$. This is to be neglected because it depends on the square of ω. The latter is therefore the only force to be considered. By Art. 262, the compound centrifugal forces on all the particles of a body are equivalent to a force at the centre of gravity and three couples. In our case these couples are easily seen to be zero. For if the plane be taken as the plane of xy, we have $\Omega_1 = 0$, $\Omega_2 = 0$, $\omega_1 = 0$, $\omega_2 = 0$. Hence L, M, N are all zero. If, therefore, m be the mass of a body, V the relative velocity of its centre of gravity, the effect of the rotation of the earth may be found according to the rule given in Art. 257, by impressing on the body a force equal to $2mV\omega \sin \alpha$, acting at the centre of gravity, in the plane of motion and perpendicular to the direction of motion of the centre of gravity.

The ratio of this force to gravity for a particle moving 32 feet per second, is at most $\dfrac{4\pi}{24.60.60}$, which is less than a five thousandth. This is so small that, except under special circumstances, its effect will be imperceptible.

271. Ex. 1. In Foucault's experiment, a long pendulum is suspended from a point over the centre of a circular table, and the arc of oscillation is seen to pass from one diameter to another. Show that the arc of the circular rim of the table described by the plane of oscillation in one day is equal to the difference in length between two parallels of latitude one through the centre and the other through the

northern or southern extremity of the rim. This theorem is due to the late Prof. Young.

Ex. 2. A heavy particle is suspended from a *fixed* point of support by a string of length a. It performs elliptic oscillations whose major and minor semi-axes are b and c. If b and c be small compared with a, prove that the apses will advance, in each complete revolution of the particle, through an angle $\frac{3}{8}\frac{bc}{a^2}2\pi$ nearly. If b and c be not small compared with a but be very nearly equal, the apse will advance through an angle

$$\left(\frac{1}{\sqrt{1-\frac{3}{4}\sin^2\alpha}}-1\right)2\pi,$$

where $\sin\alpha=\frac{b}{a}$ in each complete revolution of the particle.

Ex. 3. A pendulum, at rest relatively to the earth, is started in any direction with a small angular velocity, show that the oscillations will take place in a vertical plane turning uniformly round the vertical so that the pendulum becomes vertical once in each half oscillation.

Ex. 4. Let θ be the angle a pendulum of length l makes with the vertical, and ϕ the angle the vertical plane containing the pendulum makes with a vertical plane which turns round the vertical with uniform angular velocity $\omega\sin\lambda$ in a direction from south to west. Prove that when terms depending on ω^2 are neglected the equations of motion become

$$\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\phi}{dt}\right)^2=\frac{2g}{l}\cos\theta+A,$$

$$\frac{d}{dt}\left(\sin^2\theta\frac{d\phi}{dt}\right)=2\sin^2\theta\cos(\phi+\beta)\,\omega\cos\lambda\frac{d\theta}{dt},$$

where A is an arbitrary constant, and the other letters have the meanings given to them in Art. 267. See M. Quet in *Liouville's Journal*, 1853.

These equations will be found convenient in treating the motion of a pendulum. They may be easily obtained by transforming those given in Art. 269 to polar coordinates.

Ex. 5. A semi-circular arch ACB is fixed with its plane vertical on a horizontal wheel at A and B, and may thus be moved with any degree of rapidity from one azimuth to another. A rider slides along the inner edge of the arch which is graduated and may be fixed at any degree marked thereon. A spiral spring by means of which a slow vibration is obtained with comparatively a short length is attached at one end to a pin in the axis of the semicircle so that the point of attachment may be in the axis of rotation and at the other end it is fixed to a similar pin in a parallel position fixed to the rider. The vertical semicircle is not placed in a diameter of the horizontal wheel but parallel to it at such a distance as not to interrupt the eye of the observer from the vertical plane passing through the diameter, and in which plane the wire in all its positions remains.

If the rider be placed at an angular distance θ from the highest point of the arch and the wire set in vibration in any plane, show that the plane of vibration of the wire will make a complete revolution relatively to the arch while the arch turns round $\sec\theta$ complete revolutions. This is best observed by fixing the eye on a line

in the same plane with the wire while walking round with the wheel during its rotation. This apparatus was devised by Sir C. Wheatstone to illustrate Foucault's mechanical proof of the rotation of the earth. *Proceedings of the Royal Society*, May 22, 1851.

272. Hitherto we have considered chiefly the motion of a single particle. The effect of the rotation of the earth on the motion of a rigid body will be more easily understood when the methods to be described in the following chapters have been read. If, for example, a body be set in rotation about its centre of gravity, it will not be difficult to determine its motion as viewed by a spectator on the earth, when we know its motion in space. It seems, therefore, sufficient here to consider the peculiarities which these problems present, and to seek illustrations which do not require any extended use of the equations of motion.

273. The effect of the rotation of the earth is in general so small compared with that of gravity, that it is necessary to fix the centre of gravity in order that the effects of the former may be perceptible. Even when this is done, the friction on the points of support and the other resistances, cannot be wholly done away with. If, however, the apparatus be made with care that these resistances should be small, the effects of the rotation of the earth may be made to accumulate, and after some time to become sufficiently great to be clearly perceptible.

If a body be placed at rest relatively to the earth and free to turn about its centre of gravity as a fixed point, it is actually in rotation about an axis parallel to the axis of the earth. Unless this axis be a principal axis, the body would not continue to rotate about it, and thus a change would take place in its state of motion. By referring to Euler's equations, we see that the change in the position of the axis of rotation is due to the terms $(A-B)\,\omega_1\omega_2$, $(B-C)\,\omega_2\omega_3$, $(C-A)\,\omega_3\omega_1$. The body having been placed apparently at rest, ω_1, ω_2, ω_3 are all small quantities of the same order as the angular velocity of the earth; these terms are, therefore, all of the order of the squares of small quantities. Whether they will be great enough to produce any visible effect or not will depend on their ratio to the frictional forces which could be called into play. But since these frictional forces are just sufficient to prevent any relative motion, these terms will in general be just cancelled by the frictional couples introduced into the right-hand sides of Euler's equations. The body will, therefore, continue at rest relatively to the earth.

In order that some visible effect may be produced, it is usual to impress on the body a very great angular velocity about some axis. If this be the axis of ω_3, the terms in Euler's equations, which are due to the centrifugal forces, and which contain ω_3 as a

factor, become greater than when ω_3 had no such initial value. The greater this initial angular velocity, the greater these terms will be, and the more visible we may expect their effects on the body to be.

If the angular velocity thus communicated to the body be sufficient to turn it only once in a second, it will be still $24 \times 60 \times 60$ times as great as the angular velocity of the earth. In these problems, therefore, we may regard the angular velocity of the earth as so small, compared with the existing angular velocities of the body, that the *square* of the ratio may be neglected.

As an example of the application of these principles, we have selected one case of Foucault's pendulum, which seems to admit of an elementary solution.

274. *The centre of gravity of a solid of revolution is fixed, while the axis of figure is constrained to remain in a plane fixed relatively to the earth. The solid being set in rotation about its axis of figure, it is required to find the motion.*

Let us refer the motion to moving axes. Let the centre of gravity be the origin, the plane of yz the plane fixed relatively to the earth. Let the axis of figure be the axis of z, and let it make an angle χ with the projection of the axis of rotation of the earth on the plane of yz. Let this projection, for the sake of brevity, be called the axis of χ. Let p be the angular velocity of the earth about its axis, α the angle the normal to the plane of yz makes with the axis of the earth. The motion of the moving axes is given by

$$\theta_1 = p\cos\alpha + \frac{d\chi}{dt}, \quad \theta_2 = p\sin\alpha\sin\chi, \quad \theta_3 = p\sin\alpha\cos\chi.$$

Let ω_1, ω_2, ω_3 be the angular velocities of the body about the moving axes; A, A, C the principal moments of inertia at the centre of gravity. Let R be the reaction by which the axis of figure is constrained to remain in the fixed plane, then R acts parallel to the axis of x. Let h be the distance of its point of application from the origin. The angular momenta about the axes are respectively

$$h_1 = A\omega_1, \qquad h_2 = A\omega_2, \qquad h_3 = C\omega_3.$$

Substituting in Art. 230, the equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - A\omega_2\theta_3 + C\omega_3\theta_2 &= 0 \\ A\frac{d\omega_2}{dt} - C\omega_3\theta_1 + A\omega_1\theta_3 &= Rh \\ C\frac{d\omega_3}{dt} - A\omega_1\theta_2 + A\omega_2\theta_1 &= 0 \end{aligned}\right\}.$$

in the same plane with the wire while walking round with the wheel during its rotation. This apparatus was devised by Sir C. Wheatstone to illustrate Foucault's mechanical proof of the rotation of the earth. *Proceedings of the Royal Society*, May 22, 1851.

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Let us refer the motion to moving axes. Let the centre of gravity be the origin, the plane of yz the plane fixed relatively to the earth. Let the axis of figure be the axis of z, and let it make an angle χ with the projection of the axis of rotation of the earth on the plane of yz. Let this projection, for the sake of brevity, be called the axis of χ. Let p be the angular velocity of the earth about its axis, α the angle the normal to the plane of yz makes with the axis of the earth. The motion of the moving axes is given by

$$\theta_1 = p\cos\alpha + \frac{d\chi}{dt}, \quad \theta_2 = p\sin\alpha\sin\chi, \quad \theta_3 = p\sin\alpha\cos\chi.$$

Let ω_1, ω_2, ω_3 be the angular velocities of the body about the moving axes; A, A, C the principal moments of inertia at the centre of gravity. Let R be the reaction by which the axis of figure is constrained to remain in the fixed plane, then R acts parallel to the axis of x. Let h be the distance of its point of application from the origin. The angular momenta about the axes are respectively

$$h_1 = A\omega_1, \qquad h_2 = A\omega_2, \qquad h_3 = C\omega_3.$$

Substituting in Art. 230, the equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - A\omega_2\theta_3 + C\omega_3\theta_2 &= 0 \\ A\frac{d\omega_2}{dt} - C\omega_3\theta_1 + A\omega_1\theta_3 &= Rh \\ C\frac{d\omega_3}{dt} - A\omega_1\theta_2 + A\omega_2\theta_1 &= 0 \end{aligned}\right\}.$$

Since the axis of z is fixed in the body, we see by Art. 243, that $\omega_1 = \theta_1$, $\omega_2 = \theta_2$. The last equation of motion, therefore, shows that ω_3 is constant. It should however be remembered that ω_3 is not the apparent angular velocity of the body as viewed by a spectator on the earth. If Ω_3 be the angular velocity relatively to the moving axes, we have by Art. 243, $\Omega_3 = \omega_3 - \theta_3$, so that

$$\Omega_3 + p \sin \alpha \cos \chi = \text{constant}.$$

Thus the body, if so small a difference could be perceived, would appear to rotate quicker the nearer its axis approached the projection of the axis of the earth's rotation on the fixed plane.

The first equation of motion after substitution for $\omega_1, \omega_2, \theta_2, \theta_3$, their values in terms of χ, becomes

$$A \frac{d^2\chi}{dt^2} - Ap^2 \sin^2 \alpha \sin \chi \cos \chi + Cnp \sin \alpha \sin \chi = 0,$$

where n has been written for ω_3.

The second term may be rejected as compared with the third, since it depends on the square of the small quantity p. We have, therefore,

$$\frac{d^2\chi}{dt^2} = -\frac{C}{A} np \sin \alpha \sin \chi.$$

By Art. 92, this is the equation of motion of a pendulum under the action of a force constant in magnitude, and whose direction is along the axis of χ, *i.e.* the projection of the axis of rotation of the earth on the fixed plane. The body being set in rotation about its axis of figure, we see that that axis will immediately begin to approach one extremity or the other of the axis of χ with a continually increasing angular velocity. When the axis of figure reaches the axis of χ, its angular velocity will begin to decrease, and it will come to rest when it makes an angle on the other side of the axis of χ equal to its initial value. The oscillation will then be repeated continually.

The axis of figure will oscillate about that extremity of the axis of χ, which, when χ is measured from it, makes the coefficient on the right-hand side of the last equation negative. This extremity is such, that when the axis of figure is passing through it, the rotation n of the body is in the same direction as the resolved rotation p of the earth.

275. If we compare bodies of different form, we see that the time of oscillation depends only on the ratio $\frac{C}{A}$. It is otherwise independent of the structure or form of the body. The greater this ratio the quicker will the oscillation be. For a solid of revolution, it appears from the definitions in Art. 4, that this

ratio is greatest when $\Sigma mz^2 = 0$. In this case the ratio is equal to 2, and the body is a circular disc or ring.

276. If we compare the different planes in which the axis may be constrained to remain, we see that the motion is the same for all planes making the same angle with the axis of the earth. It is therefore independent of the inclination of the plane to the horizon at the place of observation. The time of oscillation will be least, and the motion of the axis most perceptible when $\alpha = \frac{\pi}{2}$, *i.e.* when the plane is parallel to the axis of rotation of the earth. If the plane be perpendicular to the axis of the earth, the axis of figure will not oscillate, but if the initial value of $\frac{d\chi}{dt}$ is zero, it will remain at rest in whatever position it may be placed.

277. Ex. 1. Show that a person furnished with the particular form of Foucault's pendulum just described, could, without any Astronomical observations, determine the latitude of the place, the direction of the rotation of the earth, and the length of the sidereal day. This remark is due to M. Quet, who has given a different solution of this problem in *Liouville's Journal*, vol. XVIII.

Ex. 2. If the body be a rod, and its centre of gravity supported without friction, prove that it could rest in relative equilibrium either parallel or perpendicular to the projection of the earth's axis on the plane of constraint. If it be placed in any other position, its motion will be very slow, depending on p^2, but it will oscillate about a mean position perpendicular to the projection of the earth's axis.

Ex. 3. If the axis of figure be acted on by a frictional force producing a retarding couple, whose moment about the axis of x bears a constant ratio μ to the moment of the reactional couple about the axis of y, and if the fixed plane be parallel to the axis of the earth, find the small oscillations about the position of equilibrium. Show that the position at any time t is given by

$$\chi = Le^{-\lambda t} \cos \left[\left(\frac{Cnp}{A} - \lambda^2 \right)^{\frac{1}{2}} t + M \right],$$

where $2A\lambda = \mu (Cn - 2Ap)$ and L and M are two constants depending on the initial conditions.

Ex. 4. The centre of gravity of a solid of revolution is fixed, while the axis of figure is constrained to remain in the surface of a smooth right cone fixed relatively to the earth. Show that the axis of figure will oscillate about the projection of the axis of rotation of the earth on the surface of the cone, and that the time of a complete small oscillation about the mean position will be $2\pi \sqrt{\frac{A \sin \epsilon}{Cpn \sin \beta}}$, where ϵ is the semi-angle of the cone, β the inclination of its axis to the axis of the earth, and the other letters have the same meaning as before. This result is due to M. Quet.

Ex. 5. Two equal heavy rods CA, CB are connected by a hinge at C, with a spring so that they tend to make a known angle with each other. The free ends A and B are then tied together and the whole is suspended by a string OC attached

to the hinge. The system is left to itself until it is at rest relatively to the earth. If the string which fastens A and B be now cut, the arms separate from each other. Show that the system will immediately have an apparent angular velocity round the vertical equal to $\frac{I'-I}{I'} p \sin \lambda$, where I, I' are the moments of inertia of the system about the vertical OC respectively before and after the string joining A and B was cut, p is the angular velocity of the earth about its axis and λ is the latitude of the place. In which direction will the system turn? This apparatus was devised by M. Poinsot who considered that the experiment would be so effective that the latitude of the place could be deduced from the observed angular velocity. See *Comptes Rendus*, 1851, Tome XXXII. page 206.

Ex. 6. If a river is flowing due north, prove that the pressure on the eastern bank at a depth z is increased by the change of latitude of the running water in the ratio $gz + bv\omega \sin l : gz$, where b is the breadth of the stream, v its velocity, l the latitude and ω the angular velocity of the earth about its axis. [Math. Tripos, 1875.]

CHAPTER VI.

ON MOMENTUM.

278. THE term Momentum has been given as the heading of this Chapter, though it only expresses a portion of its contents. The object of the Chapter may be enunciated in the following problem. The circumstances of the motion of a system at any time t_0 are given. At the time t_1 the system is moving under other circumstances. It is required to determine the relations which may exist between these two motions. The manner in which these changes are effected by the forces is not the subject of enquiry. We only wish to determine what changes have been effected in the time $t_1 - t_0$. If the time $t_1 - t_0$ be very small, and the forces very great, this becomes the general problem of impulses. This also will be considered in the Chapter.

Let us refer the system to any fixed axes Ox, Oy, Oz. Then the six general equations of motion may, by Art. 71, be written in the form

$$\left.\begin{aligned} \Sigma m \frac{d^2z}{dt^2} &= \Sigma mZ \\ \Sigma m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2}\right) &= \Sigma m\,(xY - yX) \end{aligned}\right\}.$$

Integrating these from $t = t_0$ to $t = t_1$, we have

$$\left[\Sigma m \frac{dz}{dt}\right]_{t_0}^{t_1} = \Sigma m \int_{t_0}^{t_1} Z\,dt,$$

$$\left[\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right)\right]_{t_0}^{t_1} = \Sigma m \int_{t_0}^{t_1} (xY - yY)\,dt.$$

Let a force P act on a moving particle m during any time $t_1 - t_0$, and let this time be divided into intervals each equal to dt. At the middle of each of these intervals let a line be drawn from the position of m at that instant, to represent, at the same instant, the value of $mPdt$ both in direction and magnitude. Then the resultant of these forces, found by the rules of Statics, may be called the *whole force* expended in the time $t_1 - t_0$. Thus $\int_{t_0}^{t_1} mZ\,dt$ is the whole force resolved parallel to the axis of Z. These equations then show that

(1) The change produced by any forces in the resolved part of the momentum of any system is equal in any time to the whole resolved force in that direction.

(2) The change produced by any forces in the moment of the momentum of the system about any straight line is, in any time, equal to the whole moment of these forces about that straight line.

When the interval $t_1 - t_0$ is very small, the "whole force" expended is the usual measure of an impulsive force, and the preceding equations are identical with those given in Art. 86.

It is not necessary to deduce these two results from the equations of motion. The following general theorem, which is really equivalent to the two theorems enunciated above, may be easily obtained by an application of D'Alembert's principle.

279. *If the momentum of any particle of a system in motion be compounded and resolved, as if it were a force acting at the instantaneous position of the particle, according to the rules of Statics, then the momenta of all the particles at any time* t_1 *are together equivalent to the momenta at any previous time* t_0 *together with the whole forces which have acted during the interval.*

In the case in which no forces act on the system, except the mutual actions of the particles, we see that the momenta of all the particles of a system at any two times are equivalent; a result which has been already enunciated in Art. 72. The two principles of the Conservation of Linear Momentum and Conservation of Areas may be enunciated as follows.

If the forces which act on a system be such that they have no component along a certain fixed straight line, then the motion is such that the linear momentum resolved along this line is constant.

If the forces be such that they have no moment about a certain fixed straight line, then the moment of the momentum or area conserved about this straight line is constant.

It is evident that these principles are only particular cases of the results proved in Art. 79.

280. Ex. Suppose that a simple particle m describes an orbit about a centre of force O. Let v, v' be its velocities at any two points P, P' of its course. Then mv' supposed to act along the tangent at P' if reversed would be in equilibrium with mv acting along the tangent at P together with the whole central force from P to P'. If p, p' be the lengths of the perpendiculars from O on the tangents at P, P', we have, by taking moments about O, $vp = v'p'$, and hence vp is constant throughout the

motion. Also if the tangents meet in T, the whole central force expended must act along the line TO, and may be found in terms of v, v' by the rules for compounding velocities.

Ex. Two particles of masses m, m' move about the same centre of force. If h, h' be the double areas described by each per unit of time, prove that $mh+m'h'$ is unaltered by an impact between the particles.

281. Ex. Suppose three particles to start from rest attracting each other, but under the action of no external forces. Then the momenta of the three particles at any instant are together equivalent to the three initial momenta and are therefore in equilibrium. Hence at any instant the tangents to their paths must meet in some point O, and if parallels to their directions of motion be drawn so as to form a triangle, the momenta of the several particles are proportional to the sides of that triangle.

If there are n particles it may be shown in the same way that the n forces represented by mv, $m'v'$, &c. are in equilibrium, and if parallels be drawn to the directions of motion and proportional to the momenta of the particles beginning at any point, they will form a closed polygon.

If F, F', F'' be the resultant attraction on the three particles, the lines of action of F, F', F'' also meet in a point. For let X, Y, Z be the actions between the particles $m'm''$, $m''m$, mm', taken in order. Then F is the resultant of $-Y$ and Z; F' of $-Z$ and X; F'' of $-X$ and Y. Hence the three forces F, F', F'' are in equilibrium*, and therefore their lines of action must meet in a point O'. Also the magnitude of each is proportional to the sine of the angle between the directions of the other two. This point is not generally fixed, and does not coincide with O.

If the law of attraction be proportional to the distance, the two points O, O' coincide with the centre of gravity G, and are fixed in space throughout the motion. For it is a known proposition in Statics that with this law of attraction, the whole attraction of a system of particles on one of the particles is the same as if the whole system were collected at its centre of gravity. Hence O' coincides with G. Also, since each particle starts from rest, the initial velocity of the centre of gravity is zero, and therefore, by Art. 79, G is a fixed point. Again, since each particle starts from rest and is urged towards a fixed point G, it will move in the straight line joining its initial position with G. Hence O coincides with G. When the law of attraction is proportional to the distance, it is proved in Dynamics of a Particle, that the time of reaching the centre of force from a position of rest is independent

* This proof is merely an amplification of the following. The three forces F, F', F'', being the internal re-actions of a system of three bodies, are in equilibrium by D'Alembert's Principle.

of the distance of that position of rest. Hence all the particles of the system will reach G at the same time, and meet there. If Σm be the sum of the masses, measured by their attractions in the usual manner, this time is known to be $\frac{1}{4}\frac{2\pi}{\sqrt{\Sigma m}}$.

282. Ex. *Three particles whose masses are* m, m′, m″, *mutually attracting each other, are so projected that the triangle formed by joining their positions at any instant remains always similar to its original form. It is required to determine the conditions of projection.*

The centre of gravity will be either at rest or will move uniformly in a straight line. We may therefore consider the centre of gravity at rest and may afterwards generalise the conditions of projection by impressing on each particle an additional velocity parallel to the direction in which we wish the centre of gravity to move. Let O be the centre of gravity, P, P', P'' the positions of the particles at any time t. Then by the conditions of the question the lengths OP, OP', OP'' are always to be proportional, and their angular velocities about O are to be equal. Since the moment of the momenta of the system about O is always the same, we have

$$mr^2n + m'r'^2n + m''r''^2n = \text{constant},$$

where r, r', r'' are the distances OP, OP', OP'', and n is their common angular velocity. Since the ratios $r : r' : r''$ are constants, it follows from this equation that mr^2n is constant, *i.e.* OP traces out equal areas in equal times. Hence by Newton, Section II, the resultant force on P tends towards O.

Let ρ, ρ', ρ'' be the sides $P'P''$, $P''P$, PP' of the triangle formed by the particles, and let the law of attraction be $\frac{\text{mass}}{(\text{dist.})^k}$. Then since the resultant attraction of m', m'' on m passes through O,

$$\frac{m'}{\rho''^k}\sin P'PO = \frac{m''}{\rho'^k}\sin P''PO,$$

but since O is the centre of gravity,

$$m'\rho''\sin P'PO = m''\rho'\sin P''PO.$$

Hence either the three particles are in one straight line or $\rho''^{k+1} = \rho'^{k+1}$. If $k = -1$ the law of attraction is "as the distance." If k be not $= -1$, we have $\rho' = \rho''$, and the triangle must be equilateral.

Conversely, suppose the particles to be projected in directions making equal angles with their distances from the centre of gravity with velocities proportional to those distances, and suppose also the resultant attractions towards the centre of gravity to

be proportional to those distances, then in all the three cases the same conditions will hold at the end of a time dt, and so on continually. The three particles will therefore describe similar orbits about the centre of gravity in a similar manner.

First, let us suppose that the three particles are to be in one straight line. To fix our ideas, let m' lie between m and m'', and O between m and m'. Then since the attraction on any particle must be proportional to the distance of that particle from O, the three attractions

$$\frac{m'}{(PP')^k}+\frac{m''}{(PP'')^k},\qquad \frac{m''}{(P'P'')^k}-\frac{m}{(PP')^k},\qquad -\frac{m}{(PP'')^k}-\frac{m'}{(P'P'')^k},$$

must be proportional to OP, OP', OP''. Since $\Sigma mOP=0$, these two equations amount to but one on the whole. Let $z=\dfrac{P'P''}{PP'}$, so that $\dfrac{OP}{PP'}=\dfrac{m'+m''(1+z)}{m+m'+m''}$, $\dfrac{OP'}{PP'}=\dfrac{-m+m''z}{m+m'+m''}$.

Then we have

$$\left(m'+\frac{m''}{(1+z)^k}\right)(-m+m''z)=\left(\frac{m''}{z^k}-m\right)\{m'+m''(1+z)\},$$

which agrees with the result given by Laplace, by whom this problem was first considered.

In the case in which the attraction follows the law of nature $k=2$ and the equation becomes

$$mz^2\{(1+z)^3-1\}-m'(1+z)^2(1-z^3)-m''\{(1+z)^3-z^3\}=0.$$

This is an equation of the fifth degree, and it has therefore always one real root. The left side of the equation has opposite signs when $z=0$ and $z=\infty$, and hence this real root is positive. It is therefore always possible to project the three masses so that they shall remain in a straight line. Laplace remarks that if m be the sun, m' the earth, and m'' the moon, we have very nearly $z=\sqrt[3]{\dfrac{m'+m''}{3m}}=\dfrac{1}{100}$. If then originally the earth and moon had been placed in the same straight line with the sun at distances from the sun proportional to 1 and $1+\dfrac{1}{100}$, and if their velocities had been initially parallel and proportional to those distances, the moon would have always been in opposition to the sun. The moon would have been too distant to have been in a state of continual eclipse, and thus would have been full every night. It has however been shown by Liouville, in the *Additions à la Connaissance des Temps*, 1845, that such a motion would be unstable.

The paths of the particles will be similar ellipses having the centre of gravity for a common focus.

Secondly. Let us suppose that the law of attraction is "as the distance." In this case the attraction on each particle is the same as if all the three particles were collected at the centre of gravity. Each particle will describe an ellipse having this point for centre in the same time. The necessary conditions of projection are that the velocities of projection should be proportional to the initial distances from the centre of gravity, and the directions of projection should make equal angles with those distances.

Thirdly. Let us suppose the particles to be at the angular points of an equilateral triangle. The resultant force on the particle m is

$$\frac{m'}{\rho''^k}\cos P'PO + \frac{m''}{\rho'^k}\cos P''PO.$$

The condition that the forces on the particles should be proportional to their distances from O shows that the ratio of this force to the distance OP is the same for all the particles. Since

$$m'\rho''\cos P'PO + m''\rho'\cos P''PO = (m + m' + m'')\, OP,$$

it is clear that the condition is initially satisfied when $\rho = \rho' = \rho''$. Hence, by the same reasoning as before, if the particles be projected with equal velocities in directions making equal angles with OP, OP', OP'' respectively, they will always remain at the angular points of an equilateral triangle.

Ex. 1. Show that if the three particles attracted each other according to the law of nature, the paths of the particles, when at the corners of an equilateral triangle, are equal ellipses having O for a common focus. Find the periodic time.

Ex. 2. If four particles be placed at the corners of a quadrilateral whose sides taken in order are a, b, c, d and diagonals ρ, ρ', then the particles could not move under their mutual attractions so as to remain always at the corners of a similar quadrilateral unless

$$(\rho^n\rho'^n - b^n d^n)(c^n + a^n) + (a^n c^n - \rho^n\rho'^n)(b^n + d^n) + (b^n d^n - a^n c^n)(\rho^n + \rho'^n) = 0,$$

where the law of attraction is the inverse $(n-1)^{\text{th}}$ power of the distance.

Show also that the mass at the intersection of b, c divided by the mass at intersection of c, d is equal to the product of the area formed by a, ρ', d divided by the area formed by a, b, ρ and the difference $\dfrac{1}{\rho'^n} - \dfrac{1}{d^n}$ divided by the difference $\dfrac{1}{\rho^n} - \dfrac{1}{b^n}$.

These results may be conveniently arrived at by reducing one angular point as A of the quadrilateral to rest. The resolved part of all the forces which act on each particle perpendicular to the straight line joining it to A will then be zero. The case of three particles may be treated in the same manner. The process is a little shorter than that given in the text, but does not illustrate so well the subject of the chapter.

283. When the system under consideration consists of rigid bodies we must use the results of Art. 75 to find the resolved part of the momentum in any direction. The moment of the momentum about any straight line may also be found by Art. 76 in Chap. II, combined with Art. 123 in Chap. IV, if the motion be in two dimensions, or Art. 240 in Chap. V, if the motion be in three dimensions.

284. Ex. *A disc of any form is moving in its own plane in any manner. Suddenly any point* O *in the disc is fixed, find the angular velocity of the disc about* O.

Let us suppose that just before O became fixed the centre of gravity G was moving with velocity V, and that p is the length of the perpendicular from O on the direction of motion. Also let ω be the angular velocity of the body about its centre of gravity. Just after O has become fixed, let the body be turning about O with angular velocity ω'. Let Mk^2 be the moment of inertia of the disc about the centre of gravity, and let $OG = r$.

The change in the motion of the disc is produced by impulsive forces acting at O during a short time $t_1 - t_0$. These forces have no moment about O. Hence the moment of the momentum about O is the same just after and just before the impact. Just before O became fixed, the moment of the momentum about G was $Mk^2\omega$ (Art. 123), and the moment of the momentum of the whole mass collected at G was MVp. Hence the whole moment of the momentum about O is the sum of these two (Art. 76). Just after O has become fixed the body is turning about O, hence by Art. 123 the moment of the momentum about O is $M(k^2 + r^2)\,\omega'$. Equating these we have

$$M(k^2 + r^2)\,\omega' = Mk^2\omega + MVp\,;$$

$$\therefore\ \omega' = \frac{k^2\omega + Vp}{k^2 + r^2}.$$

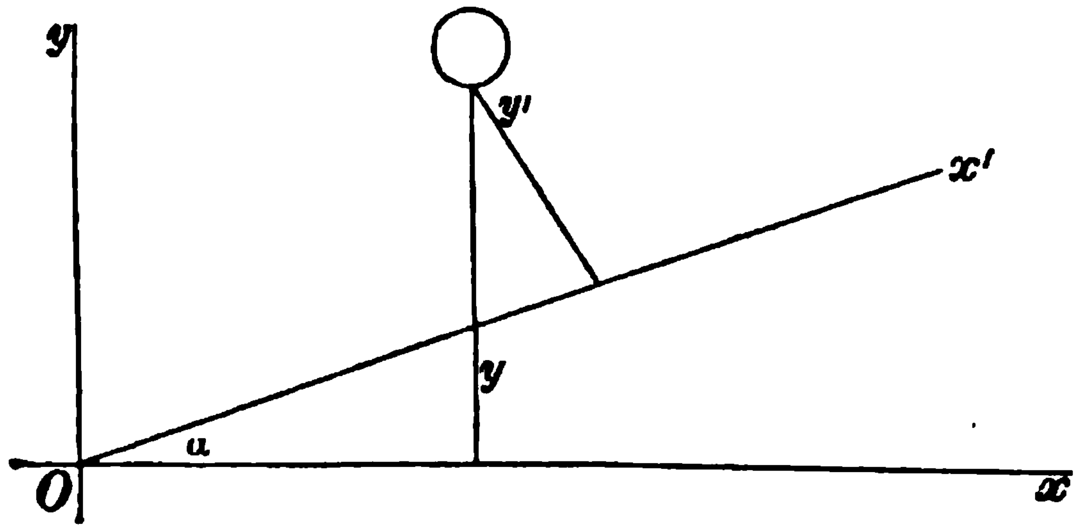
Ex. A circular area is turning about a point A on its circumference. Suddenly A is loosed and another point B also on the circumference is fixed. Show that if AB is a quadrant, the angular velocity is reduced to one-third its value, and if AB is a third of the circumference, the area will be reduced to rest.

285. Ex. *A disc of any form is turning about an axis* Ox *situated in its own plane with an angular velocity* ω. *Suddenly this axis is let free and another axis* Ox′, *also situated in the plane of the disc, becomes fixed, it is required to find the new angular velocity* ω' *about* Ox′.

The change in the motion of the disc is caused by the action of the impulsive forces due to the sudden fixing of the axis Ox'. These act at points situated in Ox' and have no moment about

Ox'. Hence the moment of the momentum about Ox' is the same just before and just after Ox' is fixed.

Let $d\sigma$ be any element of the area of the disc; y, y' its distances from Ox, Ox'. Then $y\omega$, $y'\omega'$ are the velocities of $d\sigma$ just



before and just after the impact. The moments of the momentum about Ox' just before and just after are therefore $yy'\omega d\sigma$ and $y'^2\omega' d\sigma$. Summing these for the whole area of the disc, we have

$$\omega'\Sigma y'^2 d\sigma = \omega\Sigma yy' d\sigma \dots\dots\dots\dots\dots\dots\dots(1).$$

First, let Ox, Ox' be parallel, so that the point O is at infinity. Let h be the distance between the axes, then $y' = y - h$. Hence we have

$$\omega'\Sigma y'^2 d\sigma = \omega\,\{\Sigma y^2 d\sigma - h\Sigma y d\sigma\}.$$

Let A, A' be the moments of inertia of the disc about Ox, Ox' respectively, $\bar{y}$ the distance of the centre of gravity from Ox, M the mass of the disc. Then we have

$$A'\omega' = \omega\,(A - Mh\bar{y}).$$

Secondly, let Ox, Ox' not be parallel. Let O be the origin and the angle $xOx' = \alpha$, then $y' = y\cos\alpha - x\sin\alpha$. Let F be the product of inertia of the disc about Ox, Oy where Oy is perpendicular to Ox. Then by substitution in (1) we have

$$A'\omega' = \omega\,(A\cos\alpha - F\sin\alpha).$$

Ex. 1. An elliptic area of eccentricity e is turning about one latus rectum. Suddenly this latus rectum is loosed and the other fixed. Show that the angular velocity is $\frac{1-4e^2}{1+4e^2}$ of its former value.

Ex. 2. A right-angled triangular area ACB is turning about the side AC. Suddenly AC is loosed and BC fixed. If C be the right angle, the angular velocity is $\frac{BC}{2.AC}$ of its former value.

286. *A rigid body is moving freely in space in a known manner. Suddenly either a straight line or a point in the body becomes fixed. To determine the initial subsequent motion.*

This proposition will include the last two articles as particular cases. It is obvious that all the impulsive actions on the body pass through the fixed straight line or the fixed point. Hence the moments of the momentum of the body about the fixed axis in the first case or about any axis through the fixed point in the second case are unaltered by the impulsive forces.

First. Let a straight line suddenly become fixed. Let it be taken as the axis of z.

Let MK^2 be the moment of inertia of the body about the axis of z, and Ω the angular velocity after the straight line has become fixed. Suppose that the body when moving freely was turning with angular velocities ω_x, ω_y, ω_z about three straight lines Gx', Gy' Gz' through the centre of gravity parallel to the axes of co-ordinates. And let the co-ordinates of the centre of gravity be $\bar{x}, \bar{y}, \bar{z}$.

Then

$$C'\omega_z - (\Sigma mz'x')\,\omega_x - (\Sigma mz'y')\,\omega_y + M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) = MK^2.\Omega,$$

where C' is the moment of inertia of the body about Gz', and $\Sigma mz'x'$, $\Sigma mz'y'$ are calculated with reference to the axes Gx', Gy', Gz'.

Secondly. Let a point O in the moving body be suddenly fixed in space. Take any three rectangular axes Ox, Oy, Oz, and three parallel axes Gx', Gy', Gz' through the centre of gravity G. Let ω_x, ω_y, ω_z be the known angular velocities of the body about the axes Gx', Gy', Gz' before the point O became fixed, Ω_x, Ω_y, Ω_z the unknown angular velocities about Ox, Oy, Oz after O has become fixed.

Then, following the same notation as before, we have by Art. 240,

$$A'\omega_x - (\Sigma m\,x'y')\,\omega_y - (\Sigma m\,x'z')\,\omega_z + \Sigma m\left(\bar{y}\frac{d\bar{z}}{dt} - \bar{z}\frac{d\bar{y}}{dt}\right)$$
$$= A\Omega_x - (\Sigma m\,xy)\,\Omega_y - (\Sigma m\,xz)\,\Omega_z.$$

$$B'\omega_y - (\Sigma m\,y'z')\,\omega_z - (\Sigma m\,y'x')\,\omega_x + \Sigma m\left(\bar{z}\frac{d\bar{x}}{dt} - \bar{x}\frac{d\bar{z}}{dt}\right)$$
$$= B\Omega_y - (\Sigma m\,yz)\,\Omega_z - (\Sigma m\,yx)\,\Omega_x.$$

$$C'\omega_z - (\Sigma m\,z'x')\,\omega_x - (\Sigma m\,z'y')\,\omega_y + \Sigma m\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right)$$
$$= C\Omega_z - (\Sigma m\,zx)\,\Omega_x - (\Sigma m\,zy)\,\Omega_y.$$

These equations determine Ω_x, Ω_y, Ω_z, It is obvious that they may be greatly simplified by so choosing the axes that one

of the two sets Ox, Oy, Oz or Gx', Gy', Gz' may be a set of principal axes.

287. If the body be turning about an axis GI through the centre of gravity G just before the point O is fixed, the terms containing the co-ordinates of the centre of gravity disappear from the equations. They now admit of an easy geometrical interpretation. The equation to the momental ellipsoid at the centre of gravity is

$$A'X^2 + B'Y^2 + C'Z^2 - 2\Sigma my'z'\,YZ - 2\Sigma m\,z'x'\,ZX \\ - 2\Sigma m\,x'y'\,XY = M\epsilon^4.$$

It is therefore clear that the left-hand sides of these equations are proportional to the direction-cosines of the diametral plane of a straight line whose direction-cosines are proportional to $(\omega_x, \omega_y, \omega_z)$. In the same way if we construct the momental ellipsoid at O, the right-hand sides are proportional to the direction-cosines of the diametral plane of the axis $(\Omega_x, \Omega_y, \Omega_z)$. Thus the instantaneous axes of rotation, before and after O is fixed, are so related that their diametral planes with regard to the momental ellipsoids at G and O respectively are parallel.

We may also deduce this result, without difficulty, from Art. 117. The motion of the body about the axis GI may be produced by an impulsive couple in the diametral plane of GI with regard to the momental ellipsoid at G. Let us then suppose the body at rest and O fixed, and let it be acted on by this couple. It follows from the same article, that the body will begin to turn about an axis OI' which is such that its diametral plane with regard to the momental ellipsoid at O is parallel to the plane of the couple.

The direction of the blow at O may also be easily found. The centre of gravity being at rest suddenly begins to move perpendicular to the plane containing it and the axis OI'. This is obviously the direction of the blow.

288. Ex. 1. *A sphere in co-latitude θ is hung up by a point O in its surface in equilibrium under the action of gravity. Suddenly the rotation of the earth is stopped, it is required to determine the motion of the sphere.* [Math. Tripos, 1857.]

Let G be the centre of the sphere, O its point of suspension, and a its radius. Let C be the centre of the earth. Let us suppose the figure so drawn that the sphere is moving away from the observer.

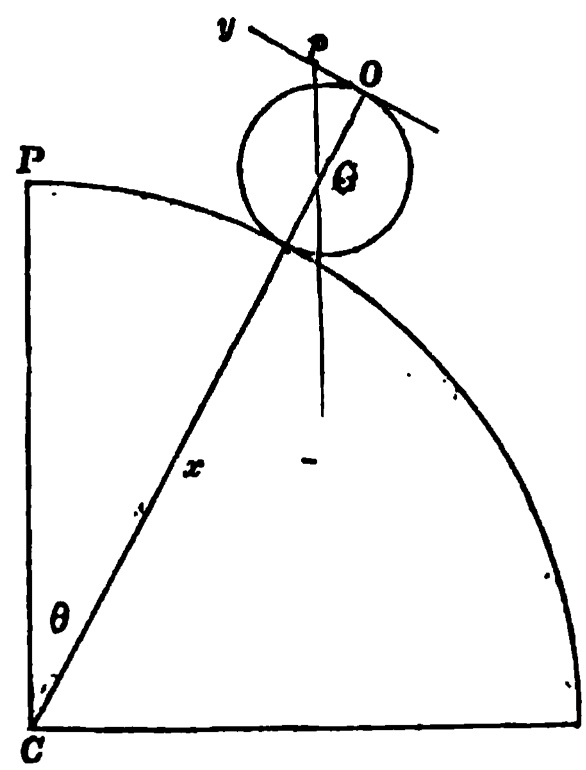
Let ω = angular velocity of the earth, then if $CG = \mu a$, the sphere is turning about an axis Gp parallel to CP, the axis of the earth with angular velocity ω, while the centre of gravity is moving with velocity $\mu a \sin\theta \,.\, \omega$.

Let OC, Op, and the perpendicular to the plane of OC, Op be taken as the axes of x, y, z respectively, and let Ω_x, Ω_y, Ω_z be the angular velocities about them just after the rotation of the earth is stopped.

By Art. 286, the angular momenta about Ox, just before and just after the rotation was stopped, are equal to each other;

$$\therefore Mk^2 \omega \cos\theta = Mk^2 \Omega_x,$$

where Mk^2 is the moment of inertia of the sphere about a diameter.



Again, the angular momenta about Oy are equal to each other;

$$\therefore -Mk^2\omega \sin\theta + M\mu a^2 \omega \sin\theta = M(k^2+a^2)\Omega_y.$$

Lastly, the angular momenta about Oz are equal; $\therefore 0 = Mk^2\Omega_z$.

Solving these equations, we get

$$\Omega_y = \omega \sin\theta \frac{-k^2+\mu a^2}{k^2+a^2} = \omega \sin\theta \frac{-2+5\mu}{7}.$$

But $\Omega_x = \omega\cos\theta$. Adding together the squares of $\Omega_x, \Omega_y, \Omega_z$ we have

$$\Omega^2 = \omega^2 \left\{\cos^2\theta + \left(\frac{-2+5\mu}{7}\right)^2 \sin^2\theta\right\},$$

where Ω is the angular velocity of the sphere about its instantaneous axis.

Ex. 2. A particle of mass M, without velocity, is suddenly attached to the surface of the earth at the extremity of a radius vector making an angle θ with the axis of the earth. If E be the mass of the earth before the addition of M, A and C its principal moments of inertia at the centre, ω the angular velocity about its axis, prove that

$$\frac{\omega}{\Omega} = 1 + \frac{EMAr^2\sin^2\theta}{(E+M)AC + EMCr^2\cos^2\theta},$$

$$\cot\phi = \cot\theta + \frac{E+M}{E} \cdot \frac{A}{Mr^2\sin\theta\cos\theta},$$

where Ω is the initial angular velocity about an axis parallel to the axis of the earth and ϕ is the angle the initial axis of rotation makes with the axis of the earth.

Ex. 3. A body having a point O fixed is turning with angular velocity ω about an axis OI whose direction cosines referred to the principal axes at O are (l, m, n). Suddenly, an axis OI' whose direction-cosines are (l', m', n') is fixed. Show that the angular velocity about OI' is given by

$$(Al'^2 + Bm'^2 + Cn'^2)\,\omega' = (All' + Bmm' + Cnn')\,\omega,$$

where A, B, C are the principal moments at O.

Ex. 4. A regular homogeneous prism whose normal section is a regular polygon of n sides rolls on a perfectly rough plane. Prove that, when the axis of rotation changes from one edge to another, the angular velocity is reduced in the ratio

$$\left(\frac{2+7\cos\frac{2\pi}{n}}{8+\cos\frac{2\pi}{n}}\right).$$

289. In these examples the changes produced in the motion were sudden, but the method of proceeding is the same if the changes are gradual.

Ex. 1. A bead of mass m slides on a circular wire of mass M and radius a, and the wire can turn freely about a vertical diameter. Prove that, if ω, Ω be the angular velocities of the wire when the bead is respectively at the extremities of a horizontal and vertical diameter, $\frac{\Omega}{\omega}=1+2\frac{m}{M}$.

Ex. 2. If the earth gradually contracted by radiation of heat, so as to be always similar to itself as regards its physical constitution and form, prove that when every radius vector has contracted an n^{th} part of its length, where n is small, the angular velocity has increased a $2n^{th}$ part of its former value.

Ex. 3. If two railway trains each of mass M were to travel in opposite directions from the pole along a meridian and to arrive at the equator at the same time, prove that the angular velocity of the earth would be decreased by $\frac{2Ma^2}{Ek^2}$, where a is the equatorial radius of the earth and Ek^2 its moment of inertia about its axis of figure.

What would be the effect if one train only were to travel from the pole to the equator?

Ex. 4. A fly alights perpendicularly on a sheet of paper lying on a smooth horizontal plane and proceeds to describe the curve $r=f(\theta)$ traced on the sheet of paper, the equation to the curve being referred to the centre of gravity of the paper as origin. Supposing the fly to be able to prevent himself from slipping on the paper, show that his angular velocity in space about the common centre of gravity of the paper and fly is equal to $\frac{(M+m)k^2}{(M+m)k^2+mr^2}\frac{d\theta}{dt}$, where M and m are the masses of the paper and the fly and k is the radius of gyration of the paper about its centre of gravity. Hence find the path of the fly in space.

Ex. 5. Suppose the ice to melt from the polar regions twenty degrees round each pole to the extent of something more than a foot thick, enough to give $1\frac{1}{10}$ feet over those areas or ·066 of a foot of water spread over the whole globe, which would in reality raise the sea-level by only some such undiscoverable difference as $\frac{3}{4}$th of an inch or an inch, then this would slacken the earth's rate as a time-keeper by one-tenth of a second per year. This and the next example are taken from the *Phil. Mag.* They are both due to Sir W. Thomson.

If E be the mass of the earth, a its radius, k its radius of gyration about the polar axis, ω its angular velocity before the melting, then we have by the principle

of angular momentum $\frac{\delta\omega}{\omega} = -\frac{Ma^2}{8Ek^2}\cos\theta\,(1+\cos\theta)$, where M is the mass of the ice melted and θ is twenty degrees. Substituting for the letters their known numerical values, the value of $\delta\omega$ is easily found.

Ex. 6. A layer of dust is formed on the earth h feet thick, where h is small, by the fall of meteors reaching the earth from all directions. Show that the change in the length of the day is nearly $\frac{5h}{a}\frac{\rho}{D}$ of a day where a is the radius of the earth in feet, ρ and D the densities of the dust and earth respectively. If the density of the dust be twice that of water and $h = \frac{1}{25}$ express this in numbers.

The Invariable Plane.

290. It is shown in Art. 72 of Chap. II, that all the momenta of the several particles of a system in motion, are together equivalent to a single resultant linear momentum at any assumed origin O, represented in direction and magnitude by a line OV, together with an angular momentum about some line passing through O, represented in direction and magnitude by a line OH. Let h_1, h_2, h_3 be the moments of the momenta of the particles about any rectangular axes Ox, Oy, Oz meeting in O, so that

$$h_1 = \Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right),$$

with similar expressions for h_2, h_3, and let

$$h^2 = h_1^2 + h_2^2 + h_3^2.$$

Then the direction-cosines of OH are $\frac{h_1}{h}$, $\frac{h_2}{h}$, $\frac{h_3}{h}$ and the angular momentum itself is represented by h.

If no external forces act on the system then by Art. 72 or Art. 279 h_1, h_2, h_3 are constant throughout the motion, hence OH is fixed in direction and magnitude. It is therefore called the invariable line at O, and a plane perpendicular to OH is called the invariable plane at O.

If any straight line OL be drawn through O making an angle θ with the invariable line OH at O, the angular momentum about OL is $h\cos\theta$. For the axis of the resultant momentum-couple is OH, and the resolved part about OL is therefore $OH\cos\theta$. Hence the invariable line at O may also be defined as that axis through O about which the moment of the momentum is greatest.

At different points of the system the position of the invariable line is different. But the rules by which they are connected are the same as those which connect the axes of the resultant couple of a system of forces when the origin of reference is varied. These

or an explosion of a planet similar to that by which Olbers supposed the planets Pallas, Ceres, Juno and Vesta, &c., to have been produced, might make a considerable change in the sum of the terms omitted. In this case there would be a change in the position of the Astronomical Invariable Plane, but the Dynamical Invariable Plane would be altogether unaffected. It might be supposed that it would be preferable to use in Astronomy the true Invariable Plane. But this is not necessarily the case, for the angular velocities and moments of inertia of the bodies forming our system are not all known, so that the position of the Dynamical Invariable Plane cannot be calculated to any near degree of approximation, while we do know that the terms into which these unknown quantities enter are all very small or nearly constant. All the terms rejected being small compared with those retained, the Astronomical Invariable Plane must make only a small angle with the Dynamical Invariable Plane. Although the plane is very nearly fixed in space, yet its intersection with the Dynamical Invariable Plane, owing to the smallness of the inclination, may undergo considerable changes in course of time.

In the *Mécanique Céleste,* Laplace calculated the position of the Astronomical Invariable Plane at the two epochs, 1750 and 1950, assuming the correctness for this period of his formulæ for the variations of the eccentricities, inclinations and nodes of the planetary orbits. At the first epoch the inclination of this plane to the ecliptic was $1^{\circ}.7689$, and longitude of the ascending node $114^{\circ}.3979$; at the second epoch the inclination will be the same as before, and the longitude of the node $114^{\circ}.3934$.

294. Ex. 1. Show that the invariable plane at any point of space in the straight line described by the centre of gravity of the solar system is parallel to that at the centre of gravity.

Ex. 2. If the invariable planes at all points in a certain straight line are parallel, then that straight line is parallel to the straight line described by the centre of gravity.

Impulsive Forces in Three Dimensions.

295. *To determine the general equations of motion of a body about a fixed point under the action of given impulses.*

Let the fixed point be taken as the origin, and let the axes of co-ordinates be rectangular. Let $(\Omega_x, \Omega_y, \Omega_z)$, $(\omega_x, \omega_y, \omega_z)$ be the angular velocities of the body just before and just after the impulse, and let the differences $\omega_x - \Omega_x$, $\omega_y - \Omega_y$, $\omega_z - \Omega_z$ be called ω_x', ω_y', ω_z'. Then ω_x', ω_y', ω_z' are the angular velocities *generated by the impulse.* By D'Alembert's Principle, see Art. 87,

the difference between the moments of the momenta of the particles of the system just before and just after the action of the impulses is equal to the moment of the impulses. Hence by Art. 240,

$$\left.\begin{aligned} A\omega_x' - (\Sigma mxy)\,\omega_y' - (\Sigma mxz)\,\omega_z' &= L \\ B\omega_y' - (\Sigma myz)\,\omega_z' - (\Sigma myx)\,\omega_x' &= M \\ C\omega_z' - (\Sigma mzx)\,\omega_x' - (\Sigma mzy)\,\omega_y' &= N \end{aligned}\right\} \ldots\ldots\ldots\ldots (1),$$

where L, M, N are the moments of the impulsive forces about the axes.

These three equations will suffice to determine the values of ω_x', ω_y', ω_z'. These being added to the angular velocities before the impulse, the initial motion of the body after the impulse is found.

296. Ex. 1. Show that these equations are independent of each other.

This follows from Art. 20 where it is shown that the eliminant of the equations cannot vanish.

Ex. 2. Deduce these equations from the general equations of motion referred to moving axes given in Art. 253.

Ex. 3. Show that if the body be acted on by a finite number of given impulses following each other at infinitely short intervals, the final motion is independent of their order.

297. It is to be observed that these equations leave the axes of reference undetermined. They should be so chosen that the values of A, Σmxy, &c. may be most easily found. If the positions of the principal axes at the fixed point are known they will in general be found the most suitable.

In that case the equations reduce to the simple form

$$\left.\begin{aligned} A\omega_x' &= L \\ B\omega_y' &= M \\ C\omega_z' &= N \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The values of ω_x', ω_y', ω_z' being known, we can find the pressures on the fixed point. For by D'Alembert's Principle the change in the linear momentum of the body in any direction is equal to the resolved part of the impulsive forces. Hence if F, G, H, be the pressures of the fixed point on the body

$$\left.\begin{aligned} \Sigma X + F &= M\,.\,\frac{d\bar{x}}{dt} \text{ by Art. 86} \\ &= M\,(\omega_y'\bar{z} - \omega_z'\bar{y}) \text{ by Art. 219} \\ \Sigma Y + G &= M\,(\omega_z'\bar{x} - \omega_x'\bar{z}) \\ \Sigma Z + H &= M\,(\omega_x'\bar{y} - \omega_y'\bar{x}) \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

298. Ex. *A uniform disc bounded by an arc* OP *of a parabola, the axis* ON, *and the ordinate* PN, *has its vertex* O *fixed. A blow* B *is given to it perpendicular*

to its plane at the other extremity P *of the curved boundary. Supposing the disc to be at rest before the application of the blow, find the initial motion.*

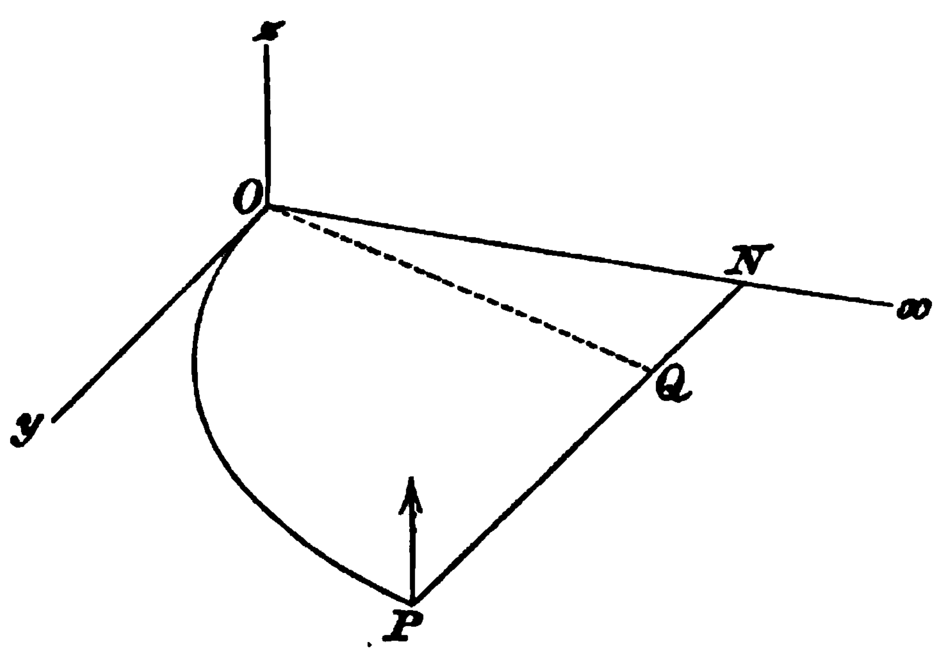
Let the equation to the parabola be $y^2=4ax$ and let the axis of z be perpendicular to its plane. Then $\Sigma mxz=0$, $\Sigma myz=0$. Let μ be the mass of a unit of area and let $ON=c$. Also

$$\Sigma mxy=\mu\iint xy\,dx\,dy=\mu\int_0^c x\frac{y^2}{2}\,dx=2\mu\int_0^c ax^2\,dx=\frac{2}{3}\mu ac^3,$$

$$A=\frac{1}{3}\mu\int_0^c y^3\,dx=\frac{16}{15}\mu a^{\frac{3}{2}}c^{\frac{5}{2}},\quad B=\mu\int_0^c x^2y\,dx=\frac{4}{7}\mu a^{\frac{1}{2}}c^{\frac{7}{2}}\text{ and } C=A+B \text{ by Art. 7.}$$

The moments of the blow B about the axes are $L=B\sqrt{4ac}$, $M=-Bc$, $N=0$. The equations of Art. 295 will become after substitution of these values

$$\left.\begin{aligned}\frac{16}{15}\mu a^{\frac{3}{2}}c^{\frac{5}{2}}\omega_x-\frac{2}{3}\mu ac^3\omega_y&=2Ba^{\frac{1}{2}}c^{\frac{1}{2}}\\ \frac{4}{7}\mu a^{\frac{1}{2}}c^{\frac{7}{2}}\omega_y-\frac{2}{3}\mu ac^3\omega_x&=-Bc\\ \omega_z&=0\end{aligned}\right\}.$$



From these ω_x, ω_y may be found. By eliminating B we have $\dfrac{\omega_y}{\omega_x}=\dfrac{7}{25}\dfrac{2\sqrt{ac}}{c}$. Hence if NQ be taken equal to $\dfrac{7}{25}NP$, the disc will begin to rotate about OQ. The resultant angular velocity will be $\dfrac{75}{26}\dfrac{B}{\mu ac^3}OQ$.

299. When a body free to turn about a fixed point is acted on by any number of impulses, each impulse is equivalent to an equal and parallel impulse acting at the fixed point together with an impulsive couple. The impulse at the fixed point can have no effect on the motion of the body, and may therefore be left out of consideration if only the motion is wanted. Compounding all the couples, we see that the general problem may be stated thus:—A body moving about a fixed point is acted on by a given impulsive *couple*, find the change produced in the motion. The analytical solution is comprised in the equations which have been written down in Art. 295. The following examples express the result in a geometrical form.

Ex. 1. Show from these equations that the resultant axis of the angular velocity generated by the couple is the diametral line of the plane of the couple with regard to the momental ellipsoid. See also Art. 117.

Ex. 2. Let G be the magnitude of the couple, p the perpendicular from the fixed point on the tangent plane to the momental ellipsoid parallel to the plane of the couple G. Let Ω be the angular velocity generated, r the radius vector of the ellipsoid which is the axis of Ω. Let $M\epsilon^4$ be the parameter of the ellipsoid. Prove that $\frac{G}{\Omega} = \frac{M\epsilon^4}{pr}$.

Ex. 3. If Ω_x, Ω_y, Ω_z be angular velocities about three conjugate diameters of the momental ellipsoid at the fixed point, such that their resultant is the angular velocity generated by an impulsive couple G, A', B', C' the moments of inertia about these conjugate diameters, prove that

$$A'\Omega_x = G\cos\alpha,\quad B'\Omega_y = G\cos\beta,\quad C'\Omega_z = G\cos\gamma,$$

where α, β, γ are the angles the axis of G makes with the conjugate diameters.

Ex. 4. If a body free to turn about a fixed point O be acted on by an impulsive couple G, whose axis is the radius vector r of the ellipsoid of gyration at O, and if p be the perpendicular from O on the tangent plane at the extremity of r, then the axis of the angular velocity generated by the blow will be the perpendicular p and the magnitude Ω is given by $G = Mpr\Omega$.

Ex. 5. Show that if a body at rest be acted on by any impulses, we may take moments about the initial axis of rotation, according to the rule given in Art. 89, as if it were a fixed axis.

300. Ex. 1. When a body turns about a fixed point the product of the moment of inertia about the instantaneous axis into the square of the angular velocity is called the Vis Viva. Let the vis viva generated from rest by any impulse be $2T$ and let the vis viva generated by the same impulse when the body is constrained to turn about a fixed axis passing through the fixed point be $2T'$. Then prove that $T' = T\cos^2\theta$, where θ is the angle between the eccentric lines of the two axes of rotation with regard to the momental ellipsoid at the fixed point.

Ex. 2. Hence deduce Lagrange's theorem, that the vis viva generated from rest by an impulse is greater when the body is free to turn about the fixed point, than when constrained to turn about any axis through the fixed point.

Ex. 3. If a body be moving in any manner about a fixed point and an axis through the fixed point be suddenly fixed, show that if the vis viva $2T$ be changed into $2T'$, we have $T' = T\cos^2\theta$, where θ has the same meaning as before.

301. *To determine the motion of a free body acted on by any given impulse.*

Since the body is free, the motion round the centre of gravity is the same as if that point were fixed. Hence the axes being any three straight lines at right angles meeting at the centre of gravity, the angular velocities of the body may still be found by equations (1) and (2) of Art. 295.

To find the motion of the centre of gravity, let (U, V, W), (u, v, w) be the resolved velocities of the centre of gravity just

before and just after the impulse. Let X, Y, Z be the components of the blow, and let M be the whole mass. Then by resolving parallel to the axes we have

$$M(u-U)=X,\ M(v-V)=Y,\ M(w-W)=Z.$$

If we follow the same notation as in Art. 295, the differences $u-U$, $v-V$, $w-W$ may be called u', v', w'.

302. Ex. 1. A body at rest is acted on by an impulse whose components parallel to the principal axes at the centre of gravity are (X, Y, Z) and the co-ordinates of whose point of application referred to these axes are (p, q, r). Prove that if the resulting motion be one of rotation only about some axis,

$$A(B-C)pYZ+B(C-A)qZX+C(A-B)rXY=0.$$

Is this condition sufficient as well as necessary? See Art. 221.

Ex. 2. A homogeneous cricket-ball is set rotating about a horizontal axis in the vertical plane of projection with an angular velocity Ω. When it strikes the ground, supposed perfectly rough and inelastic, the centre is moving with velocity V in a direction making an angle α with the horizon, prove that the direction of the motion of the ball after impact will make with the plane of projection an angle $\tan^{-1}\dfrac{2}{5}\dfrac{a\omega}{V\cos\alpha}$, where a is the radius of the ball.

303. The equations of Art. 301 completely determine the motion of a free body acted on by a given impulse, and from these by Art. 219 we may determine the initial motion of any point of the body. Let (p, q, r) be the co-ordinates of the point of application of the blow, then the moments of the blow round the axes are respectively $qZ-rY$, $rX-pZ$, $pY-qX$. These must be written on the right-hand sides of the equations of Art. 295. Let (p', q', r') be the co-ordinates of the point whose initial velocities parallel to the axes are required. Let (u_1, v_1, w_1), (u_2, v_2, w_2) be its velocities just before and just after the impulse. Let the rest of the notation be the same as that used in Art. 295. Then

$$u_2-u_1=u'+\omega_y'r'-\omega_z'q',$$

with similar equations for v_2-v_1, w_2-w_1. Substituting in these equations the value of u', v', w', ω_x', ω_y', ω_z' given by Art. 301 we see that u_2-u_1, v_2-v_1, w_2-w_1 are all linear functions of X, Y, Z of the first degree of the form

$$u_2-u_1=FX+GY+HZ,$$

where F, G, H are functions of the structure of the body and the co-ordinates of the two points.

304. When the point whose initial motion is required is the point of application of the blow, and the axes of reference the principal axes at the centre of gravity, these expressions take the simple forms

$$u_2 - u_1 = \left(\frac{1}{M} + \frac{r^2}{B} + \frac{q^2}{C}\right) X - \frac{pq}{C} Y - \frac{pr}{B} Z,$$

$$v_2 - v_1 = -\frac{pq}{C} X + \left(\frac{1}{M} + \frac{p^2}{C} + \frac{r^2}{A}\right) Y - \frac{qr}{A} Z,$$

$$w_2 - w_1 = -\frac{pr}{B} X - \frac{qr}{A} Y + \left(\frac{1}{M} + \frac{q^2}{A} + \frac{p^2}{B}\right) Z.$$

The right-hand sides of these equations are the differential coefficients of a quadratic function of X, Y, Z, which we may call E. It follows that for all blows at the same point P of the same body the resultant change in the velocity of the point P of application is perpendicular to the diametral plane of the direction of the blow with regard to a certain ellipsoid whose centre is at P, and whose equation is $E =$ constant.

The expression for E may be written in either of the equivalent forms

$$2E = \frac{X^2 + Y^2 + Z^2}{M} + \frac{1}{ABC}\{(Ap^2 + Bq^2 + Cr^2)(AX^2 + BY^2 + CZ^2) - (ApX + BqY + CrZ)^2\}$$

$$= \frac{X^2 + Y^2 + Z^2}{M} + \frac{1}{A}(qZ - rY)^2 + \frac{1}{B}(rX - pZ) + \frac{1}{C}(pY - qX)^2.$$

In this latter form we see that it is

$$= M(u'^2 + v'^2 + w'^2) + A\omega_x'^2 + B\omega_y'^2 + C\omega_z'^2,$$

which is the vis viva of the motion generated by the impulse.

Impact of Rough Elastic Bodies.

305. The problem of determining the motion of any two bodies after a collision involves some rather long analysis and yet there are some points in which it differs essentially from the same problem considered in two dimensions. We shall, therefore, first consider a special problem which admits of being treated briefly, and will then apply the same principles to the general problem in three dimensions.

306. *Two rough ellipsoids moving in any manner impinge on each other so that the extremity of a principal diameter of one strikes the extremity of a principal diameter of the other, and at that instant the three principal diameters of one are parallel to those of the other. Find the motion just after impact.*

Let us refer the motion to co-ordinate axes parallel to the principal diameters of either ellipsoid at the beginning of the impact. Then since the duration of the impact is indefinitely small and the velocities are finite, the bodies will not have time to change

their position, and therefore the principal diameters will be parallel to the co-ordinate axes throughout the impact.

Let U, V, W be the resolved velocities of the centre of gravity of one body just before impact; u, v, w the resolved velocities at any time t after the beginning of the impact, but before its termination. Let Ω_x, Ω_y, Ω_z be the angular velocities of the body just before impact about its principal diameters at the centre of gravity; ω_x, ω_y, ω_z the angular velocities at the time t. Let a, b, c be the semiaxes of the ellipsoid, and A, B, C the moments of inertia at the centre of gravity about these axes respectively. Let M be the mass of the body. Let accented letters denote the same quantities for the other body. Let the bodies impinge at the extremities of the axes of c, c'.

Let P, Q, R be the resolved parts parallel to the axes of the momentum generated in the body M by the blow during the time t. Then $-P$, $-Q$, $-R$ are the resolved parts of the momentum generated in the other body in the same time.

The equations of motion of the body M are

$$\left.\begin{aligned} A(\omega_x - \Omega_x) &= Qc \\ B(\omega_y - \Omega_y) &= -Pc \\ C(\omega_z - \Omega_z) &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

$$\left.\begin{aligned} M(u - U) &= P \\ M(v - V) &= Q \\ M(w - W) &= R \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

There will be six corresponding equations for the other body which may be derived from these by accenting all the letters on the left-hand sides and writing $-P$, $-Q$, $-R$ and $-c'$ for P, Q, R and c on the right-hand sides of these equations. Let us call these new equations respectively (3) and (4).

Let S be the velocity with which one ellipsoid slides along the other, and θ the angle the direction of sliding makes with the axis of x, then

$$S\cos\theta = u' + c'\omega_y' - u + c\omega_y \dots\dots\dots\dots\dots\dots (5),$$

$$S\sin\theta = v' - c'\omega_x' - v + c\omega_x \dots\dots\dots\dots\dots\dots (6).$$

Let C be the relative velocity of compression, then

$$C = w' - w \dots\dots\dots\dots\dots\dots\dots\dots\dots (7).$$

Substituting in these equations from the dynamical equations we have

$$S\cos\theta = S_0\cos\theta_0 - pP \dots\dots\dots\dots\dots\dots (8),$$

$$S\sin\theta = S_0\sin\theta_0 - qQ \dots\dots\dots\dots\dots\dots (9),$$

$$C = C_0 - rR \dots\dots\dots\dots\dots\dots\dots\dots (10),$$

where

$$\left.\begin{aligned} S_0 \cos\theta_0 &= U' + c'\Omega_y' - U + c\Omega_y \\ S_0 \sin\theta_0 &= V' - c'\Omega_x' - V + c\Omega_x \\ C_0 &= W' - W \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(11),$$

$$\left.\begin{aligned} p &= \frac{1}{M} + \frac{1}{M'} + \frac{c^2}{B} + \frac{c'^2}{B'} \\ q &= \frac{1}{M} + \frac{1}{M'} + \frac{c^2}{A} + \frac{c'^2}{A'} \\ r &= \frac{1}{M} + \frac{1}{M'} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(12).$$

These are the constants of the impact. S_0, C_0 are the initial velocities of sliding, and θ_0 the angle the direction of initial sliding makes with the axis of x. Let us take as the standard case that in which the body M' is sliding along and compressing the body M, so that S_0 and C_0 are both positive. The other three constants p, q, r are independent of the initial motion and are essentially positive quantities.

307. Exactly as in two dimensions we shall adopt a graphical method of tracing the changes which occur in the frictions. Let us measure along the axes of x, y, z three lengths OP, OQ, OR to represent the three reactions P, Q, R. Then if these be regarded as the co-ordinates of a point T, the motion of T will represent the changes in the forces. It will be convenient to trace the loci given by $S = 0$, $C = 0$. The locus given by $S = 0$ is a straight line parallel to the axis of R, which we may call the *line of no sliding*. The locus given by $C = 0$ is a plane parallel to the plane P, Q, which we may call the *plane of greatest compression.* At the beginning of the impact one ellipsoid is sliding along the other, so that according to Art. 144 the friction called into play is limiting. Since P, Q, R are the whole resolved momenta generated in the time t; dP, dQ, dR will be the resolved momenta generated in the time dt, the two former being due to the frictional, and the latter to the normal blow. Then the direction of the resultant of dP, dQ must be opposite to the direction in which one point of contact slides over the other, and the magnitude of the resultant must be equal to μdR, where μ is the coefficient of friction. We have therefore

$$\frac{dP}{dQ} = \cot\theta = \frac{S_0 \cos\theta_0 - pP}{S_0 \sin\theta_0 - qQ} \quad\ldots\ldots\ldots\ldots\ldots(13),$$

$$(dP)^2 + (dQ)^2 = \mu^2 (dR)^2 \quad\ldots\ldots\ldots\ldots\ldots(14).$$

The solution of these equations will indicate the manner in which the representative point T approaches the line of no sliding.

The equation (13) can be solved by separating the variables. We get

$$(S_0 \cos\theta_0 - pP)^{\frac{1}{p}} = \alpha\,(S_0 \sin\theta_0 - qQ)^{\frac{1}{q}},$$

where α is an arbitrary constant. At the beginning of the motion P and Q are zero, hence we have

$$\left(\frac{S_0\cos\theta_0 - pP}{S_0\cos\theta_0}\right)^{\frac{1}{p}} = \left(\frac{S_0\sin\theta_0 - qQ}{S_0\sin\theta_0}\right)^{\frac{1}{q}} \text{ ....(15)},$$

which may also be written

$$\left(\frac{S\cos\theta}{S_0\cos\theta_0}\right)^{\frac{1}{p}} = \left(\frac{S\sin\theta}{S_0\sin\theta_0}\right)^{\frac{1}{q}} \text{ ....(16)},$$

or

$$S = S_0\left(\frac{\sin\theta}{\sin\theta_0}\right)^{\frac{p}{q-p}}.\left(\frac{\cos\theta_0}{\cos\theta}\right)^{\frac{q}{q-p}} \text{ ....(17)}.$$

This equation gives the relation between the direction and the velocity of sliding.

308. If the direction of sliding does not change during the impact θ must be constant and equal to θ_0. We see from (16) that if $p = q$, then $\theta = \theta_0$; and conversely if $\theta = \theta_0$, S would be constant unless $p = q$. Also if $\sin\theta_0$ or $\cos\theta_0$ be zero, S would be zero or infinite unless $\theta = \theta_0$. The necessary and sufficient condition that the direction of friction should not change during the impact is therefore $p = q$ or $\sin 2\theta_0 = 0$. The former of these two conditions by (12) leads to

$$c^2\left(\frac{1}{A} - \frac{1}{B}\right) + c'^2\left(\frac{1}{A'} - \frac{1}{B'}\right) = 0 \text{..............(18)}.$$

If this condition holds, we have by (13) $P = Q\cot\theta_0$ and therefore by (14)

$$\left.\begin{aligned} P &= \mu R\cos\theta_0 \\ Q &= \mu R\sin\theta_0 \end{aligned}\right\} \text{ ....(19)}.$$

It follows from these equations that when the friction is limiting, the representative point T moves along a straight line making an angle $\tan^{-1}\mu$ with the axis of R, in such a direction as to meet the straight line of no sliding.

309. If the condition $p = q$ does not hold, we may, by differentiating (8) and (9) and eliminating P, Q, and S, reduce the determination of R in terms of θ to an integral.

By substituting for S from (17) in (8) and (9), we then have P, Q, R expressed as functions of θ. Thus we have the equations to the curve along which the representative point T travels. The curve along which T travels may more conveniently be

defined by the property that its tangent by (14) makes a constant angle $\tan^{-1}\mu$ with the axis of R and its projection on the plane of PQ is given by (15). And it follows that this curve must meet the straight line of no sliding, for the equation (15) is satisfied by $pP = S_0 \cos\theta_0$, $qQ = S_0 \sin\theta_0$.

310. The whole progress of the impact may now be traced exactly as in the corresponding problem in two dimensions. The representative point T travels along a certain known curve, until it reaches the line of no sliding. It then proceeds along the line of no sliding, in such a direction that the abscissa R increases. The complete value R_2 of R for the whole impact is found by multiplying the abscissa R_1 of the point at which T crosses the plane of greatest compression by $1+e$ so that $R_2 = R_1(1+e)$, if e be the measure of the elasticity of the two bodies. The complete values of the frictions called into play are the ordinates of the position of T corresponding to the abscissa $R = R_2$. Substituting these in the dynamical equations (1), (2), (3), (4), the motion of the two bodies just after impact may be found.

311. Let us consider an example. Since the line of no sliding is perpendicular to the plane of PQ, P and Q are constant when T travels along this line. So that when once the sliding friction has ceased, no more friction is called into play. If therefore sliding ceases at any instant before the termination of the impact as when the bodies are either very rough or perfectly rough, the whole frictional impulses are given by

$$P = \frac{S_0 \cos\theta_0}{p}, \quad Q = \frac{S_0 \sin\theta_0}{q}.$$

If σ be the arc of the curve whose equation is (15) from the origin to the point where it meets the line of no sliding, then the representative point T cuts the line of no sliding at a point whose abscissa is $R = \dfrac{\sigma}{\mu}$. If the bodies be so rough that $\dfrac{\sigma}{\mu} < \dfrac{C_0}{r}$, the point T will not cross the plane of greatest compression until after it has reached the line of no sliding. The whole normal impulse in this case is therefore given by $R = \dfrac{C_0}{r}(1+e)$. Substituting these values of P, Q, R in the dynamical equations, the motion just after impact may be found.

312. Ex. 1. If θ be the angle the direction of sliding of one ellipsoid over the other makes with the axis of x, prove that θ continually increases or continually decreases throughout the impact. And if the initial value of θ lie between 0 and $\dfrac{\pi}{2}$, then θ approaches $\dfrac{\pi}{2}$ or zero according as p is $>$ or $< q$. Show also that the representative point reaches the line of no sliding when θ has either of these values.

Ex. 2. If the bodies be such that the direction of sliding continues unchanged during the impact and the sliding ceases before the termination of the impact, the roughness must be such that $\mu > \frac{S_0 r}{C_0 p (1+e)}$.

Ex. 3. If two rough spheres impinge on each other, prove that the direction of sliding is the same throughout the impact. This proposition was first given by Coriolis. *Jeu de billard*, 1835.

Ex. 4. If two inelastic solids of revolution impinge on each other, the vertex of each being the point of contact, prove that the direction of sliding is the same throughout the impact. This and the next proposition have been given by M. Phillips in the fourteenth volume of *Liouville's Journal*.

Ex. 5. If two bodies having their principal axes at their centres of gravity parallel impinge so that these centres of gravity are in the common normal at the point of contact and if the initial direction of sliding be parallel to a principal axis at either centre of gravity, then the direction of sliding will be the same throughout the impact.

Ex. 6. If two ellipsoids of equal masses impinge on each other at the extremities of their axes of c, c', and if $aa' = bb'$ and $ca' = bc'$, prove that the direction of friction is constant throughout the impact.

313. *Two rough bodies moving in any manner impinge on each other. Find the motion just after impact.*

Let us refer the motion to co-ordinate axes, the axes of x, y being in the tangent plane at the point of impact and the axis of z along the normal. Let U, V, W be the resolved velocities of the centre of gravity of one body just before impact, u, v, w the resolved velocities at any time t after the beginning, but before the termination of the impact. Let Ω_x, Ω_y, Ω_z be the angular velocities of the same body just before impact about axes parallel to the co-ordinate axes, meeting at the centre of gravity; ω_x, ω_y, ω_z the angular velocities at the time t. Let A, B, C, D, E, F be the moments and products of inertia about axes parallel to the co-ordinate axes meeting at the centre of gravity. Let M be the mass of the body. Let accented letters denote the same quantities for the other body.

Let P, Q, R be the resolved parts parallel to the axes of the momentum generated in the body M from the beginning of the impact, up to the time t. Then $-P$, $-Q$, $-R$ are the resolved parts of the momentum generated in the other body in the same time.

Let (x, y, z) (x', y', z') be the co-ordinates of the centres of gravity of the two bodies referred to the point of contact as origin. The equations of motion are therefore

$$\left.\begin{aligned} A(\omega_x - \Omega_x) - F(\omega_y - \Omega_y) - E(\omega_z - \Omega_z) &= -yR + zQ \\ -F(\omega_x - \Omega_x) + B(\omega_y - \Omega_y) - D(\omega_z - \Omega_z) &= -zP + xR \\ -E(\omega_x - \Omega_x) - D(\omega_y - \Omega_y) + C(\omega_z - \Omega_z) &= -xQ + yP \end{aligned}\right\} \quad \text{..............(1)}.$$

$$\left.\begin{aligned} M(u - U) &= P \\ M(v - V) &= Q \\ M(w - W) &= R \end{aligned}\right\} \quad \text{..................................(2)}.$$

We have six similar equations for the other body, which differ from these in having all the letters, except P, Q, R, accented, and in having the signs of P, Q, R changed. These we shall call equations (3) and (4).

Let S be the velocity with which one body slides along the other and let θ be the angle the direction of sliding makes with the axis of x. Also let C be the relative velocity of compression, then

$$\left.\begin{aligned} S\cos\theta &= u' - \omega_y' z' + \omega_z' y' - u + \omega_y z - \omega_z y \\ S\sin\theta &= v' - \omega_z' x' + \omega_x' z' - v + \omega_z x - \omega_x z \\ C &= w' - \omega_x' y' + \omega_y' x' - w + \omega_x y - \omega_y x \end{aligned}\right\} \quad \text{......................} \quad (5).$$

If we substitute from (1) (2) (3) (4) in (5) we find

$$\left.\begin{aligned} S_0\cos\theta_0 - S\cos\theta &= aP + fQ + eR \\ S_0\sin\theta_0 - S\sin\theta &= fP + bQ + dR \\ C_0 - C &= eP + dQ + cR \end{aligned}\right\} \quad \text{.........................} \quad (6),$$

where S_0, θ_0, C_0 are the initial values of S, θ, C and are found from (5) by writing for the letters their initial values. The expressions for a, b, c, d, e, f are rather complicated, but it is unnecessary to calculate them.

314. We may now trace the whole progress of the impact by the use of a graphical method. Let us measure from the point of contact O, along the axes of co-ordinates, three lengths OP, OQ, OR to represent the three reactions P, Q, R. Then if, as before, these be regarded as the co-ordinates of a point T, the motion of T will represent the changes in the forces. The equations to the line of no sliding are found by putting $S=0$ in the first two of equations (6). We see that it is a straight line.

The equation to the plane of greatest compression is found by putting $C=0$ in the third of equations (6).

At the beginning of the impact one body is sliding along the other, so that the friction called into play is limiting. The path of the representative point as it travels from O is given, as before, by

$$\frac{dP}{\cos\theta} = \frac{dQ}{\sin\theta} = \mu dR \text{.................................} (7).$$

When the representative point T reaches the line of no sliding, the sliding of one body along the other ceases for the instant. After this, only so much friction is called into play as will suffice to prevent sliding, provided this amount is less than the limiting friction. If therefore the angle the line of no sliding makes with the axis of R be less than $\tan^{-1}\mu$, the point T will travel along it. But if the angle be greater than $\tan^{-1}\mu$, more friction is necessary to prevent sliding than can be called into play. Accordingly the friction will continue to be limiting, but its direction will be changed if S changes sign. The point T will then travel along a curve given by equations (7) with θ increased by π.

The complete value R_2 of R for the whole impact is found by multiplying the abscissa R of the point at which T crosses the plane of greatest compression by $1+e$, where e is the measure of elasticity, so that $R_2 = R_1(1+e)$. The complete values of P and Q are represented by the ordinates corresponding to the abscissa R_2. Substituting in the dynamical equations, the motion just after impact may be found.

315. The path of the representative point before it reaches the line of no sliding must be found by integrating (7). By differentiating (6) we have

$$\frac{d(S\cos\theta)}{d(S\sin\theta)} = \frac{adP + fdQ + edR}{fdP + bdQ + ddR} = \frac{a\mu\cos\theta + f\mu\sin\theta + e}{f\mu\cos\theta + b\mu\sin\theta + d},$$

which reduces to

$$\frac{1}{S}\frac{dS}{d\theta}=\frac{\frac{a+b}{2}+\frac{a-b}{2}\cos 2\theta+f\sin 2\theta+\frac{e}{\mu}\cos\theta+\frac{d}{\mu}\sin\theta}{-\frac{a-b}{2}\sin 2\theta+f\cos 2\theta+\frac{d}{\mu}\cos\theta-\frac{e}{\mu}\sin\theta}.$$

From this equation we may find S as a function of θ in the form $S=Af(\theta)$, the constant A being determined from the condition that $S=S_0$ when $\theta=\theta_0$. Differentiating the first of equations (6) and substituting from (7) we get

$$-Ad\{\cos\theta f(\theta)\}=(\mu a\cos\theta+\mu f\sin\theta+e)\,dR,$$

whence we find $R=AF(\theta)+B$, the constant B being determined from the condition that R vanishes when $\theta=\theta_0$. By substituting these values of S and R in the first two equations of (6) we find P and Q in terms of θ. The three equations giving P, Q, R as functions of θ are the equations to the path of the representative point. It should be noticed that the tangent to the path at any point makes with the axis of R an angle equal to $\tan^{-1}\mu$.

316. If the direction of friction does not change during the impact, θ is constant and equal to θ_0, so that θ cannot be chosen as the independent variable. In this case $P=\mu R\cos\theta_0$, $Q=\mu R\sin\theta_0$ and the representative point moves along a straight line making with the axis of R an angle $\tan^{-1}\mu$. Substituting these values of P and Q in the first two of equations (6) we have

$$-\frac{a-b}{2}\sin 2\theta_0+f\cos 2\theta_0+\frac{d}{\mu}\cos\theta_0-\frac{e}{\mu}\sin\theta_0=0,$$

as a necessary condition that the direction of friction should not change. Conversely if this condition is satisfied the equations (6) and (7) may all be satisfied by making θ constant. In this case it is also easy to see that the path of the representative point intersects the line of no sliding. If S_0 be zero, and if more friction is necessary to prevent sliding than can be called into play, the initial value of θ is unknown. But if θ_0 be taken equal to that root of the above equation which makes S positive, and if θ be supposed constant, the equations (6) and (7) are all satisfied.

317. Ex. 1. Let $G=\begin{vmatrix} A & -F & -E & yR-zQ \\ -F & B & -D & zP-xR \\ -E & -D & C & xQ-yP \\ yR-zQ & zP-xR & xQ-yP & 0 \end{vmatrix}$

and let Δ be the determinant obtained by leaving out the last row and last column. Let G', Δ' be the corresponding expressions for the other body. Then a, b, c, d, e, f are the coefficients of P^2, Q^2, R^2, $2QR$, $2RP$, $2PQ$ in the quadric

$$\left(\frac{1}{M}+\frac{1}{M'}\right)(P^2+Q^2+R^2)+\frac{1}{\Delta}G+\frac{1}{\Delta'}G'=2E,$$

where $2E$ is a constant, which may be shown to be the sum of the vires vivæ of the motions generated in the two bodies, as explained in Art. 304.

This quadric may be shown to be an ellipsoid by comparing its equation with that given in Art. 28, Ex. 3.

Show also that a, b, c are necessarily positive and $ab>f^2$, $bc>d^2$, $ca>e^2$.

Show that by turning the axes of reference round the axis of R through the proper angle we can make f zero.

Ex. 2. Prove that the line of no sliding is parallel to the conjugate diameter of the plane containing the frictions P, Q. And the plane of greatest compression is the diametral plane of the reaction R.

Ex. 3. The line of no sliding is the intersection of the polar planes of two points situated on the axes of P and Q and distant respectively from the origin $\dfrac{2E}{S_0 \cos\theta_0}$ and $\dfrac{2E}{S_0 \sin\theta_0}$. The plane of greatest compression is the polar plane of a point on the axis of R distant $\dfrac{2E}{C_0}$ from the origin.

Ex. 4. The plane of PQ cuts the ellipsoid of Ex. 1 in an ellipse, whose axes divide the plane into four quadrants; the line of no sliding cuts the plane of PQ in that quadrant in which the initial sliding S_0 occurs.

Ex. 5. A parallel to the line of no sliding through the origin cuts the plane of greatest compression, in a point whose abscissa R has the same sign as C_0. Hence show, from geometrical considerations, that the representative point T must cross the plane of greatest compression.

EXAMPLES*.

1. A cone revolves round its axis with a known angular velocity. The altitude begins to diminish and the angle to increase, the volume being constant. Show that the angular velocity is proportional to the altitude.

2. A circular disc is revolving in its own plane about its centre; if a point in the circumference become fixed, find the new angular velocity.

3. A uniform rod of length $2a$ lying on a smooth horizontal plane passes through a ring which permits the rod to rotate freely in the horizontal plane. The middle point of the rod being indefinitely near the ring any angular velocity is impressed on it, show that when it leaves the ring the radius vector of the middle point will have swept out an area equal to $\dfrac{a^2}{6}$.

4. An elliptic lamina is rotating about its centre on a smooth horizontal table. If ω_1, ω_2, ω_3 be its angular velocities respectively when the extremities of its major axis, its focus, and the extremity of the minor axis become fixed, prove

$$\frac{7}{\omega_1} = \frac{6}{\omega_2} + \frac{5}{\omega_3}.$$

5. A rigid body moveable about a fixed point O at which the principal moments are A, B, C is struck by a blow of given magnitude at a given point. If the angular velocity thus impressed on the body be the greatest possible, prove that (a, b, c) being the co-ordinates of the given point referred to the principal axes at O, and (l, m, n) the direction cosines of the blow, then

$$al + bm + cn = 0,$$

$$\frac{a}{l}\left(\frac{1}{B^2} - \frac{1}{C^2}\right) + \frac{b}{m}\left(\frac{1}{C^2} - \frac{1}{A^2}\right) + \frac{c}{n}\left(\frac{1}{A^2} - \frac{1}{B^2}\right) = 0.$$

* These examples are taken from the Examination Papers which have been set in the University and in the Colleges.

6. Any triangular lamina ABC has the angular point C fixed and is capable of free motion about it. A blow is struck at B perpendicular to the plane of the triangle. Show that the initial axis of rotation is that trisector of the side AB which is furthest from B.

7. A cone of mass m and vertical angle 2α can move freely about its axis, and has a fine smooth groove cut along its surface so as to make a constant angle β with the generating lines of the cone. A heavy particle of mass P moves along the groove under the action of gravity, the system being initially at rest with the particle at a distance c from the vertex. Show that if θ be the angle through which the cone has turned when the particle is at any distance r from the vertex, then

$$\frac{mk^2 + Pr^2 \sin^2 \alpha}{mk^2 + Pc^2 \sin^2 \alpha} = \epsilon^{2\theta \sin \alpha . \cot \beta},$$

k being the radius of gyration of the cone about its axis.

8. A body is turning about an axis through its centre of gravity, a point in the body becomes suddenly fixed. If the new instantaneous axis be a principal axis with respect to the point, show that the locus of the point is a rectangular hyperbola.

9. A cube is rotating with angular velocity ω about a diagonal, when one of its edges which does not meet the diagonal suddenly becomes fixed. Show that the angular velocity about this edge as axis $= \dfrac{\omega}{4\sqrt{3}}$.

10. Two masses m, m' are connected by a fine smooth string which passes round a right circular cylinder of radius a. The two particles are in motion in one plane under no impressed forces, show that if A be the sum of the absolute areas swept out in a time t by the two unwrapped portions of the string,

$$\frac{d^2A}{dt^2} = \frac{1}{2} a \left(\frac{1}{m} + \frac{1}{m'}\right) T,$$

T being the tension of the string at any time.

11. A piece of wire in the form of a circle lies at rest with its plane in contact with a smooth horizontal table, when an insect on it suddenly starts walking along the arc with uniform relative velocity. Show that the wire revolves round its centre with uniform angular velocity while that centre describes a circle in space with uniform angular velocity.

12. A uniform circular wire of radius a, moveable about a fixed point in its circumference, lies on a smooth horizontal plane. An insect of mass equal to that of the wire crawls along it, starting from the extremity of the diameter opposite to the fixed point, its velocity relative to the wire being uniform and equal to V. Prove that after a time t the wire will have turned through an angle

$$\frac{Vt}{2a} - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{Vt}{2a}\right).$$

13. A small insect moves along a uniform bar of mass equal to itself, and length $2a$, the extremities of which are constrained to remain on the circumference of a fixed circle, whose radius is $\dfrac{2a}{\sqrt{3}}$. Supposing the insect to start from the middle

point of the bar, and its velocity relatively to the bar to be uniform and equal to V; prove that the bar in time t will turn through an angle $\frac{1}{\sqrt{3}}\tan^{-1}\frac{Vt}{a}$.

14. A rough circular disc can revolve freely in a horizontal plane about a vertical axis through its centre. An equiangular spiral is traced on the disc having the centre for pole. An insect whose mass is an n^{th} that of the disc crawls along the curve starting from the point at which it cuts the edge. Show that when the insect reaches the centre the disc will have revolved through an angle $\frac{\tan\alpha}{2}\log\left(1+\frac{2}{n}\right)$, where α is the angle between the tangent and radius vector at any point of the spiral.

15. A uniform circular disc moveable about its centre in its own plane (which is horizontal) has a fine groove in it cut along a radius, and is set rotating with an angular velocity ω. A small rocket whose weight is an nth of the weight of the disc is placed at the inner extremity of the groove and discharged; and when it has left the groove, the same is done with another equal rocket, and so on. Find the angular velocity after n of these operations, and if n be indefinitely increased, show that the limiting value of the same is ωe^{-2}.

16. A rigid body is rotating about an axis through its centre of gravity, when a certain point of the body becomes suddenly fixed, the axis being simultaneously set free; find the equations of the new instantaneous axis; and prove that, if it be parallel to the originally fixed axis, the point must lie in the line represented by the equations $a^2lx+b^2my+c^2nz=0$, $(b^2-c^2)\frac{x}{l}+(c^2-a^2)\frac{y}{m}+(a^2-b^2)\frac{z}{n}=0$; the principal axes through the centre of gravity being taken as axes of co-ordinates, a, b, c the radii of gyration about these lines, and l, m, n the direction-cosines of the originally fixed axis referred to them.

17. A solid body rotating with uniform velocity ω about a fixed axis contains a closed tubular channel of small uniform section filled with an incompressible fluid in relative equilibrium; if the rotation of the solid body were suddenly destroyed the fluid would move in the tube with a velocity $\frac{2A\omega}{l}$, where A is the area of the projection of the axis of the tube on a plane perpendicular to the axis of rotation and l is the length of the tube.

18. A gate without a latch in the form of a rectangular lamina is fitted with a universal joint at the upper corner and at the lower corner there is a short bar normal to the plane of the gate and projecting equally on both sides of it. As the gate swings to either side from its stable position of rest, one or other end of the bar becomes a fixed point. If h be the height of the gate, $h\tan\alpha$ its length and 2β the angle which the bar subtends at the upper corner, show that the angular velocity of the gate as it passes through the position of rest is impulsively diminished in the ratio $\frac{\sin^2\alpha-\tan^2\beta}{\sin^2\alpha+\tan^2\beta}$, and the time between successive impacts when the oscillations become small decreases in the same ratio, the weights of the bar and joint being neglected.

CHAPTER VII.

VIS VIVA.

The Force-function and Work.

318. If a particle of mass m be projected along the axis of x with an initial velocity V and be acted on by a force F in the same direction, the motion is given by the equation $m\dfrac{d^2x}{dt^2}=F$.

Integrating this with regard to t, if v be the velocity after a time t, we have,

$$m(v-V)=\int_0^t F\,dt.$$

If we multiply both sides of the differential equation of the second order by $\dfrac{dx}{dt}$ and integrate, we get*

$$\frac{1}{2}m(v^2-V^2)=\int_0^x F\,dx.$$

* It is seldom that Mathematicians can be found engaged in a controversy such as that which raged for forty years in the last century. The object of the dispute was to determine how the force of a body in motion was to be measured. Up to the year 1686, the measure taken was the product of the mass of the body into its velocity. Leibnitz, however, thought he perceived an error in the common opinion, and undertook to show that the proper measure should be, the product of the mass into the square of the velocity. Shortly all Europe was divided between the rival theories. Germany took part with Leibnitz and Bernoulli; while England, true to the old measure, combated their arguments with great success. France was divided, an illustrious lady, the Marquise du Chatelet, being first a warm supporter and then an opponent of Leibnitzian opinions. Holland and Italy were in general favourable to the German philosopher. But what was most strange in this great dispute was, that the same problem, solved by geometers of opposite opinions, had the same solution. However the force was measured, whether by the first or the second power of the velocity, the result was the same. The arguments and replies advanced on both sides are briefly given in Montucla's *History*, and are most interesting. For this however we have no space. The controversy was at last closed by D'Alembert, who showed in his treatise on Dynamics that the whole dispute was a mere question of words. When we speak, he says, of the force of a moving body, we either attach no clear meaning to the word or we understand only the property that certain resistances can be overcome by the moving body. It

The first of these integrals shows that the change of the momentum is equal to the time-integral of the force. By applying similar reasoning to the motion of a dynamical system we have been led in the last chapter to the general principle enunciated in Art. 279, and afterwards to its application to determine the changes produced by very great forces acting for a very short time. The second integral shows that half the change of the vis viva is equal to the space-integral of the force. It is our object in this chapter to extend this result also, and to apply it to the general motion of a system of bodies.

319. For the purposes of description it will be convenient to give names to the two sides of this equation. * Twice the left-hand side is usually called the vis viva of the particle, a term introduced by Leibnitz about the year 1695. Half the vis viva is also called the kinetic energy of the particle. Many names have been given to the right-hand side at various times. It is now commonly called the work of the force F. When the force does not act in the direction of the motion of its point of application the term "work" will require a more extended definition. This we shall discuss in the next article.

320. Let a force F act at a point A of a body in the direction AB, and let us suppose the point A to move into any other position A' very near A. If ϕ be the angle the direction AB of the force makes with the direction AA' of the displacement of the point of application, then the product $F.AA'.\cos\phi$ is called the work done by the force. If for ϕ we write the angle the direction AB of the force makes with the direction $A'A$ opposite to the displacement, the product is called the work done against the force. If we drop a perpendicular $A'M$ on AB, the work done by the force is also equal to the product $F.AM$, where AM is to be estimated as positive when in the direction of the force. If F' be the resolved part of F in the direction of the displacement, the work is also equal to $F'.AA'$. If several forces act, we can in the same way find the work done by each. The sum of all these is the work done by the whole system of forces.

is not then by any simple considerations of merely the mass and the velocity of the body that we must estimate this force, but by the nature of the obstacles overcome. The greater the resistance overcome, the greater we may say is the force; provided we do not understand by this word a pretended existence inherent in the body, but simply use it as an abridged mode of expressing a fact. D'Alembert then points out that there are different kinds of obstacles and examines how their different kinds of resistances may be used as measures. It will perhaps be sufficient to observe, that the resistance may in some cases be more conveniently measured by a space-integral and in others by a time-integral. See Montucla's *History*, Vol. III. and Whewell's *History*, Vol. II.

Thus defined, the work done by a force, corresponding to any indefinitely small displacement, is the same as the virtual moment of the force. In Statics, we are only concerned with the small hypothetical displacements, we give the system in applying the principle of Virtual Velocities, and this definition is therefore sufficient. But in Dynamics the bodies are in motion, and we must extend our definition of work to include the case of a displacement of any magnitude. When the points of application of the forces receive finite displacements we must divide the path of each into elements. The work done in each element may be found by the definition given above. The sum of all these is the whole work.

It should be noticed that the work done by given forces as the body moves from one given position to another, is independent of the time of transit. As stated in Art. 318, the work is a space-integral and not a time-integral.

321. *If two systems of forces be equivalent, the work done by one in any small displacement is equal to that done by the other.* This follows at once from the principle of Virtual Velocities in Statics. For if every force in one system be reversed in direction without altering its point of application or its magnitude, the two systems will be in equilibrium, and the sum of their virtual moments will therefore be zero. Restoring the system of forces to its original state, we see that the virtual moments of the two systems are equal. If the displacements are finite the same remark applies to each successive element of the displacement, and therefore to the whole displacement.

322. We may now find an analytical expression for the work done by a system of forces. Let (x, y, z) be the rectangular co-ordinates of a particle of the system and let the mass of this particle be m. Let (X, Y, Z) be the accelerating forces acting on the particle resolved parallel to the axes of co-ordinates. Then mX, mY, mZ are the dynamical measures of the acting forces. Let us suppose the particle to move into the position $x+dx$, $y+dy$, $z+dz$; then according to the definition the work done by the forces will be

$$\Sigma\,(mXdx + mYdy + mZdz)\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the summation extending to all the forces of the system. If the bodies receive any finite displacements, the whole work will be

$$\Sigma m\int(Xdx + Ydy + Zdz)\;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

the limits of the integral being determined by the extreme positions of the system.

323. When the forces are such as generally occur in nature, it will be proved that the summation (1) of the last Article is a complete differential, *i.e.* it can be integrated independently of any relation between the co-ordinates x, y, z. The summation (2) can therefore be expressed as a function of the co-ordinates of the system. When this is the case the *indefinite integral* of the summation (2) is called the *force-function.* This name was given to the function by Sir W. R. Hamilton and Jacobi independently of each other.

If the force-function be called U, the work done by the forces when the bodies move from one given position to another is the definite integral $U_2 - U_1$, where U_1 and U_2 are the values of U, corresponding to the two given positions of the bodies. It follows that the work is independent of the mode in which the system moves from the first given position to the second. In other words, the work depends on the co-ordinates of the two given extreme positions, and not on the co-ordinates of any intermediate position. When the forces are such as to possess this property, *i.e.* when they possess a force-function, they have been called a conservative system of forces. This name was given to the system by Sir W. Thomson.

324. *There will be a force-function, first, when the external forces tend to fixed centres at finite distances and are functions of the distances from those centres; and secondly, when the force due to the mutual attractions or repulsions of the particles of the system are functions of the distances between the attracting or repelling particles.*

Let $m\phi(r)$ be the action of any fixed centre of force on a particle m distant r, estimated positive in the direction in which r is measured, *i.e. from* the centre of force. Then the summation (1) in Art. 322 is clearly $\Sigma m\phi(r)\,dr$. This is a complete differential. Thus the force-function exists and is equal to $\Sigma m\int\phi(r)dr$.

Let $mm'\phi(r)$ be the action between two particles m, m' whose distance apart is r, and as before let this force be considered positive when repulsive. Then the summation (1) becomes $\Sigma mm'\phi(r)\,dr$. The force-function therefore exists, and is equal to $\Sigma mm'\int\phi(r)\,dr$.

If the law of attraction be the inverse square of the distance, $\phi(r) = -\dfrac{1}{r^2}$ and the integral is $\dfrac{1}{r}$. Thus the force-function differs from the Potential by a constant quantity.

325. It is clear that there is nothing in the definition of the force-function to compel us to use Cartesian Co-ordinates. If

P, Q, &c. be forces acting on a particle, dp, dq, &c. their virtual velocities, m the mass of the particle, then the force-function is

$$U = \Sigma m \int (Pdp + Qdq + \&c.),$$

the summation extending to all the forces of the system.

Ex. 1. If (ρ, ϕ, z) be the cylindrical or semi-polar co-ordinates of the particle m; P, Q, Z the resolved parts of the forces respectively along and perpendicular to ρ and along z, prove that $dU = \Sigma m\,(Pd\rho + Q\rho d\phi + Zdz)$.

Ex. 2. If (r, θ, ϕ) be the polar co-ordinates of the particle m; P, Q, R the resolved parts of the forces respectively along the radius vector, perpendicular to it in the plane of θ and perpendicular to that plane, prove that

$$dU = \Sigma m\,(Pdr + Qrd\theta + Rr\sin\theta d\phi).$$

Ex. 3. If (x, y, z) be the oblique Cartesian co-ordinates of m; X, Y, Z the components along the axes, prove that

$$dU = \Sigma m \{X(dx + \nu dy + \mu dz) + Y(\nu dx + dy + \lambda dz) + Z(\mu dx + \lambda dy + dz)\},$$

where (λ, μ, ν) are the cosines of the angles between the axes yz, zx, xy respectively. This example is due to Poinsot.

Ex. 4. If the system be referred to rectangular axes moving about a fixed origin, show that the force-function may be found by writing for dx, dy, dz, in Art. 322 the values of udt, vdt, wdt given in Art. 244.

326. *If a system receive any small displacement ds parallel to a given straight line and an angular displacement $d\theta$ round that line, then the partial differential coefficients $\frac{dU}{ds}$ and $\frac{dU}{d\theta}$ represent respectively the resolved part of all the forces along the line and the moment of the forces about it.*

Since dU is the sum of the virtual moments of all the forces due to any displacement, it is independent of any particular co-ordinate axes. Let the straight line along which ds is measured be taken as the axis of z. Taking the same notation as before,

$$dU = \Sigma m\,(Xdx + Ydy + Zdz).$$

But $dx = 0$, $dy = 0$, and $dz = ds$, hence we have

$$dU = ds\,.\,\Sigma mZ;\ \therefore \frac{dU}{ds} = \Sigma mZ.$$

Here dU means the change produced in U by the single displacement of the system, taken as one body, parallel to the given straight line, through a space ds.

Again, the moment of all the forces about the axis of z is $\Sigma m\,(xY - yX)$, but $dx = -yd\theta$, $dy = xd\theta$, and $dz = 0$. Hence the above moment

$$= \Sigma m \frac{Ydy + Xdx + Zdz}{d\theta} = \frac{dU}{d\theta}.$$

Here dU is the change produced in U by the single rotation of the system, taken as one body, round the given axis through an angle $d\theta$.

327. As considerable use will be made of the force-function, the student will find it advantageous to acquire a facility in writing down its form. The following examples have therefore been given.

Ex. An elastic string whose unstretched length is l is stretched, find the work done by the tension when the string is stretched from a length r to a length r'.

Let ρ be any length intermediate between r and r' and let E be the coefficient of elasticity. The tension is $T=E\dfrac{\rho-l}{l}$ and acts opposite to the direction in which ρ is measured. The work done while ρ becomes $\rho+d\rho$ is therefore equal to $-Td\rho$. The force-function is therefore $-\int Td\rho$. If this be integrated and taken between the limits r to r', we find the required work equal to $-\dfrac{E}{2l}\{(r'-l)^2-(r-l)^2\}$.

It follows from this that the work required to stretch an elastic string from one length to another is the product of the arithmetic mean of the initial and final tensions into the extension of the string.

328. Ex. 1. A system of bodies falls under the action of gravity. If M be the whole mass, h the space descended by the centre of gravity of the whole system, the work done by gravity is Mgh.

Let the axis of z be vertical and let the positive direction be downwards. Then in the summation (1) of Art. 322, $X=0$, $Y=0$ and $Z=g$. Hence $dU=\Sigma mgdz$. If $\bar{z}$ be the depth of the centre of gravity below the plane of xy, and C be any constant, we find $U=Mg\bar{z}+C$. Taking this between limits we easily obtain the result given.

The theoretical unit of work is the work done by a dynamical unit of force acting through a unit of space. We may use the result of this example to supply a practical unit. The work required to raise the centre of gravity of a given mass a given height at a given place may be taken as the unit of work. English engineers use a pound for the mass and a foot for the height, and the unit is then called a *foot-pound*. The term *Horse-power* is used to express the work done per unit of time. The unit of horse-power is usually taken to be 33000 foot-pounds per minute. The *duty* of a steam-engine is the actual work done by the consumption of a unit quantity, usually a bushel, of coal.

Ex. 2. A force communicates to a particle whose mass is equal to that of a cubic foot of water a velocity of one foot per minute. Find the work done in foot-pounds.

Ex. 3. Prove that the amount of work required to raise to the surface of the earth the homogeneous contents of a very small conical cavity whose vertex is at the centre of the earth, is equal to that which would be expended in raising the whole mass of the contents, through a space equal to one-fifth of the earth's radius from the surface, supposing the force of gravity to remain constant. [Coll. Exam.]

329. Ex. 1. If m, m' be the masses of two particles attracting each other with a force $\dfrac{mm'}{r^2}$ where r is the distance between them, show that the work done when they have moved from an infinite distance apart to a distance r is $\dfrac{mm'}{r}$.

This follows from Art. 324.

Ex. 2. If the particles composing any mass were separated from each other, work might be obtained from their mutual attractions by allowing the particles to approach each other. The work thus obtained is greatest when the particles are collected together from infinite distances. If dv be an element of volume of a solid mass attracting according to the law of nature, ρ the density of the element, V the potential of the solid mass at the element dv, prove that the work performed in collecting the particles composing the mass from infinite distances is $\frac{1}{2}\int V\rho dv$.

Let m_1, m_2, m_3, &c. be the masses of any particles, r_{12}, r_{13}, &c. the distances between the masses m_1, m_2, m_2, m_3, &c. in any arrangement. Then as before the work done in collecting them from infinite distances is $U=\frac{m_1m_2}{r_{12}}+\frac{m_2m_3}{r_{23}}+$ &c., which may be written $U=\Sigma\frac{mm'}{r}$. Now if V_1 be the potential at the particle m_1 of all the particles except m_1 in the given arrangement, $V_1=\frac{m_2}{r_{12}}+\frac{m_3}{r_{13}}+\ldots$ If V_2, V_3, &c. have similar meanings we may write the work in the form

$$U=\frac{1}{2}(V_1m_1+V_2m_2+\ldots)=\frac{1}{2}\Sigma Vm.$$

In finding the potential of any solid mass at any point P we may omit the matter within any indefinitely small element enclosing P if its density be finite. For, since "potential is mass divided by distance," and the mass varies as the cube of the linear dimensions, it follows that the potential of similar figures at points similarly situated must vary as the square of the linear dimensions and must vanish when the mass becomes elementary and the distance indefinitely small. In applying, therefore, the form $U=\frac{1}{2}\Sigma Vm$ to a solid body we may write ρdv for m and take V to be the potential of the whole mass at the element dv.

The problem of determining how much work can be obtained from the bodies forming the solar system by allowing them to consolidate into a solid mass has been considered by several philosophers. Sir W. Thomson has calculated that the potential energy or the work which can be obtained from the existing solar system is $380{,}000\times10^{33}$ foot-pounds. *Edin. Trans.* 1854.

Ex. 3. The particles composing a homogeneous sphere of mass M and radius r were originally at infinite distances from each other. Prove that the work done by their mutual attractions is $\frac{3}{5}\frac{M^2}{r}$.

Ex. 4. The particles of a homogeneous ellipsoid whose mass is M and semiaxes a, b, c are collected from infinite distances, show that the work done is

$$\frac{3}{10}M^2\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

330. Ex. 1. An envelope of any shape and whose volume is v, contains gas at a uniform pressure p. Assuming that the pressure of the gas per unit of area is some function of the volume occupied by it, prove that the work done by the pressures when the volume increases from $v=a$ to $v=b$ is $\int_a^b pdv$.

Divide the surface into elementary areas each equal to $d\sigma$, then $pd\sigma$ is the pressure on $d\sigma$. When the volume has increased to $v+dv$, let any element $d\sigma$ take the position $d\sigma'$ and let dn be the length of the perpendicular drawn from the central point of $d\sigma'$ on the plane of $d\sigma$, then $pd\sigma dn$ is the work done by the pressure on $d\sigma$ and $p\int d\sigma dn$ is the work done over the whole area. But $d\sigma dn$ is the volume of the oblique cylinder whose base is $d\sigma$ and opposite face $d\sigma'$; so that $\int d\sigma dn$ is the whole increment of volume. The whole work done when the volume increases by dv is therefore pdv.

Ex. 2. A spherical envelope of radius a contains gas at pressure P, assuming that the pressure of the gas per unit of area is inversely proportional to the volume occupied by it, prove that the work required to compress the envelope into a sphere of radius b is $4\pi a^3 P \log \frac{a}{b}$.

Ex. 3. An envelope of any shape contains gas and the shape is altered without altering the volume. Show that the work done over the whole surface is zero.

331. Ex. 1. An impulsive force acts on a body in a fixed direction in space. Show that if F be the whole momentum communicated by the force; u_0, u_1 the velocities of the point of application resolved in the direction of the force, just before and just after the impulse, then the work done by the impulse is $\frac{u_0+u_1}{2}F$. This proposition is given in Thomson and Tait's *Natural Philosophy*.

Let us regard the impulse as the limit of a finite force acting in the fixed direction for a very short time T. Let the direction of the axis of x be taken parallel to the fixed direction and let X be the whole momentum communicated during a time t measured from the commencement of the impulse. Here t is any time less than T and X varies from zero to F as t varies from 0 to T. Also, since X is the whole momentum up to the time t, $\frac{dX}{dt}$ is the moving force on the body at the time t. Let u be the resolved velocity of the point of application at the time t, then u_0 and u_1 are the values of u when $t=0$ and $t=T$. Since udt is the space described in the time dt by the point of application of the force $\frac{dX}{dt}$, the work done in the time T is $\int_0^T \frac{dX}{dt} udt$. This is the same as $\int_0^F udX$. Now, when the time t is indefinitely small, the velocity u is known by Art. 303 to be a linear function of X, so that we may write $u=u_0+LX$ where L is a constant depending on the nature of the body. Substituting this value of u, we have the work equal to $\int_0^F (u_0+LX)\,dX = u_0F + L\frac{F^2}{2}$. But $u_1=u_0+LF$. Eliminating L we find that the work $=\frac{1}{2}(u_0+u_1)F$.

Ex. 2. Find the work done by an impulse whose direction is not necessarily the same during the indefinitely short duration of the force.

Let X, Y, Z be the components of the whole momentum given to the body in any time t measured from the commencement of the impulse. Let u, v, w be the resolved velocities of the point of application at the time t. Then, by the same reasoning as before, the work done $=\int_0^T \left(\frac{dX}{dt}u + \frac{dY}{dt}v + \frac{dZ}{dt}w\right)dt$. But by Art. 304

when T is indefinitely small $u=u_0+\frac{dE}{dX}$, $v=v_0+\frac{dE}{dY}$, $w=w_0+\frac{dE}{dZ}$, where E is a known quadratic function of (X, Y, Z) depending on the nature of the body. Substituting we have

$$\text{work}=u_0X_1+v_0Y_1+w_0Z_1+\int\left(\frac{dE}{dX}dX+\frac{dE}{dY}dY+\frac{dE}{dZ}dZ\right)$$
$$=u_0X_1+v_0Y_1+w_0Z_1+E_1,$$

where X_1, Y_1, Z_1, E_1 are the values of X, Y, Z, E when $t=T$.

We may eliminate the form of the body and express the work in terms of the resolved velocities of the point of application just after the termination of the impulse. Since E_1 is a homogeneous quadratic function of X_1, Y_1, Z_1 we have

$$2E_1=\frac{dE_1}{dX_1}X_1+\frac{dE_1}{dY_1}Y_1+\frac{dE_1}{dZ_1}Z_1=(u_1-u_0)X_1+(v_1-v_0)Y_1+(w_1-w_0)Z.$$

Substituting we find

$$\text{work}=\frac{u_0+u_1}{2}X_1+\frac{v_0+v_1}{2}Y_1+\frac{w_0+w_1}{2}Z_1.$$

332. A spherical membrane is stretched into a sphere whose radius is r. Let Tds be the tension across any elementary arc ds when the membrane is stretched, where T is a known function of r depending on the nature of the material. Then the work done by the tensions, when the membrane is stretched into a sphere of radius b is $8\pi\int_a^b Trdr$.

Let the centre of the sphere be taken as origin and let us refer any point on the sphere to polar co-ordinates (r, θ, ϕ). The adjacent sides of an elementary area are $rd\theta$, $r\sin\theta d\phi$. The tensions across $rd\theta$ and the opposite side are each equal to $Trd\theta$. When the radius r increases by dr, the distance between these sides is increased by $dr\sin\theta d\phi$, this being the differential of an adjacent side. Hence the work done by these tensions is $Trd\theta\,.\,dr\sin\theta\,d\phi$. Let us now consider the remaining two sides of the element. The tensions across $r\sin\theta\,d\phi$ and the opposite side are each equal to $Tr\sin\theta d\phi$. When the radius r increases by dr, the distance between these sides is increased by $dr\,d\theta$. Hence the work done by these tensions is $Tr\sin\theta d\phi\,.\,dr\,d\theta$. The work done by the tensions on the four sides of the element is therefore $2Trdr\sin\theta d\theta d\phi$. Integrating this from $\phi=0$ to 2π, $\theta=0$ to π, we find that the work done over the whole sphere when the radius increases by dr is $8\pi Trdr$.

If the membrane be such that we may apply Hooke's law to the tension T, we have $T=E\frac{r-a}{a}$, where a is the natural radius of the membrane and E is the coefficient of elasticity. Substituting this value of T we find that the work done by the tensions when the radius increases from a to b is $\frac{4}{3}\frac{E}{a}(b-a)^2(2b+a)$.

If we assume that for a soap-bubble T is constant, we find that the work done when the radius increases from a to b is $4\pi T(b^2-a^2)$.

If we suppose the spherical membrane to be slowly stretched by filling it with gas at a pressure p, we have by a theorem in Hydrostatics $pr=2T$. In this case the work required has been shown to be $\int pdv$, and since $v=\frac{4}{3}\pi r^3$ this leads to the same result as before.

333. Ex. 1. A rod originally straight is bent in one plane, if L be the stress couple at any point, ρ the radius of curvature, it is known both by experiment and theory that $L = \frac{E}{\rho}$ where E is a constant depending on the nature of the material and the form of a section of the rod. Assuming this prove that the work done in bending the rod is $\frac{1}{2}\int \frac{L^2}{E}\, ds$.

Let PQ be any element of the rod and let its length be ds. As PQ is being bent, let ψ be the indefinitely small angle between the tangents at its extremities, then the stress couple is $E\frac{\psi}{ds}$. As ψ increases from 0 to $\frac{ds}{\rho}$ the work done is $\frac{E}{ds}\int \psi d\psi$, which is the same as $\frac{Eds}{2\rho^2}$. The work done on the whole rod is therefore $\frac{1}{2}\int \frac{E}{\rho^2}\, ds$.

Ex. 2. A uniform heavy rod of length l and weight w is supported at its two extremities so as to be horizontal. Show the work done by gravity in bending it is $\frac{w^2 l^3}{240E}$.

Conservation of Vis Viva and Energy.

334. Def. The *Vis Viva* of a particle is the product of its mass into the square of its velocity.

If a system be in motion under the action of finite forces, and if the geometrical relations of the parts of the system be expressed by equations which do not contain the time explicitly, the change in the vis viva of the system in passing from any one position to any other is equal to twice the corresponding work done by the forces.

In determining the force-function all forces may be omitted which would not appear in the equation of Virtual Velocities.

Let x, y, z be the co-ordinates of any particle m, and let X, Y, Z be the resolved parts in the directions of the axes of the impressed accelerating forces acting on the particle.

The effective forces acting on the particle m at any time t are

$$m\frac{d^2x}{dt^2},\quad m\frac{d^2y}{dt^2},\quad m\frac{d^2z}{dt^2}.$$

If the effective forces on all the particles be reversed, they will be in equilibrium with the whole group of impressed forces by Art. 67. Hence, by the principle of virtual velocities,

$$\Sigma m\left\{\left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z\right\} = 0,$$

where δx, δy, δz are any small arbitrary displacements of the particle m consistent with the geometrical relations at the time t.

Now if the geometrical relations be expressed by equations which do not contain the time explicitly, the geometrical relations which hold at the time t will hold throughout the time δt; and, therefore, we can take the *arbitrary* displacements δx, δy, δz to be respectively equal to the *actual* displacements $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t$, $\frac{dz}{dt}\delta t$ of the particle in the time δt.

Making this substitution, the equation becomes

$$\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)=\Sigma m\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right).$$

Integrating, we get

$$\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=C+2\Sigma m\int(Xdx+Ydy+Zdz),$$

where C is a constant to be determined by the initial conditions of motion.

Let v and v' be the velocities of the particle m at the times t and t'. Also let U_1, U_2 be the values of the force-function for the system in the two positions which it has at the times t and t'. Then

$$\Sigma mv'^2-\Sigma mv^2=2\,(U_2-U_1).$$

335. The following illustration, taken from Poisson, may show more clearly why it is necessary that the geometrical relations should not contain the time explicitly. Let, for example,

$$\phi\,(x,\,y,\,z,\,t)=0\text{.............................}(1)$$

be any geometrical relation connecting the co-ordinates of the particle m. This may be regarded as the equation to a moving surface on which the particle is constrained to rest. The quantities δx, δy, δz are the projections on the axes of any arbitrary displacement of the particle m consistent with the geometrical relations which hold at the time t. They must therefore satisfy the equation

$$\frac{d\phi}{dx}\delta x+\frac{d\phi}{dy}\delta y+\frac{d\phi}{dz}\delta z=0.$$

The quantities $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t$, $\frac{dz}{dt}\delta t$ are the projections on the axes of the displacement of the particle due to its motion in the time δt. They must therefore satisfy the equation

$$\frac{d\phi}{dx}\frac{dx}{dt}\delta t+\frac{d\phi}{dy}\frac{dy}{dt}\delta t+\frac{d\phi}{dz}\frac{dz}{dt}\delta t+\frac{d\phi}{dt}\delta t=0.$$

Hence unless $\frac{d\phi}{dt}$ is zero throughout the whole motion we cannot take δx, δy, δz to be respectively equal to $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t$, $\frac{dz}{dt}\delta t$. The equation $\frac{d\phi}{dt}=0$ expresses the condition that the geometrical equation (1) should not contain the time explicitly.

336. If a system be under the action of no external forces, we have $X=0$, $Y=0$, $Z=0$, and hence the vis viva of the system is constant.

If, however, the mutual reactions between the particles of the system are such as would appear in the equation of virtual moments, then the vis viva of the system will not be constant. Thus, even if the solar system were not acted on by any external forces, yet its vis viva would not be constant. For the mutual attractions between the several planets are reactions between particles whose distance does not remain the same, and hence the sum of the virtual moments will not be zero.

Again, if the earth be regarded as a body rotating about an axis and slowly contracting from loss of heat in course of time, the vis viva will not be constant, for the same reason as before. The increase of angular velocity produced by this contraction can be easily found by the conservation of areas.

337. Let gravity be the only force acting on the system. Let the axis of z be vertical, then we have $X=0$, $Y=0$, $Z=-g$. Hence the equation of vis viva becomes

$$\Sigma mv'^2 - \Sigma mv^2 = -2Mg(z'-z).$$

Thus the vis viva of the system depends only on the altitude of the centre of gravity. If any horizontal plane be drawn, the vis viva of the system is the same whenever the centre of gravity passes through the plane.

338. The equation of Virtual Velocities in Statics is known to contain in one formula all the conditions of equilibrium. In the same way the general equation

$$\Sigma m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right)=\Sigma m\,(X\delta x+Y\delta y+Z\delta z),$$

may be made to give all the equations of motion by properly choosing the arbitrary displacements δx, δy, δz. In Article 334 we made one choice of these displacements and thus obtained an equation in an integrable form.

If we give the whole system a displacement parallel to the axis of z we have $\delta x=0$, $\delta y=0$, and δz is arbitrary. The equation then becomes $\Sigma m\frac{d^2z}{dt^2}=\Sigma mZ$, which represents any one of the three first general equations of motion in Art. 71.

If we give the whole system a displacement round the axis of z through an angle $\delta\theta$, we have $\delta x=-y\delta\theta$, $\delta y=x\delta\theta$, $\delta z=0$.

The equation then becomes $\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\Sigma m\,(xY-yX)$, which represents any one of the three last general equations of motion in Art. 71.

339. The principle of Vis Viva was first used by Huyghens in his determination of the centre of oscillation of a body, but in a form different from that now used. *See the note to page* 69. The principle was extended by John Bernoulli and applied by his son, Daniel Bernoulli, to the solution of a great variety of problems, such as the motion of fluids in vases, and the motion of rigid bodies under certain given conditions. See Montucla, *Histoire de Mathématique*, Tome III.

The great advantage of this principle is that it gives at once a relation between the velocities of the bodies considered and the variables or co-ordinates which determine their positions in space, so that when, from the nature of the problem, the position of all the bodies may be made to depend on one variable, the equation of vis viva is sufficient to determine the motion. In general the principle of vis viva will give a first integral of the equations of motion of the second order. If, at the same time, some of the other principles enunciated in Art. 278 may be applied to the bodies under consideration, so that the whole number of equations thus obtained is equal to the number of independent co-ordinates of the system, it becomes unnecessary to write down any equations of motion of the second order.

340. Ex. If a system in motion pass through a position of equilibrium, *i. e.* a position in which it would remain in equilibrium under the action of the forces if placed at rest, prove that the vis viva of the system is either a maximum or a minimum. Courtivron's Theorem, *Mém. de l'Acad.* 1748 and 1749.

341. Suppose a weight mg to be placed at any height h above the surface of the earth. As it falls through a height z, the force of gravity does work which is measured by mgz. The weight has acquired a velocity v, half of its vis viva is $\frac{1}{2}mv^2$ which is known to be equal to mgz. If the weight fall through the remainder of the height h, gravity may be made to do more work measured by $mg(h-z)$. When the weight has reached the ground, it has fallen as far as the circumstances of the case permit, and no more work can be done by gravity until the weight has been lifted up again. Throughout the motion we see that when the weight has descended any space z, half its vis viva, together with the work that can be done during the rest of the descent is constant and equal to the work done by gravity during the whole descent h.

If we complicate the motion by making the weight work some machine during its descent, the same theorem is still true.

By the principle of vis viva, proved in Art. 334, half the vis viva of the particle, when it has descended any space z, is equal to the work mgz which has been done by gravity during this descent, diminished by the work done on the machine. Hence, as before, half the vis viva together with the difference between the work done by gravity and that done on the machine during the remainder of the descent is constant and equal to the excess of the work done by gravity over that done on the machine during the whole descent.

Let us now extend this principle to the general case of a system of bodies acted on by any conservative system of forces.

342. Let us select some position of a moving system of bodies as a position of reference. This may be an actual final position passed through by the system in its motion, or any position which it may be convenient to choose, into which the system could be moved. Suppose the system to start from some position which we may call A, and at the time t, to occupy some position P. Then at the time t, half the vis viva generated is equal to the work done from A to P. Hence half the vis viva at P together with the work which can be done from P to the position of reference is constant for all positions of P.

To express this, the word *energy* has been used. Half the vis viva is called the *kinetic energy* of the system. The work which the forces can do as the system is moved from its existing position to the position of reference is called the *potential energy* of the system. The sum of the kinetic and potential energies is called the energy of the system. The principle of the conservation of energy may be thus enunciated:—

When a system moves under any conservative forces, the sum of the kinetic and potential energies is constant throughout the motion.

343. The distinction between work done and potential energy may be analytically stated thus. The force-function has been defined in Art. 323 to be the indefinite integral of the virtual moment of the forces. As the system moves the work done is the definite integral taken with its lower limit fixed and its upper limit determined by the instantaneous position of the system. The potential energy is the definite integral taken with its upper limit fixed and its lower limit determined by the instantaneous

* Coriolis, Helmholtz and others have suggested that it would be more convenient if the *Vis Viva* were defined to be half the sum of the products of the masses into the squares of the velocities. See *Phil. Trans.* 1851, p. 89. But this change in the meaning of a term so widely established in Europe would be very likely to cause some confusion. It seems better for the present to use another name, such as kinetic energy.

position of the system. The terms potential energy and actual energy are due to Prof. Rankine.

344. Ex. 1. *A particle describes an ellipse freely about a centre of force in its centre. Find the whole energy of its motion.*

Let m be the mass of the particle, r its distance at any time from the centre, μr the accelerating force on the particle. If coincidence of the particle with the centre of force be taken as the position of reference, the potential energy by Art. 343 is $=\int_r^0 (-m\mu r)\,dr = \frac{1}{2}m\mu r^2$. If r' be the semi-conjugate of r, the velocity of the particle is $\sqrt{\mu r'}$ and the kinetic energy is therefore $\frac{1}{2}m\mu r'^2$. As the particle describes its ellipse round the centre of force, the sum of the potential and kinetic energies is equal to $\frac{1}{2}m\mu(a^2+b^2)$ where a and b are the semi-axes of the ellipse.

Ex. 2. A particle describes an ellipse freely about a centre of force in the centre. Show that the mean kinetic energy during a complete revolution is equal to the mean potential energy; the means being taken with regard to time.

Ex. 3. If in the last example the means be taken with regard to the angle described round the centre, the difference of the means is $\frac{1}{2}m\mu(a-b)^2$.

Ex. 4. A mass M of fluid is running round a circular channel of radius a with velocity u, another equal mass of fluid is running round a channel of radius b with velocity v, the radius of one channel is made to increase and the other to decrease until each has the original value of the other, show that the work required to produce the change is $\frac{1}{2}\left(\frac{v^2}{a^2}-\frac{u^2}{b^2}\right)(b^2-a^2)M$. [Math. Tripos, 1866.]

345. In applying the principle of vis viva to any actual cases, it will be important to know beforehand what forces and internal reactions may be disregarded in forming the equation. The general rule is that all forces may be neglected which do not appear in the equation of Virtual Velocities. These forces may be enumerated as follows:

A. Those reactions whose virtual velocities are zero.

1. Those whose line of action passes through an instantaneous axis; as *rolling friction*, but not sliding friction nor the resistance of any medium.

2. Those whose line of action is perpendicular to the direction of motion of the point of application; as the reactions of *smooth fixed surfaces*, but not those of moving surfaces.

B. Those reactions whose virtual velocities are not zero and which therefore would enter into the equation, but which disappear when joined to other reactions.

1. The reactions between particles whose distance apart remains the same; as the tensions of *inextensible strings*, but not those of elastic strings.

2. The reaction between two rigid bodies, parts of the same system, which roll on each other. It is necessary however to include both these bodies in the *same* equation of vis viva.

C. All tensions which act along inextensible strings, even though the strings are bent by passing through smooth fixed rings.

For let a string whose tension is T connect the particles m, m', and pass through a ring distant respectively r, r' from the particles. The virtual velocity is clearly $T\delta r + T\delta r'$, because the tension acts along the string. But since the string is inextensible $\delta r + \delta r' = 0$; therefore the virtual velocity is zero.

346. *To determine the vis viva of a rigid body in motion.*

If a body move in any manner its vis viva at any instant is equal to the vis viva of the whole mass collected at its centre of gravity, together with the vis viva round the centre of gravity considered as a fixed point: or

The vis viva of a body = vis viva due to translation
+ vis viva due to rotation.

Let x, y, z be the co-ordinates of a particle whose mass is m and velocity v, and let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity G of the body. Let $x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$. Then by a property of the centre of gravity $\Sigma m\xi = 0$, $\Sigma m\eta = 0$, $\Sigma m\zeta = 0$. Hence $\Sigma m \frac{d\xi}{dt} = 0$, $\Sigma m \frac{d\eta}{dt} = 0$, $\Sigma m \frac{d\zeta}{dt} = 0$. Now the vis viva of a body is

$$\Sigma m v^2 = \Sigma m \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}.$$

Substituting for x, y, z, this becomes

$$\Sigma m \left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2\right\} + \Sigma m \left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\}$$
$$+ 2\frac{d\bar{x}}{dt}\Sigma m \frac{d\xi}{dt} + 2\frac{d\bar{y}}{dt}\Sigma m \frac{d\eta}{dt} + 2\frac{d\bar{z}}{dt}\Sigma m \frac{d\zeta}{dt}.$$

All the terms in the last line vanish as they should, by Art. 14. The first term in the first line is the vis viva of the whole mass Σm, collected at the centre of gravity. The second term is the vis viva due to rotation round the centre of gravity.

This expression for the vis viva may be put into a more convenient shape.

347. First. *Let the motion be in two dimensions.* Let $\bar{v}$ be the velocity of the centre of gravity, $\bar{r}$, $\bar{\theta}$ its polar co-ordinates referred to any origin in the plane of motion. Let r_1 be the distance of any particle whose mass is m from the centre of gravity, and let v_1 be its velocity relatively to the centre of gravity. Let ω be the angular velocity of the whole body about the centre of gravity, and Mk^2 its moment of inertia about the same point.

The vis viva of the whole mass collected at G is $M\bar{v}^2$, which may by the Differential Calculus be put into either of the forms

$$M\bar{v}^2 = M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2\right\} = M\left\{\left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}^2\left(\frac{d\bar{\theta}}{dt}\right)^2\right\}.$$

The vis viva about G is $\Sigma m v_1^2$. But since the body is turning about G, we have $v_1 = r_1\omega$. Hence $\Sigma m v_1^2 = \omega^2 . \Sigma m r_1^2 = \omega^2 . Mk^2$.

The whole vis viva of the body is therefore

$$\Sigma m v^2 = M\bar{v}^2 + Mk^2\omega^2.$$

If the body be turning about an instantaneous axis, whose distance from the centre of gravity is r, we have $\bar{v} = r\omega$. Hence

$$\Sigma m v^2 = M\omega^2 (r^2 + k^2) = Mk'^2\omega^2,$$

where Mk'^2 is the moment of inertia about the instantaneous axis.

348. Secondly. *Let the body be in motion in space of three dimensions.*

Let $\bar{v}$ be the velocity of G; $\bar{r}$, $\bar{\theta}$, $\bar{\phi}$ its polar co-ordinates referred to any origin. Let ω_x, ω_y, ω_z be the angular velocities of the body about any three axes at right angles meeting in G, and let A, B, C be the moments of inertia of the body about the axes. Let ξ, η, ζ be the co-ordinates of a particle m referred to these axes.

The vis viva of the whole mass collected at G is $M\bar{v}^2$, which may be put equal to

$$M\left\{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2\right\} \text{or } M\left\{\left(\frac{d\bar{r}}{dt}\right)^2 + \bar{r}^2\sin^2\bar{\theta}\left(\frac{d\bar{\phi}}{dt}\right)^2 + \bar{r}^2\left(\frac{d\bar{\theta}}{dt}\right)^2\right\},$$

according as we wish to use cartesian or polar co-ordinates.

The vis viva due to the motion about G is

$$\Sigma m v_1^2 = \Sigma m\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\}.$$

But $\dfrac{d\xi}{dt} = \omega_y\zeta - \omega_z\eta$, $\dfrac{d\eta}{dt} = \omega_z\xi - \omega_x\zeta$, $\dfrac{d\zeta}{dt} = \omega_x\eta - \omega_y\xi$.

Substituting these values, we get, since $A = \Sigma m(\eta^2 + \zeta^2)$, $B = \Sigma m(\zeta^2 + \xi^2)$, $C = \Sigma m(\xi^2 + \eta^2)$,

$$\Sigma m v_1^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2$$
$$- 2(\Sigma m\xi\eta)\,\omega_x\omega_y - 2(\Sigma m\eta\zeta)\,\omega_y\omega_z - 2(\Sigma m\zeta\xi)\,\omega_z\omega_x.$$

If the axes of co-ordinates be the principal axes at G, this reduces to

$$\Sigma m v_1^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2.$$

If the body be turning about a point O, whose position is fixed for the moment, the vis viva may be proved in the same way to be

$$\Sigma mv^2 = A'\omega_x^2 + B'\omega_y^2 + C'\omega_z^2,$$

where A', B', C' are the principal moments of inertia at the point O, and ω_x, ω_y, ω_z are the angular velocities of the body about the principal axes at O.

349. Ex. 1. A rigid body of mass M is moving in space in any manner and its position is determined by the co-ordinates of its centre of gravity and the angles θ, ϕ, ψ which the principal axes at the centre of gravity make with some fixed axes in the manner explained in Art. 235. Show that its vis viva is given by

$$2T = M(x'^2 + y'^2 + z'^2) + C(\phi' + \psi' \cos\theta)^2 + (A\sin^2\phi + B\cos^2\phi)\,\theta'^2$$
$$+ \sin^2\theta\,(A\cos^2\phi + B\sin^2\phi)\,\psi'^2 + 2(B - A)\sin\theta\sin\phi\cos\phi\,\theta'\psi',$$

where accents denote differential coefficients with regard to the time.

Show also that when two of the principal moments A and B are equal, this takes the simpler form

$$2T = M(x'^2 + y'^2 + z'^2) + C(\phi' + \psi'\cos\theta)^2 + A(\theta'^2 + \sin^2\theta\,\psi'^2).$$

This result will be often found useful.

Ex. 2. A body moving freely about a fixed point is expanding under the influence of heat so that in structure and form the body is always similar to itself. If the law of expansion be that the distance between any two particles at the temperature θ is equal to their distance at temperature zero multiplied by $f(\theta)$, show that the vis viva of the body $= A\omega_x^2 + B\omega_y^2 + C\omega_z^2 + \frac{1}{2}(A + B + C)\left(\frac{d\log f(\theta)}{dt}\right)^2$, where A, B, C are the principal moments at the fixed point.

Ex. 3. A body is moving about a fixed point and its vis viva is given by the equation

$$2T = A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y.$$

Show that the angular momenta about the axes are $\dfrac{dT}{d\omega_x}$, $\dfrac{dT}{d\omega_y}$, $\dfrac{dT}{d\omega_z}$.

Let the body be moving freely and let $2T_0$ be the vis viva of translation. Prove that if x, y, z be the co-ordinates of the centre of gravity referred to any rectangular axes fixed or moving about a fixed point, and if accents denote differential coefficients with regard to the time, then the linear momenta parallel to the axes will be

$$\frac{dT_0}{dx'},\ \frac{dT_0}{dy'},\ \frac{dT_0}{dz'}.$$

Thus the vis viva, like the force-function, is a scalar function whose differential coefficients are the components of vectors. See Art. 240 and 326. In the case of the semi vis viva, these are the resultant linear momentum and angular momentum round the centre of gravity.

Ex. 4. A body is moving about a fixed point and its vis viva is given by the same expression as in the last example. Show that if the axes are fixed in space and the origin is at the fixed point, the equations of motion may be written in the form

$$\frac{d}{dt}\frac{dT}{d\omega_x} = L,$$

with two similar equations for the axes of y and z. In these equations A, B, &c. will generally be variable.

If the axes move in the manner explained in Art. 243, the equations of motion are

$$\frac{d}{dt}\frac{dT}{d\omega_x} - \frac{dT}{d\omega_y}\theta_z + \frac{dT}{d\omega_z}\theta_y = L,$$

with two similar equations. See Art. 253.

If the centre of gravity of a body moving freely be referred to axes moving about a fixed origin and if $2T_0$ be the vis viva of translation, show that the equations of motion of Art. 245 may be written

$$\frac{d}{dt}\frac{dT_0}{dx'} - \frac{dT_0}{dy'}\theta_z + \frac{dT_0}{dz'}\theta_y = X,$$

with two similar equations.

350. Ex. 1. *A circular wire can turn freely about a vertical diameter as a fixed axis, and a bead can slide freely along it under the action of gravity. The whole system being set in rotation about the vertical axis, find the subsequent motion.*

Let M and m be the masses of the wire and bead, ω their common angular velocity about the vertical. Let a be the radius of the wire, Mk^2 its moment of inertia about the diameter. Let the centre of the wire be the origin, and let the axis of y be measured vertically downwards. Let θ be the angle the radius drawn from the centre of the wire to the bead makes with the axis of y.

It is evident, since gravity acts vertically and since all the reactions at the fixed axis must pass through the axis, that the moment of all the forces about the vertical diameter is zero. Hence, taking moments about the vertical, we have

$$Mk^2\omega + ma^2\sin^2\theta\omega = h.$$

And by the principal of vis viva,

$$Mk^2\omega^2 + m\left\{a^2\left(\frac{d\theta}{dt}\right)^2 + a^2\sin^2\theta\omega^2\right\} = C + 2mga\cos\theta.$$

These two equations will suffice for the determination of $\frac{d\theta}{dt}$ and ω. Solving them, we get

$$\frac{h^2}{Mk^2 + ma^2\sin^2\theta} + ma^2\left(\frac{d\theta}{dt}\right)^2 = C + 2mga\cos\theta.$$

This equation cannot be integrated, and hence θ cannot be found in terms of t. To determine the constants h and C we must recur to the initial conditions of motion. Supposing that initially $\theta=\pi$, and $\frac{d\theta}{dt}=0$ and $\omega=\alpha$, then $h=Mk^2\alpha$ and $C=2mga+Mk^2\alpha^2$.

Ex. 2. A lamina of any form rolls on a perfectly rough straight line under the action of no forces; prove that the velocity v of the centre of gravity G is given by $v^2=c^2\dfrac{r^2}{r^2+k^2}$, where r is the distance of G from the point of contact, and k is the radius of gyration of the body about an axis through G perpendicular to its plane, and c is some constant.

Ex. 3. Two equal beams connected by a hinge at their centres of gravity so as to form an X are placed symmetrically on two smooth pegs in the same horizontal

line, the distance between which is b. Show that, if the beams be perpendicular to each other at the commencement of the motion, the velocity of their centre of gravity, when in the line joining the pegs, is equal to $\sqrt{\dfrac{b^2g}{b^2+4k^2}}$, where k is the radius of gyration of either beam about a line perpendicular to it through its centre of gravity.

Ex. 4. A uniform rod is moving on a horizontal table about one extremity, and driving before it a particle of mass equal to its own, which starts from rest indefinitely near to the fixed extremity; show that when the particle has described a distance r along the rod, its direction of motion makes with the rod an angle $\tan^{-1}\dfrac{k}{\sqrt{r^2+k^2}}$. [Christ's Coll.]

Ex. 5. A thin uniform smooth tube is balancing horizontally about its middle point, which is fixed; a uniform rod such as just to fit the base of the tube is placed end to end in a line with the tube, and then shot into it with such a horizontal velocity that its middle point shall only just reach that of the tube; supposing the velocity of projection to be known, find the angular velocity of the tube and rod at the moment of the coincidence of their middle points. [Math. Tripos.]

Result. If m be the mass of the rod, m' that of the tube, and $2a$, $2a'$ their respective lengths, v the velocity of the rod's projection, ω the required angular velocity, then $\omega^2=\dfrac{3mv^2}{ma^2+m'a'^2}$.

Ex. 6. The centre C of a circular wheel is fixed and the rim is constrained to roll in a uniform manner on a perfectly rough horizontal plane so that the plane of the wheel makes a constant angle α with the vertical. Round the circumference there is a uniform smooth canal of very small section, and a heavy particle which just fits the canal can slide freely along it under the action of gravity. If m be the particle, B the point where the wheel touches the plane and $\theta=\angle BCm$, and if n be the angular rate at which B describes the circular trace on the horizontal plane, prove that $\left(\dfrac{d\theta}{dt}\right)^2=\dfrac{2g}{a}\cos\alpha\cos\theta-n^2\cos^2\alpha\cos^2\theta+\text{const.}$ where a is the radius of the wheel. *Annales de Gergonne*, Tome XIX.

Ex. 7. If an elastic string, whose natural length is that of a uniform rod, be attached to a rod at both ends and suspended by the middle point, prove by means of vis viva that the rod will sink until the strings are inclined to the horizon at an angle θ, which satisfies the equation $\cot^3\dfrac{\theta}{2}-\cot\dfrac{\theta}{2}-2n=0$, where the tension of the string, when stretched to double its length, is n times the weight. [Math. Tripos.]

Ex. 8. A regular homogeneous prism, whose normal section is a regular polygon of n sides, the radius of the circumscribing circle being a, rolls down a perfectly rough inclined plane whose inclination to the horizon is α. If ω_n be the angular velocity just before the n^{th} edge becomes the instantaneous axis then

$$\omega_n^2-\frac{g\sin\alpha}{a\sin\frac{\pi}{n}}\frac{8+\cos\frac{2\pi}{n}}{5+4\cos\frac{2\pi}{n}}=\left(\frac{2+7\cos\frac{2\pi}{n}}{8+\cos\frac{2\pi}{n}}\right)^2\left(\omega^2_{n-1}-\frac{g\sin\alpha}{a\sin\frac{\pi}{n}}\frac{8+\cos\frac{2\pi}{n}}{5+4\cos\frac{2\pi}{n}}\right).$$

351. The equation of Vis Viva may be applied to the case of relative motion in the following manner*. *Suppose the system at any instant to become fixed to the set of moving axes relative to which the motion is required, and calculate what would then be the effective forces on the system. If we apply these as additional impressed forces to the system but reversed in direction, we may use the equation of Vis Viva to determine the relative motion as if the axes were fixed in space.*

We may reduce the origin O of the moving axes to rest by applying to every particle an acceleration equal and opposite to that of O, in the manner explained in Art. 174. As these will be included as part of the additional forces mentioned in the enunciation it will be sufficient to prove the theorem for axes moving about a fixed point.

If we follow the notation of Art. 259, the accelerations of any point P resolved parallel to rectangular moving axes having a fixed origin are

$$\frac{d^2x}{dt^2}-2\frac{dy}{dt}\theta_3+2\frac{dz}{dt}\theta_2+Ax+By+Cz,$$

with two similar expressions for y and z. The three last terms, with the corresponding terms in the other expressions, are the resolved accelerations of a point P_0 rigidly attached to the axes but occupying the instantaneous position of P. Let us call these X_0, Y_0, Z_0.

Recurring to the proof of the principle of vis viva given in Art. 334 we see that we have to substitute these expressions for $\frac{d^2x}{dt^2}$, &c. in the general equation of virtual velocities. After substitution for δx, δy, δz, it is clear that the terms containing $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ all disappear. The equation after integration then becomes, as before,

$$\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=2\Sigma m\int\{(X-X_0)\,dx+(Y-Y_0)\,dy+(Z-Z_0)\,dz\}+C.$$

The theorem of Coriolis really follows at once from that of Clairaut given in Art. 257. The above mode of proof has the advantage of recurring to first principles.

352. Ex. 1. *A sphere rolls on a perfectly rough plane which turns with a uniform angular velocity* n *about a horizontal axis in its own plane. Supposing the motion of the sphere to take place in a vertical plane perpendicular to the axis of rotation, find the motion of the sphere relative to the plane.*

Let Ox be the trace described by the sphere as it rolls on the plane, and let Oy be drawn through the axis of rotation perpendicular to Ox in the plane of motion of the sphere. Let nt be the angle Ox makes with a horizontal plane through the axis of rotation. Let ϕ be the angle that radius of the sphere which was initially perpendicular to the plane makes with the axis of y. Let (x, y) be the co-ordinates of P the centre of the sphere, and Mk^2 the moment of inertia of the sphere about a diameter.

If the sphere were fixed relatively to the plane its effective forces would be Mn^2x and Mn^2y parallel to the axes, and $Mk^2\frac{dn}{dt}=0$ round the centre of gravity. Also the

* This theorem is due to Coriolis, see the *Journal Polytech.* 1831.

impressed force, gravity, is equivalent to $g\sin nt$ and $-g\cos nt$ parallel to the moving axes. Hence the equation of Vis Viva for relative motion becomes

$$\frac{1}{2}\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\phi}{dt}\right)^2\right\}=n^2x\frac{dx}{dt}+n^2y\frac{dy}{dt}+g\sin nt\frac{dx}{dt}-g\cos nt\frac{dy}{dt}.$$

Here $\frac{dx}{dt}=a\frac{d\phi}{dt}$ and $\frac{dy}{dt}=0$. We have therefore

$$\left(1+\frac{k^2}{a^2}\right)\frac{d^2x}{dt^2}=n^2x+g\sin nt.$$

This equation might also have been derived from the formulæ for moving axes given in Art. 179. If $k^2=\frac{2}{5}a^2$, this equation leads to

$$x=-\frac{5}{12}\frac{g}{n^2}\sin nt+Ae^{n\sqrt{\frac{5}{7}}t}+Be^{-n\sqrt{\frac{5}{7}}t},$$

where A, B are two constants which depend on the initial conditions of the question.

353. *To determine the change in the vis viva of a moving system produced by any collisions between the bodies or by any explosions.* (Carnot's Theorem.)

Let $v_x, v_y, v_z, v_x', v_y', v_z'$ be the resolved parts of the velocities of any particle m of the system before and after the impulse.

Then the momenta $m(v_x'-v_x)$, $m(v_y'-v_y)$, $m(v_z'-v_z)$, being reversed and taken throughout the whole system, are by D'Alembert's Principle in equilibrium with the impulses. But these last are themselves in equilibrium. Hence the former set are also in equilibrium. Therefore by Virtual Velocities,

$$\Sigma m\left\{(v_x'-v_x)\,\delta x+(v_y'-v_y)\,\delta y+(v_z'-v_z)\,\delta z\right\}=0,$$

where δx, δy, δz are any small arbitrary displacements of the particles impinging on each other, which are consistent with the geometrical conditions of the system during the time of action of the impulse.

During the impact, it is one geometrical condition that the particles impinging on each other have no velocity of separation normal to the common surface of the bodies of which they form a part.

First. Let the bodies be devoid of elasticity. Then the above geometrical conditon will hold just *after* the moment of greatest compression as well as during the impact. Hence we can put $\delta x=v_x'\delta t$, $\delta y=v_y'\delta t$, $\delta z=v_z'\delta t$. The equation now becomes

$$\Sigma m\left\{(v_x'-v_x)\,v_x'+(v_y'-v_y)\,v_y'+(v_z'-v_z)\,v_z'\right\}=0\,;$$
$$\therefore\ \Sigma m\left(v_x'^2+v_y'^2+v_z'^2\right)=\Sigma m\left(v_xv_x'+v_yv_y'+v_zv_z'\right).$$

This may be put into the form

$$\Sigma m\,(v_x'^2 + v_y'^2 + v_z'^2) - \Sigma m\,(v_x^2 + v_y^2 + v_z^2)$$
$$= -\Sigma m\,\{(v_x' - v_x)^2 + (v_y' - v_y)^2 + (v_z' - v_z)^2\}.$$

Therefore in the impact of inelastic bodies vis viva is always lost.

Secondly. Let an explosion take place in any body of the system. Then the geometrical equation above spoken of will hold just before the impulse begins as well as during the explosion, but it will not hold after the particles of the body have separated. Hence we must now put $\delta x = v_x \delta t$, $\delta y = v_y \delta t$, $\delta z = v_z \delta t$. As before, we have

$$\Sigma m\,(v_x v_x' + v_y v_y' + v_z v_z') = \Sigma m\,(v_x^2 + v_y^2 + v_z^2),$$

and

$$\Sigma m\,(v_x'^2 + v_y'^2 + v_z'^2) - \Sigma m\,(v_x^2 + v_y^2 + v_z^2)$$
$$= +\Sigma m\,\{(v_x' - v_x)^2 + (v_y' - v_y)^2 + (v_z' - v_z)^2\}.$$

Therefore in cases of explosion vis viva is always gained.

Thirdly. Let the particles of the system be perfectly elastic. Then the whole action consists of two parts, a force of compression as if the particles were inelastic, and a force of restitution of the nature of an explosion. The circumstances of these two forces are exactly equal and opposite to each other. By examining these two expressions it is easy to see that the vis viva lost in the compression is exactly balanced by the vis viva gained in the restitution.

354. It should be noticed that Carnot's demonstration does not exclusively apply to collisions, but to all impulses which are such as do not appear in the equation of Virtual Velocities.

Let a system be moving in any way, and let us suddenly introduce some new restraints, by which some of the particles are compelled to take new courses. The impulses which produce this change of motion are of the nature of reactions, and are such that in the subsequent path their virtual moments are zero. It follows from Carnot's first theorem, that vis viva will be lost, and the amount of vis viva lost is equal to the vis viva of the relative motion.

Let there be two systems at rest, in all respects the same except that one is subject to some restraints from which the other is free. Let both these be set in motion by equal impulses, and let X, Y, Z be the components of these. Then, if

accented letters refer to the more free system and twice accented letters to the other, we have

$$\left.\begin{aligned}\Sigma m\,(v_x'\delta x + \&c.) = \Sigma\,(X\delta x + \&c.)\\ \Sigma m\,(v_x''\delta x + \&c.) = \Sigma\,(X\delta x + \&c.)\end{aligned}\right\},$$

where δx, δy, δz are any arbitrary displacements consistent with the geometrical conditions. Since both systems may be displaced in the manner in which the less free system actually begins to move, we may put $\delta x = v_x''\delta t$, &c. We therefore have

$$\Sigma m\,(v_x'v_x'' + \&c.) = \Sigma m\,(v_x''^2 + \&c.).$$

It again follows from Carnot's first demonstration that the vis viva of the constrained system is less than that of the free. Generally, *the greater the constraints impressed on a system at rest, the less will be the vis viva generated by any given impulses.* This theorem is in part due to Lagrange, it has been generalized by Bertrand in his notes to the *Mécanique Analytique.*

355. Let two systems be in all respects the same and moving in the same manner. Let us suppose that suddenly some of the constraints are removed from one system and at the same instant let both be acted on by equal impulses. Then following the same notation as before, we have

$$\Sigma m\,\{(v_x' - v_x)\,\delta x + \&c.\} = \Sigma\,(X\delta x + \&c.),$$

$$\Sigma m\,\{(v_x'' - v_x)\,\delta x + \&c.\} = \Sigma\,(X\delta x + \&c.).$$

If we make $\delta x = v_x''\delta t$, &c. we obtain

$$\Sigma m\,(v_x'v_x'' + \&c.) = \Sigma m\,(v_x''^2 + \&c.),$$

and we may deduce from this equation theorems similar to those of the last article.

Let us now give these two systems any other displacement which is permitted by the geometrical relations common to both. Let this displacement be represented by $\delta x = v_x'''\delta t$, &c. Then as before we have

$$\Sigma m\,(v_x'v_x''' + \&c.) = \Sigma m\,(v_x''v_x''' + \&c.).$$

From this and the last equation we easily find

$$\Sigma m\,\{(v_x' - v_x''')^2 + \&c.\} = \Sigma m\,\{(v_x' - v_x'')^2 + \&c.\} + \Sigma m\,\{(v_x'' - v_x''')^2 + \&c.\}.$$

Let a_1, a_2, &c. be the positions of the particles m_1, m_2, &c. just before the action of the impulses; a_1', a_2', &c., a_1'', a_2'', &c. their positions just after, in the more free and constrained systems respectively, a_1''', a_2''', &c. their positions after any hypothetical displacement. Then

$$\Sigma m\,(a'a''')^2 = \Sigma m\,(a'a'')^2 + \Sigma m\,(a''a''')^2.$$

Hence we infer that the motion of the more constrained system is such that $\Sigma m\,(a'a'')^2$ is less than if the particles took any other courses, consistent with all the geometrical relations.

If we suppose the systems to be acted on by a series of indefinitely small impulses, these impulses may be regarded as finite forces. We may therefore infer the following theorem, which is called *Gauss' principle of least constraint.*

The motion of a system of material points connected by any geometrical relations is always as nearly as possible in accordance with free motion; *i.e.* if the

constraint during any time dt is measured by the sum of the products of the mass of each particle into the square of its distance at the end of that time from the position it would have taken if it had been free, then the actual motion during the time dt is such that the constraint is less than if the particles had taken any other positions.

M. Gauss remarks that the free motions of the particles when they are incompatible with the geometrical conditions of the system are modified in exactly the same way as geometers modify results, which have been obtained by observation, by applying the method of least squares so as to render them compatible with the geometrical conditions of the question.

356. *To determine the mean vis viva of a system of material points in stationary motion.* Clausius' Theorem*.

By stationary motion is meant any motion in which the points do not continually remove further and further from their original position, and the velocities do not alter continuously in the same direction, but the points move within a limited space and the velocities only fluctuate within certain limits. Of this nature are all periodic motions, such as those of the planets about the sun and the vibrations of elastic bodies, and further, such irregular motions as are attributed to the atoms and molecules of a body in order to explain its heat.

Let x, y, z be the co-ordinates of any particle in the system and let its mass be m. Let X, Y, Z be the components of the forces on this particle. Then $m\frac{d^2x}{dt^2}=X$. We have by simple differentiation,

$$\frac{d^2(x^2)}{dt^2}=2\frac{d}{dt}\left(x\frac{dx}{dt}\right)=2\left(\frac{dx}{dt}\right)^2+2x\frac{d^2x}{dt^2},$$

and therefore

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2=-\frac{1}{2}xX+\frac{m}{4}\frac{d^2(x^2)}{dt^2}.$$

Let this equation be integrated with regard to the time from 0 to t and let the integral be divided by t, we thereby obtain

$$\frac{m}{2t}\int_0^t\left(\frac{dx}{dt}\right)^2dt=-\frac{1}{2t}\int xX\,dt+\frac{m}{4t}\left[\frac{d(x^2)}{dt}-\left(\frac{d(x^2)}{dt}\right)_0\right],$$

in which the application of the suffix zero to any quantity implies that the initial value of that quantity is to be taken.

The left-hand side of this equation and the first term on the right-hand side are evidently the mean values of $\frac{m}{2}\left(\frac{dx}{dt}\right)^2$ and $-\frac{1}{2}xX$ during the time t. For a periodic motion the duration of a period may be taken for the time t; but for irregular motions (and if we please for periodic ones also) we have only to consider that the time t, in proportion to the times during which the point moves in the same direction in respect of any one of the directions of co-ordinates is very great, so that in the course of the time t many changes of motion have taken place, and the above expressions of the mean values have become sufficiently constant. The last term of the equation, which has its factor included in square brackets, becomes, when the time is periodic, equal to zero at the end of each period. When the motion is

* This and the next article are an abridgement of Clausius' paper in the *Phil. Mag.*, August, 1870.

not periodic, but irregularly varying, the factor in brackets does not so regularly become zero, yet its value cannot continually increase with the time, but can only fluctuate within certain limits; and the divisor t, by which the term is affected, must accordingly cause the term to become vanishingly small with very great values of t. The same reasoning will apply to the motions parallel to the other co-ordinates. Hence adding together our results for each particle, we have, if v be the velocity of the particle m,

$$\text{mean}\,\frac{1}{2}\Sigma mv^2 = -\text{mean}\,\frac{1}{2}\Sigma\,(Xx + Yy + Zz).$$

The mean value of the expression $-\frac{1}{2}\Sigma\,(Xx + Yy + Zz)$ has been called by Clausius the *virial* of the system. His theorem may therefore be stated thus, *the mean semi vis viva of the system is equal to its virial.*

357. In order to apply this theorem to heat, let us consider a body as a system of material particles in motion. The forces which act on the system will in general consist of two parts. In the first place, the elements of the body exert on each other attractive or repulsive forces, and secondly, forces may act on the system from without. The virial will therefore consist of two parts, which are called the *internal* and *external virial.*

If $\phi(r)$ be the law of repulsion between two particles whose masses are m and m', we have $Xx + X'x' = -\phi(r)\frac{x'-x}{r}x - \phi(r)\frac{x-x'}{r}x' = \phi(r)\frac{(x'-x)^2}{r}$. And since for the two other co-ordinates corresponding equations may be formed, we have for the internal virial $-\frac{1}{2}\Sigma\,(Xx + Yy + Zz) = -\Sigma r\phi(r)$.

As to the external forces, the case most frequently to be considered is where the body is acted on by a uniform pressure normal to the surface. If p be this pressure, $d\sigma$ an element of the surface, l the cosine of the angle the normal makes with the axis of x, $-\frac{1}{2}\Sigma Xx = \frac{1}{2}\int xp\,l\,d\sigma = \frac{p}{2}\iint x\,dy\,dz$. If V be the volume of the body this is $\frac{1}{2}pV$, and therefore the whole external virial is $\frac{3}{2}pV$.

Ex. Show that the virial of a system of forces is independent of the origin and the directions of the axes supposed rectangular.

The first result is clear, since in stationary motion $\Sigma X = 0$, &c. The second follows from the equality $Xx + Yy + Zz = R\rho$, where R is the resultant of X, Y, Z, and ρ is the projection of the radius vector on the direction of R.

Newton's Principle of Similitude.

358. What are the conditions necessary that two systems of particles which are initially geometrically similar should also be mechanically similar, *i.e.* the relative positions of the particles in one system at time t should also be similar to the relative positions in the other system at time t', where t' bears to t a constant ratio?

In other words, a model is made of a machine, and is found to work satisfactorily, what are the conditions that a machine made according to the model should work as satisfactorily?

Since all the equations of motion may be deduced from the principle of Virtual Velocities, that principle seems to afford the simplest method of investigating any general theorem in Dynamics. It has also the advantage of not requiring us to consider the unknown reactions, if there be any in the system. This mode of proof is given by M. Bertrand in *Cahier xxxii.* of the *Journal de l'école Polytechnique.*

359. Let (x, y, z) be the co-ordinates of any particle of mass m in one system referred to any rectangular axes fixed in space, and let (X, Y, Z) be the resolved part of the impressed *moving* forces on that particle. Let accented letters refer to corresponding quantities in the other system.

Then the principle of Virtual Velocities supplies the two following equations:

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \&c.\right\} = 0,$$

$$\Sigma\left\{\left(X' - m'\frac{d^2x'}{dt'^2}\right)\delta x' + \&c.\right\} = 0.$$

It is evident that one of these equations will be changed into the other if we put $X' = FX$, $Y' = FY$, &c., $x' = lx$, $y' = ly$, &c., $m' = \mu m$, &c., $t' = \tau t$, &c., where F, l, μ, τ are all constants, provided $\mu l = F\tau^2$. In two geometrically similar systems, we have but one ratio of similarity, viz. that of the linear dimensions, but in two mechanically similar systems we have three other ratios, viz. that of the masses of the particles, that of the forces which act on them, and that of the times at which the systems are to be compared. It is clear that if the relation just established hold between these four ratios of similitude, the motion of the two systems will be similar.

Suppose then the two systems to be initially geometrically similar, and that the masses of corresponding particles are proportional each to each, and that they begin to move in parallel directions with like motions and in proportional times, then they will continue to move with like motions and in proportional times provided the external moving forces in either system are proportional to $\dfrac{\text{mass} \times \text{linear dimensions}}{(\text{time})^2}$. Since the resolved velocities of any particle are $\dfrac{dx}{dt}$, &c., it is clear that in two similar systems the velocities of corresponding points at corresponding times are

proportional to $\dfrac{\text{linear dimensions}}{\text{time}}$. If we eliminate the time between these two relations, we may state, briefly, that the condition of similitude between two systems is that the moving forces must be proportional to $\dfrac{\text{mass} \times (\text{velocity})^2}{\text{linear dimensions}}$.

360. M. Bertrand remarks, that in comparing the working of a model with that of a large machine, we must take care that all the forces bear their proper ratios. Supposing the model to be made of the same material as the machine, the weights of the several parts will vary as their masses, and therefore as the cubes of the linear dimensions. Hence we infer that the velocity of working the model must be made to be proportional to the square root of its linear dimensions. The times of describing corresponding arcs will also be in the same ratio.

If there be any forces besides gravity which act on the model, these must bear the same ratio to the corresponding forces in the machine, if the model is to be similar to the machine. Hence the impressed forces must be made to vary as the cubes of the linear dimensions. For example, in the case of a model of a steam-engine, the pressure of the steam on the piston varies as the product of the area of the piston into the elastic force. Hence, the elastic force of the steam used must be proportional to the linear dimensions of the model.

Supposing the impressed forces in the two systems to have, each to each, the proper ratio, the mutual reactions between the parts of the systems will, of themselves, assume the same ratio. For if, by giving proper displacements according to the principle of Virtual Velocities, we form equations of motion to find these reactions, it is easy to see that they will be, each to each, in the same ratio as the forces. Since sliding friction varies as the normal pressure, and is independent of the areas in contact, these frictions will bear their proper ratio in the model and machine. This, however, is not the case with rolling friction. Recurring to Art. 150, we see that the rolling friction varies inversely as the diameter of the wheel, and will, therefore, bear a greater ratio to the other forces in the model than in the machine. If the resistance of the air be proportional to the product of the area exposed into the square of the velocity, this resistance will bear the proper ratio in the model and the machine.

361. As an example, let us apply the principle to the case of a rigid body oscillating about a fixed axis under the action of gravity. That the motions of two pendulums may be similar they must describe equal angles, corresponding times are therefore proportional to their times of oscillation. Since the forces vary as the mass into gravity, we see that when a pendulum oscillates through a given angle,

the square of the time of oscillation must vary as the ratio of the linear dimensions to gravity.

As a second example consider the case of a particle describing an orbit round the centre of attraction whose force is equal to the product of the inverse square of the distance into some constant μ. The principle at once shows that the square of the periodic time must vary as the cube of the distance directly and as μ inversely. This is Kepler's third law.

362. In the twenty-ninth volume of the *Annales de Chimie* (Paris, 1825) Savart describes numerous experiments which he made on the notes sounded by similar vessels containing air. He says that if we construct cubical boxes and set the air in motion as is ordinarily done in organ pipes we find that the number of vibrations in a given time is proportional to the reciprocals of the linear dimensions of the masses of air. This law was verified between extreme limits, and its truth tested with a great many notes. He says he frequently used the law during his researches, and never once found it led him wrong. This result having been obtained for cubes, it was natural to examine if the same law held for prismatic tubes on square bases. After a great many experiments he found the same law to be true.

He then tested the law with conical pipes in which the opening was always of the same solid angle, then with cylindrical pipes, then with pipes whose bases were equilateral triangles. These he made to sound in different ways, putting the mouth-piece for instance at different points of the length of the tube. In all cases the same law was found to hold, for tubes whose diameters were very small compared with their lengths as well as for those whose diameters were very great. This law he again found applicable to masses of air set in motion by communication from other vibrating bodies. Hence he infers this general law which he enunciates as an experimental fact.

When masses of air are contained in two similar vessels, the number of vibrations in a given time [*i.e.* the pitch of the note sounded] is proportional inversely to the linear dimensions of the vessel.

This theorem of Savart's follows at once from the principle of Similarity. Divide the similar vessels into corresponding elements, then the motion of these elements will be similar each to each if the forces vary as $\dfrac{\text{mass} \times \text{lin. dim.}}{(\text{time})^2}$. But by Mariotte's law the force between two elements varies as the product of the area of contact into the density. Hence the times of oscillation of corresponding particles of air must vary as the linear dimensions of the vessel.

363. The first person who gave a theoretical explanation of Savart's law was Cauchy, who showed, in a *Memoire* presented to the Academy of Sciences in 1829, that it followed from the linearity of the equations of motion. He refers to the general equations of motion of an elastic body whose particles are but slightly displaced even though the elasticity is different in different directions. These equations which serve to determine the displacements (ξ, η, ζ) of a particle in terms of the time t and the co-ordinates (x, y, z) are of two kinds. One applies to all points of the interior of the elastic body and the other to all points on its surface. These are to be found in all treatises on elasticity. An inspection of these equations shows that they will continue to exist if we replace $\xi, \eta, \zeta, x, y, z, t$ by $\kappa\xi, \kappa\eta, \kappa\zeta, \kappa x, \kappa y, \kappa z, \kappa t$, where κ is any constant provided we alter the accelerating forces in the ratio κ to 1. Hence if these accelerating forces are zero, it will be sufficient to

increase the dimensions of the elastic body and the initial values of the displacements in the ratio 1 to κ, in order that the general values of ξ, η, ζ and the durations of the vibrations should vary in the same ratio. Hence we deduce Cauchy's extension of Savart's law, viz. if we measure the pitch of the note given by a body, by a plate or an elastic rod, by the number of vibrations produced in a unit of time; the pitch will vary inversely as the linear dimensions of the body, plate or rod, supposing all its dimensions altered in a given ratio.

364. These results may be also deduced from the theory of dimensions. Following the notation of Art. 318, a force F is measured by $m\frac{d^2x}{dt^2}$. We may then state the general principle, that all dynamical equations must be such that the dimensions of terms added together are the same in space, time and mass, the dimensions of force being taken to be $\frac{\text{mass . space}}{(\text{time})^2}$.

Let us apply this to the case of a single pendulum of length l, oscillating through a given angle α, under the action of gravity. Let m be the mass of the particle, F the moving force of gravity, then the time τ of oscillation can be a function only of F, l, m and α. Let this function be expanded in a series of powers of F, l and m. Thus

$$\tau = \Sigma A F^p l^q m^r,$$

where A being a function of α only is a number. Since τ is of no dimension in space, we have $p + q = 0$. Also τ is of one dimension in time; $\therefore -2p = 1$. Finally τ is of no dimensions in mass; $\therefore p + r = 0$. Hence $p = -\frac{1}{2}$, $q = r = \frac{1}{2}$, and since p, q, r have each only one value, there is but one term in the series. We infer that in any simple pendulum $\tau = A\sqrt{\frac{ml}{F}}$ where A is an undetermined number.

365. Ex. 1. A particle moves from rest towards a centre of force whose attraction varies as the distance in a medium resisting as the velocity, show by the theory of dimensions that the time of reaching the centre of force is independent of the initial position of the particle.

Ex. 2. A particle moves from rest in vacuo towards a centre of force whose attraction varies inversely as the n^{th} power of the distance, show that the time of reaching the centre of force varies as the $\frac{n+1}{2}$th power of the initial distance of the particle.

Lagrange's Equations.

366. Our object in this section is to form the general equation of motion of a dynamical system freed from all the unknown reactions and expressed, as far as is possible, in terms of any kind of co-ordinates which may be convenient in the problem under consideration.

In order to eliminate the reactions we shall use the principle of Virtual Velocities. This principle has already been applied to obtain the equation of Vis Viva by giving the system that particular displacement which it would have taken if it had been left to itself. But since every dynamical problem can, by D'Alembert's principle, be reduced to one in statics, it is clear that by giving the system proper displacements, we must be able to deduce, as in Art. 338, not Vis Viva only, but all the equations of motion.

367. Let (x, y, z) be the co-ordinates of any particle m of the system referred to any fixed rectangular axes. These are not independent of each other, being connected by the geometrical relations of the system. But they may be expressed in terms of a certain number of independent variables whose values will determine the position of the system at any time. Extending the definition given in Art. 73, we shall call these the co-ordinates of the system. Let these be called θ, ϕ, ψ, &c. Then x, y, z, &c. are functions of θ, ϕ, &c. Let

$$x = f(t, \theta, \phi, \text{\&c.}) \quad\text{.......................}(1),$$

with similar equations for y and z. It should be noticed that these equations are not to contain $\dfrac{d\theta}{dt}, \dfrac{d\phi}{dt}$, &c. The independent variables in terms of which the motion is to be found may be any we please, with this restriction, that the co-ordinates of every particle of the body could, if required, be expressed in terms of them by means of equations which do not contain any differential coefficients with regard to the time.

The number of independent co-ordinates to which the position of a system is reduced by its geometrical relations, is sometimes spoken of as the number of the *degrees of freedom of that body.* Sometimes it is referred to as being the *number of independent motions* which the system admits of.

In the following investigations total differential coefficients with regard to t will be denoted by accents. Thus $\dfrac{dx}{dt}$ and $\dfrac{d^2x}{dt^2}$ will be written x' and x''.

If $2T$ be the vis viva of the system, we have

$$2T = \Sigma m\,(x'^2 + y'^2 + z'^2)\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

we also have, since the geometrical equations do not contain θ', ϕ', &c.,

$$x' = \frac{dx}{dt} + \frac{dx}{d\theta}\theta' + \frac{dx}{d\phi}\phi' + \&c.\ldots\ldots\ldots\ldots\ldots(3),$$

with similar equations for y' and z'. In these the differential coefficients $\frac{dx}{dt}$, $\frac{dx}{d\theta}$, &c. are all partial. Substituting these in the expression for $2T$, we find

$$2T = F(t, \theta, \phi, \&c.\ \theta', \phi', \&c.).$$

When the system of bodies is given, the form of F will be known. It will appear presently that it is only through the form of F that the effective forces depend on the nature of the bodies considered; so that two dynamical systems which have the same F are dynamically equivalent.

It should be noticed that no powers of θ', ϕ', &c. above the second enter into this function, and when the geometrical equations do not contain the time explicitly, it is a homogeneous function of θ', ϕ', &c. of the second order.

368. *To find the virtual moments of the momenta of a system, and also of the effective forces corresponding to a displacement produced by varying one co-ordinate only.*

Let this co-ordinate be θ, and let us follow the notation already explained. Let all differential coefficients be partial, unless it be otherwise stated, excepting those denoted by accents. Since x', y', z' are the components of the velocity, the virtual moment of the momenta will be $\Sigma m\,(x'\delta x + y'\delta y + z'\delta z)$, where δx, δy, δz are the small changes produced in the co-ordinates of the particle m by a variation $\delta\theta$ of θ. This is the same as

$$\Sigma m\left(x'\frac{dx}{d\theta} + y'\frac{dy}{d\theta} + z'\frac{dz}{d\theta}\right)\delta\theta.$$

If $2T$ be the vis viva given by (2) of the last article

$$\frac{dT}{d\theta'} = \Sigma m\left(x'\frac{dx'}{d\theta'} + \&c.\right).$$

But differentiating (3) partially with regard to θ', we see that $\frac{dx'}{d\theta'} = \frac{dx}{d\theta}$. Hence the virtual moment of the momenta is equal to $\frac{dT}{d\theta'}\delta\theta$.

The virtual moment of the effective forces will be

$$\Sigma m \left(x'' \frac{dx}{d\theta} + y'' \frac{dy}{d\theta} + z'' \frac{dz}{d\theta} \right) \delta\theta.$$

This may be written in the form

$$\frac{d}{dt} \Sigma m \left(x' \frac{dx}{d\theta} + \&c. \right) - \Sigma m \left(x' \frac{d}{dt} \frac{dx}{d\theta} + \&c. \right),$$

where the $\frac{d}{dt}$ represents a total differential coefficient with regard to t. We have already proved that the first of these terms is $\frac{d}{dt}\frac{dT}{d\theta'}$. It remains to express the second term also as a differential coefficient of T. Differentiating the expression for $2T$ partially with regard to θ

$$\frac{dT}{d\theta} = \Sigma m \left(x' \frac{dx'}{d\theta} + \&c. \right).$$

But differentiating the expression for x' with regard to θ

$$\frac{dx'}{d\theta} = \frac{d^2x}{d\theta\, dt} + \frac{d^2x}{d\theta^2}\theta' + \frac{d^2x}{d\theta\, d\phi}\phi' + \&c.$$

and this is the same as $\frac{d}{dt}\frac{dx}{d\theta}$. Hence the second term may be written $\frac{dT}{d\theta}$, and the virtual moment* of the effective forces is therefore $\left(\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} \right) \delta\theta$.

* The following explanation will make the argument clearer. The virtual moment of the effective forces is clearly the ratio to dt of the difference between the virtual moments of the momenta of the particles of the system at the times $t+dt$ and t, the displacements being the same at each time. The virtual moment of the momenta at the time t is first shown to be $\frac{dT}{d\theta'}\delta\theta$. Hence $\left(\frac{dT}{d\theta'} + \frac{d}{dt}\frac{dT}{d\theta'} dt \right) \delta\theta$ is the virtual moment of the momenta at the time $t+dt$ corresponding to a displacement $\delta\theta$ consistent with the positions of the particles at that time. To make the displacements the same, we must subtract from this the virtual moment of the momenta for a displacement which is the difference between the two displacements at the times t and $t+dt$. Since $\delta x = \frac{dx}{d\theta}\delta\theta$, this difference for an abscissa is $\frac{d}{dt}\left(\frac{dx}{d\theta}\right) dt\, \delta\theta$. We therefore subtract on the whole $\Sigma m \left\{ x' \frac{d}{dt}\left(\frac{dx}{d\theta}\right) dt + \&c. \right\} \delta\theta$, and this is shown to be $\frac{dT}{d\theta} dt\, \delta\theta$.

369. *To deduce the general equations of motion referred to any co-ordinates.*

Let U be the force-function, then U is a function of θ, ϕ, &c. and t. The virtual moment of the impressed forces corresponding to a displacement produced by varying θ only is $\frac{dU}{d\theta}\delta\theta$. But by D'Alembert's principle this must be the same as the virtual moment of the effective forces. Hence

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta}.$$

Similarly we have $\frac{d}{dt}\frac{dT}{d\phi'}-\frac{dT}{d\phi}=\frac{dU}{d\phi}$,

$$\&c. = \&c.$$

It may be remarked that if V be the potential energy we must write $-V$ for U. We then have

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}+\frac{dV}{d\theta}=0,$$

with similar equations for ϕ, ψ, &c.

In using these equations, it should be remembered that all the differential coefficients are partial except that with regard to t.

These are called Lagrange's general equations of motion. Lagrange only considers the case in which the geometrical equations do not contain the time explicitly, but it has been shown by Vieille, in *Liouville's Journal*, 1849, that the equations are still true when this restriction is removed. In the proof given above we have included Vieille's extension, and adopted in part Sir W. Hamilton's mode of proof, *Phil. Trans.*, 1834. It differs from Lagrange's in these respects; firstly, he makes the arbitrary displacement such that only one co-ordinate varies at a time, and secondly, he operates directly on T instead of $\Sigma mx'^2$.

370. *To deduce the general equations of motion for Impulsive forces.*

Let δU_1 be the virtual moment of the impulsive forces produced by any displacement of the system. Then from the geometry of the system, we can express δU_1 in the form

$$\delta U_1 = P\delta\theta + Q\delta\phi + \dots$$

The virtual moment of the momenta imparted to the particles of the system is

$$\Sigma m\{(x_1'-x_0')\,\delta x+(y_1'-y_0')\,\delta y+(z_1'-z_0')\,\delta z\},$$

where (x_0', y_0', z_0'), (x_1', y_1', z_1') are the values of (x', y', z') just before and just after the action of the impulsive forces.

Let θ_0', ϕ_0', &c. θ_1', ϕ_1', &c. be the values of θ', ϕ', &c. just before and just after the impulse, and let T_0, T_1 be the values of T when these are substituted for θ', ϕ', &c. Then as in Art. 368 the virtual moment of the momenta is $= \left(\frac{dT_1}{d\theta_1'} - \frac{dT_0}{d\theta_0'}\right)\delta\theta$. The Lagrangian equations of impulses may therefore be written

$$\frac{dT_1}{d\theta_1'} - \frac{dT_0}{d\theta_0'} = P,$$

with similar equations for ϕ, and ψ, &c.

371. If we compare this equation with the general principle of Art. 295, viz. that the momenta of the particles just after an impulse compounded with the reversed momenta just before are equivalent to the impulse, we see that it will be convenient to call $\frac{dT}{d\theta'}$ the component of the momenta with regard to θ, a name only slightly altered from that suggested in Thomson and Tait's *Natural Philosophy*. More briefly we may say that the θ-component of the momentum is $\frac{dT}{d\theta'}$. In the same way we may define the θ component of the effective forces to be $\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta}$.

372. These equations for impulsive forces are not given by Lagrange. They seem to have been first deduced by Prof. Niven from the Lagrangian equation

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta}.$$

We may regard an impulse as the limit of a very large force acting for a very short time. Let t_0, t_1 be the times at which the force begins and ceases to act. Let us integrate this equation between the limits $t = t_0$ to $t = t_1$. The integral of the first term is $\left[\frac{dT}{d\theta'}\right]_{t_0}^{t_1}$ which is the difference between the initial and final values of $\frac{dT}{d\theta'}$. The integral of the second term is zero. For $\frac{dT}{d\theta}$ is a function of θ, ϕ, &c. θ', ϕ', &c. which though variable remains finite during the time $t_1 - t_0$. If A be its greatest value during this time, then the integral is less than $A(t_1 - t_0)$ which ultimately vanishes. Hence the Lagrangian equation becomes $\left[\frac{dT}{d\theta'}\right]_{t_0}^{t_1} = \frac{dU_1}{d\theta}$. See a paper in the *Mathematical Messenger* for May, 1867.

373. Other expressions for the virtual moments of the momenta and of the effective forces may be found when T is expressed in terms of the angular velocities of the bodies of the system instead of the co-ordinates. Thus taking any one body, if (x, y, z) be the co-ordinates of its centre of gravity, ω_x, ω_y, ω_z the angular velocities about rectangular axes meeting at the centre of gravity, M its mass, A, B, C, &c. its moments and products of inertia, v the velocity of its centre of gravity, then by Art. 348,

$$2T = Mv^2 + A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y.$$

The virtual moment of the momenta will then be by Ex. 3. of Art. 349

$$\frac{dT}{dx'}\delta x+\frac{dT}{dy'}\delta y+\frac{dT}{dz'}\delta z+\frac{dT}{d\omega_x}\delta\theta+\frac{dT}{d\omega_y}\delta\phi+\frac{dT}{d\omega_z}\delta\psi,$$

and by Ex. 4 the virtual moment of the effective forces will be if the directions of the axes are *fixed in space*

$$\frac{d}{dt}\frac{dT}{dx'}\delta x+\&c.+\frac{d}{dt}\frac{dT}{d\omega_x}\delta\theta+\&c.,$$

where δx, δy, δz are the linear displacements of the centre of gravity and $\delta\theta$, $\delta\phi$, $\delta\psi$ the angular displacements of the body about the axes of ω_x, ω_y, ω_z. If the axes be *moving* we have merely to substitute for the coefficients of δx, &c. the corresponding expressions given in the example just referred to.

374. Before proceeding to discuss some properties of Lagrange's equations, let us illustrate their use by the following problems.

A body, two of whose principal moments at the centre of gravity are equal, turns about a fixed point O situated in the axis of unequal moment under the action of gravity. To determine the conditions that there may be a simple equivalent pendulum.

Def. If a body be suspended from a fixed point O under the action of gravity, and if the angular motion of the straight line joining O to the centre of gravity be the same as that of a string of length l to the extremity of which a heavy particle is attached, then l is called the *length of the simple equivalent pendulum.* This is an extension of the definition in Art. 92.

Let OC be the axis of unequal moment, A, A, C the principal moments at the fixed point, and let the rest of the notation be the same as in Art. 349, Ex. 1. Then

$$2T=A(\theta'^2+\sin^2\theta\psi'^2)+C(\phi'+\psi'\cos\theta)^2,$$

$$U=Mgh\cos\theta+\text{constant},$$

where h is the distance of the centre of gravity from the fixed point, and gravity is supposed to act in the positive direction of the axis of z. Lagrange's equations will be found to become

$$\frac{d}{dt}(A\theta')-A\sin\theta\cos\theta\psi'^2+C\psi'(\phi'+\psi'\cos\theta)\sin\theta=-Mgh\sin\theta,$$

$$\frac{d}{dt}\{C(\phi'+\psi'\cos\theta)\}=0,$$

$$\frac{d}{dt}\{C(\phi'+\psi'\cos\theta)\cos\theta+A\sin^2\theta\psi'\}=0.$$

Integrating the second of Lagrange's equations we have

$$\phi'+\psi'\cos\theta=n,$$

where n is some constant expressing the angular velocity about the axis of unequal moment. Integrating the third we have

$$Cn\cos\theta+A\sin^2\theta\frac{d\psi}{dt}=a,$$

where a is another constant expressing the moment of the momentum about the vertical through O.

There is an error, sometimes made in using Lagrange's equations, which we should here guard against. If ω_3 be the angular velocity about OC, we know by Euler's equations, Art. 230, that ω_3 is constant. If n be this constant, the Vis Viva of the body might have been correctly written in the form

$$2T = A(\theta'^2 + \sin^2\theta\psi'^2) + Cn^2.$$

But if this value of T be substituted in Lagrange's equations, we should obtain results altogether erroneous. The reason is, that, in Lagrange's equations, all the differential coefficients except those with regard to t are partial. Though ω_3 is constant, and therefore its total differential coefficient with regard to t is zero, yet its partial differential coefficients with regard to θ, ϕ, &c. are not zero. In writing down the value of T, preparatory to using it in Lagrange's equation, no properties of the motion are to be assumed which involve differential coefficients of the co-ordinates as indicated in Art. 367. But we must introduce into the expression any geometrical relations which exist between the co-ordinates and which therefore reduce the number of independent variables.

Instead of the first equation, we may use the equation of vis viva, which gives

$$A\left\{\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right\} = \beta + 2Mgh\cos\theta.$$

To determine the arbitrary constants α and β we must have recourse to the initial values of θ and ψ. Let θ_0, ψ_0, $\frac{d\theta_0}{dt}$, $\frac{d\psi_0}{dt}$ be the initial values of θ, ψ, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, then the above equations become

$$\left.\begin{aligned}\sin^2\theta\frac{d\psi}{dt} + \frac{Cn}{A}\cos\theta &= \sin^2\theta_0\frac{d\psi_0}{dt} + \frac{Cn}{A}\cos\theta_0\\ \sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 &= \sin^2\theta_0\left(\frac{d\psi_0}{dt}\right)^2 + \left(\frac{d\theta_0}{dt}\right)^2 + 2\frac{Mgh}{A}(\cos\theta - \cos\theta_0)\end{aligned}\right\}\ldots\ldots(1).$$

These equations, when solved, give θ and ψ in terms of t, and thus determine the motion of the line OG. The corresponding equations for the motion of the simple equivalent pendulum OL are found by making $C=0$, $A=Ml^2$, and $h=l$, where l is the length of the pendulum. This gives

$$\left.\begin{aligned}\sin^2\theta\frac{d\psi}{dt} &= \sin^2\theta_0\frac{d\psi_0}{dt}\\ \sin^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 &= \sin^2\theta_0\left(\frac{d\psi_0}{dt}\right)^2 + \left(\frac{d\theta_0}{dt}\right)^2 + 2\frac{g}{l}(\cos\theta - \cos\theta_0)\end{aligned}\right\}\ldots\ldots\ldots.(2).$$

In order that the motions of the two lines OG and OL may be the same, the two equations (1) and (2) must be the same. This will be the case if either $Cn=0$, or $\theta=\theta_0$. In the first case, we must have $n=0$, or $C=0$, so that the body must either have no rotation about OG, or else the body must be a rod. In the second case, we must have throughout the motion θ and $\frac{d\psi}{dt}$ constant, so that the body must be moving in steady motion making a constant angle with the vertical. In either case, the two sets of equations are identical if $l=\frac{A}{Mh}$. This is the same formula which was obtained in Art. 92.

375. Ex. 1. *Show how to deduce Euler's equations, Art. 280, from Lagrange's equations.*

Taking as axes of reference the principal axes at the fixed point,

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2.$$

We cannot take $(\omega_1, \omega_2, \omega_3)$ as the independent variables because the co-ordinates of every particle of the body cannot be expressed in terms of them without introducing differential coefficients into the geometrical equations. Let us therefore express $\omega_1, \omega_2, \omega_3$ in terms of θ, ϕ, ψ. By Art. 235, we have

$$\left.\begin{aligned} \omega_1 &= \theta' \sin\phi - \psi' \sin\theta \cos\phi \\ \omega_2 &= \theta' \cos\phi + \psi' \sin\theta \sin\phi \\ \omega_3 &= \phi' + \psi' \cos\theta \end{aligned}\right\}.$$

As it will be only necessary to establish one of Euler's equations, the others following by symmetry, we need only use that one of Lagrange's equations which gives the simplest result. Since ϕ' does not enter into the expressions for ω_1, ω_2, it will be most convenient to use the equation

$$\frac{d}{dt}\frac{dT}{d\phi'} - \frac{dT}{d\phi} = \frac{dU}{d\phi}.$$

Now $\frac{dT}{d\phi'} = C\omega_3 \frac{d\omega_3}{d\phi'} = C\omega_3$, and $\frac{dT}{d\phi} = A\omega_1 \frac{d\omega_1}{d\phi} + B\omega_2 \frac{d\omega_2}{d\phi} = A\omega_1\omega_2 - B\omega_2\omega_1$, as may be seen by differentiating the expressions for ω_1, ω_2. Also by Art. 326, if N be the moment of the forces about the axis of C, $\frac{dU}{d\phi} = N$.

Substituting we have

$$\frac{d}{dt}(C\omega_3) - (A - B)\omega_1\omega_2 = N,$$

which is Euler's equation.

Ex. 2. A body turns about a fixed point and its vis viva is given by

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2D\omega_2\omega_3 - 2E\omega_3\omega_1 - 2F\omega_1\omega_2.$$

Show that if the axes are fixed in the body, Euler's equations of motion may be generalized into

$$\frac{d}{dt}\frac{dT}{d\omega_1} - \frac{dT}{d\omega_2}\omega_3 + \frac{dT}{d\omega_3}\omega_2 = L,$$

with two similar equations. This result is given by Lagrange.

376. Ex. *Deduce the equation of Vis Viva from Lagrange's equations.*

If the geometrical equations do not contain the time explicitly, T is a homogeneous function of θ', ϕ', &c. of the second degree. Hence $2T = \frac{dT}{d\theta'}\theta' + \frac{dT}{d\phi'}\phi' + \ldots$

Differentiating this totally, we have $2\frac{dT}{dt} = \theta'\frac{d}{dt}\left(\frac{dT}{d\theta'}\right) + \frac{dT}{d\theta'}\theta'' + \text{\&c.}$,

where the &c. implies similar expressions for ϕ, ψ, &c. If we now substitute on the right-hand side from Lagrange's equations, we have

$$2\frac{dT}{dt} = \frac{dT}{d\theta}\theta' + \frac{dT}{d\theta'}\theta'' + \frac{dU}{d\theta}\theta' + \text{\&c.}$$

But since T is a function of θ, θ', ϕ, ϕ', &c., $\frac{dT}{dt}=\frac{dT}{d\theta}\theta'+\frac{dT}{d\theta'}\theta''+$&c., subtracting this from the last expression we have

$$\frac{dT}{dt}=\frac{dU}{d\theta}\theta'+\frac{dU}{d\phi}\phi'+\dots$$

Integrating, we have the equation of Vis Viva

$$T-U=h,$$

where h is an arbitrary constant, sometimes called the constant of Vis Viva.

377. Ex. As an illustration of the application of Lagrange's equations to impulsive forces, let us consider the example already discussed in Art. 154.

Let x be the altitude of the centre of gravity of the rhombus at any time, then x and α may be taken as the independent variables.

We have

$$T=2\{x'^2+(k^2+a^2)\,\alpha'^2\}.$$

Let P be the impulsive action between the rhombus and the plane, then the virtual moment of the impulsive forces is

$$\delta U=P\delta\,(x-2a\cos\alpha)=P\delta x+2a\sin\alpha\,P\delta\alpha.$$

The Lagrangian equations are therefore

$$\left.\begin{array}{l}4\,(x_1'-x_0')=P\\ 4\,(k^2+a^2)\,(\alpha_1'-\alpha_0')=2a\,P\sin\alpha\end{array}\right\}.$$

Now the initial and final values of x' are $x_0'=-V$, $x_1'=-2a\sin\alpha\omega$; those of α' are $\alpha_0'=0$, $\alpha_1'=\omega$. Hence eliminating P we have

$$\omega'=\frac{3}{2}\frac{V}{a}\frac{\sin\alpha}{1+3\sin^2\alpha},$$

the same result as before.

378. Sir W. R. Hamilton has put the general equations of Lagrange into another form, which is found to be more convenient for the investigation of the general properties of a dynamical system. This transformation may be made to depend on the following lemma.

Let T_1 be a function of θ, ϕ, &c., θ', ϕ', &c., such that there are no powers of the accented letters above the second. Let $\frac{dT_1}{d\theta'}=u$, $\frac{dT_1}{d\phi'}=v$, &c., then θ', ϕ', &c. may be found in terms of θ, ϕ, &c. and u, v, &c. from these equations of the first order. Let

$$T_2=-T_1+u\theta'+v\phi'+\text{\&c.},$$

and let T_2 be expressed in terms of θ, ϕ, &c., u, v, &c. only, θ', ϕ', &c. being eliminated. Then $\frac{dT_2}{d\theta}=-\frac{dT_1}{d\theta}$, $\frac{dT_2}{du}=\theta'$, with similar equations for ϕ, ψ, &c.

To prove this let us take the total differential of T_2, we have

$$dT_2 = -\frac{dT_1}{d\theta}d\theta + \left(-\frac{dT_1}{d\theta'} + u\right)d\theta' + \theta' du + \&c.$$

By the conditions of the lemma, the quantity in brackets vanishes, and therefore $\frac{dT_2}{d\theta} = -\frac{dT_1}{d\theta}$, $\frac{dT_2}{du} = \theta'$.

It should be noticed that if T_1 be a homogeneous quadratic function of $(\theta', \phi', \&c.)$ then $u\theta' + v\phi' + \&c. = 2T_1$, and therefore $T_2 = T_1$, but differently expressed, T_1 being a function of θ', ϕ', &c. and θ, ϕ, &c., T_2 a function of u, v, &c. and θ, ϕ, &c. In this case T_2 is a homogeneous quadratic function of u, v, &c.

As this process of eliminating θ', ϕ', &c. and introducing u, v, &c. will have to be frequently performed, it will be convenient to have a name for the result. We shall call T_2 the *reciprocal function* of T_1, because T_1 may be derived from T_2 by a nearly similar process.

If T_1 be the vis viva of a dynamical system, this process is equivalent to changing from the component velocities to the component momenta and conversely.

379. Ex. If (θ', ϕ', ψ'), (u, v, w) be regarded as the Cartesian co-ordinates of two points and T_1 be a homogeneous quadratic function of (θ', ϕ', ψ'), then $T_1 = h$ is the equation to a quadric. Prove that its polar reciprocal, with regard to a sphere whose radius is $\sqrt{h}$, may be found by eliminating (θ', ϕ', ψ') by means of the equations $\frac{dT_1}{d\theta'} = u$, $\frac{dT_1}{d\phi'} = v$, $\frac{dT_1}{d\psi'} = w$. Hence show geometrically that, if $T_2 = h$ be the reciprocal quadric, $\frac{dT_2}{du} = \theta'$, $\frac{dT_2}{dv} = \phi'$, $\frac{dT_2}{dw} = \psi'$.

380. *To express the Lagrangian equations in the Hamiltonian form.*

If a system be acted on by any impulses, the Lagrangian equations of motion may be written in the typical form $\left(\frac{dT}{d\theta'}\right) = P$, where the bracket implies that $\theta_1' - \theta_0'$, $\phi_1' - \phi_0'$, &c. are to be written for θ', ϕ', &c. after differentiation, using the same notation as before. Let H be the reciprocal function of T. Then these equations take the typical form $\theta_1' - \theta_0' = \left(\frac{dH}{du}\right)$, where the bracket on the right-hand side implies that $(P, Q, \&c.)$ are to be written for $(u, v, \&c.)$ after differentiation.

381. If a system be acted on by any finite forces, the Lagrangian equations of motion may be written in the typical form

$$\frac{d}{dt}\frac{dL}{d\theta'}-\frac{dL}{d\theta}=0,$$

where $L = T + U$, so that L is the *difference* between the kinetic and potential energies. Since U does not contain $(\theta', \phi', \&c.)$ the equations of transformation may be written in the form

$$u=\frac{dL}{d\theta'}=\frac{dT}{d\theta'}, \quad v=\frac{dL}{d\phi'}=\frac{dT}{d\phi'}, \&c.$$

Also Lagrange's equations may be written in the form

$$u'=\frac{dL}{d\theta}, \quad v'=\frac{dL}{d\phi}, \&c.$$

Let H be the reciprocal function of L, then these equations change into

$$\theta'=\frac{dH}{du}, \quad \phi'=\frac{dH}{dv}, \&c.$$

$$-u'=\frac{dH}{d\theta}, \quad -v'=\frac{dH}{d\phi}, \&c.,$$

which are called the Hamiltonian equations.

When the geometrical equations do not contain the time explicitly, T is a homogeneous quadratic function of $(\theta', \phi', \&c.)$, and therefore

$$u\theta' + v\phi' + \&c. = 2T.$$

Hence $$H=-L+u\theta'+v\phi'+\&c.=T-U.$$

Thus H is the *sum* of the kinetic and potential energies, and is therefore the whole energy of the system.

382. Ex. To deduce the equation of Vis Viva from the Hamiltonian equations.

Since H is a function of $(\theta, \phi, \&c.)$, $(u, v, \&c.)$ we have, if accents denote total differential coefficients with regard to the time,

$$H'=\frac{dH}{dt}+\frac{dH}{d\theta}\theta'+\frac{dH}{du}u'+\&c.=\frac{dH}{dt},$$

so that the total differential coefficient of H with regard to t is always equal to the partial differential coefficient. If the geometrical equations do not contain the time explicitly, this latter vanishes and therefore we have $H=h$, where h is a constant.

383. Ex. 1. To deduce Euler's equations of motion from the Hamiltonian equations.

Taking the same notation as in the corresponding proposition for Lagrange's equations, Art. 375, we have

$$u=\frac{dT}{d\theta'}=A\omega_1\sin\phi+B\omega_2\cos\phi,\quad v=\frac{dT}{d\phi'}=C\omega_3,$$

$$w=\frac{dT}{d\psi'}=(-A\omega_1\cos\phi+B\omega_2\sin\phi)\sin\theta+C\omega_3\cos\theta.$$

To express T in terms of (u, v, w) we must find $(\omega_1, \omega_2, \omega_3)$. We have

$$A\omega_1=u\sin\phi+(v\cos\theta-w)\frac{\cos\phi}{\sin\theta},$$

$$B\omega_2=u\cos\phi-(v\cos\theta-w)\frac{\sin\phi}{\sin\theta}.$$

Also $$H=\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2)-U.$$

As we only require one of Euler's equations, let us use $\dfrac{dH}{d\phi}=-v'$, $\dfrac{dH}{dv}=\phi'$.

The former of these gives $A\omega_1\dfrac{d\omega_1}{d\phi}+B\omega_2\dfrac{d\omega_2}{d\phi}-\dfrac{dU}{d\phi}=-C\dfrac{d\omega_3}{dt}$,

which is the same as $A\omega_1\dfrac{B\omega_2}{A}-B\omega_2\dfrac{A\omega_1}{B}-\dfrac{dU}{d\phi}=-C\dfrac{d\omega_3}{dt}$,

and this leads at once to the third Euler's equation in Art. 230. The latter of the two Hamiltonian equations leads to one of the geometrical equations of Art. 235. Thus the six Hamiltonian equations are equivalent to all the three dynamical and the three geometrical Eulerian equations.

384. Ex. 1. The position in space of a body, of mass M, is given by (x, y, z) the rectangular co-ordinates of its centre of gravity, and (θ, ϕ, ψ) the angular co-ordinates of its principal axes at the centre of gravity, as used in Art. 235. If two of its principal moments are equal and if $(\xi, \eta, \zeta, u, v, w)$ be the $(x, y, z, \theta, \phi, \psi)$ components of the momentum, prove that the Hamiltonian function H is given by

$$2H=\frac{\xi^2+\eta^2+\zeta^2}{M}+\frac{u^2}{A}+\frac{v^2}{C}+\frac{(w-v\cos\theta)^2}{A\sin^2\theta}-2U.$$

Ex. 2. If the vis viva be given by the general expression

$$2T_1=A_{11}\theta'^2+2A_{12}\theta'\phi'+\ldots\ldots$$

show that the reciprocal function of T_1 may be written in the form

$$T_2=-\frac{1}{2\Delta}\begin{vmatrix} 0 & u & v & \cdot \\ u & A_{11} & A_{12} & \cdot \\ v & A_{12} & A_{22} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

where Δ is the discriminant of T_1. Thus the coefficients of u^2, v^2, $2uv$, &c. in T_2 are the minors, after division by Δ, of the corresponding terms in T_1. See also Art. 28, Ex. 3.

385. *To explain how Lagrange's equations are to be used when some of the forces are non-conservative.*

Lagrange's equations in the form given in Art. 369 can only be used when the forces which act on the system have a force-function. If however $P\delta\theta$ be the virtual moment of the impressed forces obtained by varying θ only, $Q\delta\phi$ the vir-

tual moment obtained by varying ϕ only and so on, it is clear that Lagrange's equations may be written in the typical form $\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = P$.

386. It will often be convenient to separate the forces which act on the system into two sets. *Firstly* those which are conservative. The parts of P, Q, &c. due to these forces may be found by differentiating the force-function with regard to θ, ϕ, &c. *Secondly* those which are non-conservative, such as friction, some kinds of resistances, &c. The parts of P, Q, &c. due to these must be found by the usual methods given in Statics for writing down virtual moments.

Though these non-conservative forces do not admit of a force-function, yet sometimes their virtual moments may be represented by a differential coefficient of another kind. Thus suppose some of the forces acting on any particle of a body to be such that their resolved parts parallel to three rectangular axes fixed in space are proportional to the velocities of the particle in those directions. The virtual moment of these forces is

$$\Sigma (\mu_1 x' \delta x + \mu_2 y' \delta y + \mu_3 z' \delta z),$$

where μ_1, μ_2, μ_3 are three constants which are negative if the forces are resistances. For example, if the particles be moving in a medium whose resistance is equal to the velocity multiplied by a constant κ, then μ_1, μ_2, μ_3 are each equal to $-\kappa$. Put

$$F = \frac{1}{2}\Sigma (\mu_1 x'^2 + \mu_2 y'^2 + \mu_3 z'^2).$$

Then
$$\frac{dF}{d\theta'} = \Sigma\left(\mu_1 x' \frac{dx'}{d\theta'} + \text{\&c.}\right) = \Sigma\left(\mu_1 x' \frac{dx}{d\theta} + \text{\&c.}\right),$$

by Art. 368. Hence

$$\frac{dF}{d\theta'}\delta\theta + \frac{dF}{d\phi'}\delta\phi + \text{\&c.} = \Sigma\left\{\mu_1 x'\left(\frac{dx}{d\theta}\delta\theta + \frac{dx}{d\phi}\delta\phi + \dots\right) + \text{\&c.}\right\}$$
$$= \Sigma (\mu_1 x' \delta x + \text{\&c.}).$$

In this case, therefore, if U be the force-function of the conservative forces, F the function just defined, $\Theta\delta\theta$, $\Phi\delta\phi$, &c. the virtual moments of the remaining forces, Lagrange's equations may be written

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} + \frac{dF}{d\theta'} + \Theta,$$

with similar equations for ϕ, ψ, &c. The use of this function was suggested by Lord Rayleigh in the Proceedings of the London Mathematical Society, June, 1873. The function F was called by him the *Dissipation Function.*

387. Ex. 1. If any two particles of a dynamical system act and react on each other with a force whose resolved parts in three fixed directions at right angles are proportional to the relative velocities of the particles in those directions, show that these may be included in the dissipation function F. If V_x, V_y, V_z be the components of the velocities, $\mu_1 V_x$, $\mu_2 V_y$, $\mu_3 V_z$ the components of the force of repulsion, the part of F due to these is $\frac{1}{2}\Sigma (\mu_1 V_x^2 + \mu_2 V_y^2 + \mu_3 V_z^2)$. This example is taken from the paper just referred to.

Ex. 2. A solid body moves in a medium which acts on *every* element of the surface with a resisting force partly frictional and partly normal to the surface.

Each of these when referred to a unit of area is equal to the velocity resolved in its own direction multiplied by the same constant κ. Show that these resistances may be included in a dissipation function F,

$$F = -\frac{\kappa}{2}\{\sigma(u^2+v^2+w^2) + A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y\},$$

where σ is the area; A, B, &c. the moments and products of inertia of the *surface* of the body and (u, v, w) the resolved velocities of the centre of gravity of σ.

388. *To explain how Lagrange's equations can be used in some cases when the geometrical equations contain differential coefficients with regard to the time.*

It has been pointed out in Art. 367, that the independent variables θ, ϕ, &c. used in Lagrange's equations must be so chosen that all the co-ordinates of the bodies in the system can be expressed in terms of them without introducing θ', ϕ', &c. But when we have to discuss a motion like that of a body rolling on a perfectly rough surface, the condition that the relative velocity of the points in contact is zero may sometimes be expressed by an equation which, like that given in Art. 127, may necessarily involve differential coefficients of the co-ordinates. In some cases the equation expressing this condition is integrable. For example; when a sphere rolls on a rough plane, as in Art. 133, the condition is $x' - a\theta' = 0$, which by integration becomes $x - a\theta = b$ where b is some constant. In such cases we may use the condition as one of the geometrical relations of the motion, thus reducing by one the number of independent variables.

But when the conditions cannot easily be cleared of differential coefficients, it will be often convenient to introduce the reactions and frictions into the equations among the non-conservative forces in the manner explained in Art. 386. Each reaction will have an accompanying equation of condition, and thus we shall always have sufficient equations to eliminate the reactions and determine the co-ordinates of the system.

The elimination of the reactions may generally be most easily effected by recurring to the general equation of Virtual Velocities, and giving only such displacements to the system as may make the virtual moments of these forces disappear. Suppose, to fix our ideas, a body is rolling on a perfectly rough surface. Let θ, ϕ, &c. be the six co-ordinates of the body, then by Art. 127, there will be three equations of the form

$$L_1 = A_1\theta' + B_1\phi' + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the other two being derived from this by writing 2 and 3 for the suffix. These three equations express the fact that the resolved

velocities in three directions of the point of contact are zero. The equation of virtual velocities may be written

$$\left(\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}\right)\delta\theta + \&c. = \frac{dU}{d\theta}\delta\theta + \&c. \quad\text{...............}\quad (2),$$

where U is the force-function of the impressed forces. Since the virtual moments of the reactions at the point of contact have been omitted, this equation is not true for all variations of θ, ϕ, &c., but only for such as make the body roll on the rough surface. But the geometrical equations L_1, L_2, L_3 express the fact that the body rolls in some manner, hence $\delta\theta$, $\delta\phi$, &c. are connected by three equations of the form

$$A_1\delta\theta + B_1\delta\phi + \ldots = 0 \quad\text{....................}\quad (3).$$

If we use the method of indeterminate multipliers*, the equations of virtual velocities will be transformed in the usual manner into

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta}+\lambda\frac{dL_1}{d\theta'}+\mu\frac{dL_2}{d\theta'}+\nu\frac{dL_3}{d\theta'} \quad\text{.........}\quad (4),$$

with similar equations for the other co-ordinates ϕ, ψ, &c. These joined to the three equations L_1, L_2, L_3 are sufficient to determine the co-ordinates of the body and λ, μ, ν.

This process will be very much simplified, if we prepare the geometrical equations L_1, L_2, L_3 by elimination, so that one differential coefficient, as θ', is absent from all but the first equation, another, as ϕ', absent from all but the second, and so on. When this has been done, the equation for θ becomes

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta}+\lambda\frac{dL_1}{d\theta'} \quad\text{..................}\quad (5).$$

Thus λ is found at once. The values of μ and ν may be found from the corresponding equations for ϕ, ψ. We may then substitute their values in the remaining equations.

389. The method of indeterminate multipliers is really an introduction of the unknown reactions into Lagrange's equations.

* If we multiply the geometrical equations (3) by λ, μ, ν respectively and subtract them from (2) we get

$$\Sigma\left[\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}-\frac{dU}{d\theta}-\lambda\frac{dL_1}{d\theta'}-\mu\frac{dL_2}{d\theta'}-\nu\frac{dL_3}{d\theta'}\right]\delta\theta=0.$$

Now there will be as many indeterminate multiples λ, μ, ν as there are geometrical equations (3) connecting the quantities $\delta\theta$, $\delta\phi$, &c., *i.e.* there are as many multipliers as there are dependent variations. By properly choosing λ, μ, ν the coefficients of these variations may be made to vanish, and then the coefficients of the independent variations must vanish of themselves. Hence the coefficient of each variation in this summation will be separately zero.

Thus let R_1, R_2, R_3 be the resolved parts of the reaction at the point of contact in the directions of the three straight lines used in forming the equations L_1, L_2, L_3. Then L_1, L_2, L_3 are proportional to the resolved relative velocities of the points of contact. Let these velocities be $\kappa_1 L_1$, $\kappa_2 L_2$, $\kappa_3 L_3$. Then if θ only be varied the virtual velocity of R_1 is $\kappa_1 A_1 \delta\theta$ which may be written $\kappa_1 \frac{dL_1}{d\theta'}\delta\theta$. Similarly the virtual velocities of R_2 and R_3 are $\kappa_2 \frac{dL_2}{d\theta'}\delta\theta$ and $\kappa_3 \frac{dL_3}{d\theta'}\delta\theta$. Hence, by Art. 385, Lagrange's equations are

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} + \kappa_1 R_1 \frac{dL_1}{d\theta'} + \kappa_2 R_2 \frac{dL_2}{d\theta'} + \kappa_3 R_3 \frac{dL_3}{d\theta'}.$$

Comparing this with the equations obtained by the method of indeterminate multipliers we see that λ, μ, ν are proportional to the resolved parts of the reactions. The advantage of using the method of indeterminate multipliers is that the reactions are introduced with the least amount of algebraic calculation, and in just that manner which is most convenient for the solution of the problem.

The method of indeterminate multipliers may sometimes be used with advantage when the geometrical equations do not contain θ', ϕ', &c., but are too complicated to be conveniently solved. Thus if

$$f(t, \theta, \phi, \ldots) = 0$$

be a geometrical equation, connecting θ, ϕ, &c., we have, as in Art. 335,

$$\frac{df}{d\theta}\delta\theta + \frac{df}{d\phi}\delta\phi + \ldots = 0.$$

This may be treated in the same manner as the equations L_1, L_2, L_3 in the preceding theory. We thus obtain the equation

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} + \lambda\frac{df}{d\theta} + \ldots$$

with similar equations for ϕ, ψ, &c.

390. Ex. *Form by Lagrange's method the equations of motion of a homogeneous sphere rolling on an inclined plane under the action of gravity.*

Let the axis of x be taken down the plane along the line of greatest slope and let the axis of y be horizontal and that of z normal to the plane. Let (x, y, a) be the co-ordinates of the centre of gravity of the sphere, θ, ϕ, ψ the angular co-ordinates of three diameters at right angles fixed in the sphere in the manner explained in Art. 235. Then, if the mass be taken as unity, the Vis Viva is by Art. 349

$$2T = x'^2 + y'^2 + k^2\{(\phi' + \psi' \cos\theta)^2 + \theta'^2 + \sin^2\theta\psi'^2\}.$$

The resolved velocities parallel to the axes of x and y of the point of the sphere in contact with the plane are to be zero. These conditions will be found to lead to the equations
$$L_1 = x' - a\theta' \cos\phi - a\psi' \sin\theta \sin\phi = 0,$$
$$L_2 = y' + a\theta' \sin\phi - a\psi' \sin\theta \cos\phi = 0.$$
Also if g be the resolved part of gravity along the plane and C any constant
$$U = gx + C.$$

The general equation of motion is
$$\frac{d}{dt}\frac{dT}{dq'} - \frac{dT}{dq} = \frac{dU}{dq} + \lambda \frac{dL_1}{dq'} + \mu \frac{dL_2}{dq'},$$
where q stands for any one of the five co-ordinates x, y, θ, ψ, ϕ. Taking these in turn we have
$$x'' = g + \lambda, \quad y'' = \mu,$$
$$\left.\begin{aligned} &k^2(\theta'' + \phi'\psi' \sin\theta) = -\lambda a \cos\phi + \mu a \sin\phi \\ &k^2 \frac{d}{dt}(\phi' \cos\theta + \psi') = -\lambda a \sin\theta \sin\phi - \mu a \sin\theta \cos\phi \\ &k^2 \frac{d}{dt}(\phi' + \psi' \cos\theta) = 0 \end{aligned}\right\}.$$

The last equation shows that $\phi' + \psi' \cos\theta$ is constant. From this we infer that the angular velocity of the sphere about a normal to the plane is constant throughout the motion. Eliminating μ from the two preceding equations and substituting for ψ'' from the last, we find
$$-\frac{\lambda a}{k^2} = \theta'' \cos\phi + \psi'' \sin\theta \sin\phi - \theta'\phi' \sin\phi + \phi'\psi' \sin\theta \cos\phi + \theta'\psi' \cos\theta \sin\phi.$$

But this is $\dfrac{x''}{a}$. In the same way we find $-\dfrac{\mu a}{k^2} = \dfrac{y''}{a}$. Substituting these values of λ and μ in the first two of Lagrange's equations, we have
$$x''\left(1 + \frac{k^2}{a^2}\right) = g, \text{ and } y''\left(1 + \frac{k^2}{a^2}\right) = 0.$$

These are the equations of motion of a projectile. Hence the centre of gravity describes a parabola as if it were under a constant acceleration equal to $\dfrac{a^2 g}{a^2 + k^2}$ tending along the line of greatest slope.

If we had used some of the other expressions for the virtual moments given in Art. 373, the solution of this problem would have been much simplified. Thus let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the sphere about axes meeting at the centre of gravity parallel to the co-ordinate axes. Then
$$2T = x'^2 + y'^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2),$$
and the equations of condition are
$$x' - a\omega_y = 0, \qquad y' + a\omega_x = 0.$$

Displace the sphere by rolling it along a small arc parallel to the axis of x through an angle $\delta\theta$. Then we have
$$\frac{d}{dt}\frac{dT}{dx'} a\delta\theta + \frac{d}{dt}\frac{dT}{d\omega_y}\delta\theta = \frac{dU}{dx} a\delta\theta, \qquad \therefore\ ax'' + k^2 \frac{d\omega_y}{dt} = ga.$$

Similarly rolling the sphere parallel to the axis of y and twisting it round the axis of ω_z, we have
$$-ay'' + k^2 \frac{d\omega_x}{dt} = 0, \text{ and } k^2 \frac{d\omega_z}{dt} = 0.$$

These, by elimination of $\omega_x, \omega_y, \omega_z$, lead to the same result as before.

Principles of Least Action and Varying Action.

391. Let (q_1, q_2, q_3, &c.) be the co-ordinates of a system of bodies, and let q stand for any one of these. Let $2T$ be the vis viva of the whole system and U the force-function, and let $L = T + U$. As before let accents denote differential coefficients with regard to the time.

Let us imagine the system to be moving in some manner, which we will call the actual motion. Then q_1, q_2, &c. are all functions of t, and it is generally our object to find the form of these functions. Let us suppose the system to move in some slightly different manner, i.e. let q_1, q_2, &c. be functions of t slightly different from their actual forms. Let us call the motion thus represented a neighbouring motion. We may pass, in our minds, from the actual motion to any neighbouring motion by the process called *variation* in the calculus of that name. By the fundamental theorem in that calculus

$$\delta \int_{t_0}^{t_1} L dt = [L\delta t]_{t_0}^{t_1} + \int_{t_0}^{t_1} \Sigma \left(\frac{dL}{dq} - \frac{d}{dt}\frac{dL}{dq'}\right)(\delta q - q'\delta t)\, dt$$
$$+ \left[\Sigma \frac{dL}{dq'}(\delta q - q'\delta t)\right]_{t_0}^{t_1},$$

where the letter Σ implies summation for all the co-ordinates q_1, q_2, &c. and, as implied by the square brackets, the terms outside the integral sign are to be taken between limits.

The co-ordinates being independent of each other, each separate term under the integral sign vanishes by Lagrange's equations, and we have therefore

$$\delta \int_{t_0}^{t_1} L\, dt = \left[\left(L - \Sigma \frac{dT}{dq'} q'\right)\delta t + \Sigma \frac{dT}{dq'}\delta q\right]_{t}^{t_1}$$
$$= \left[-H\delta t + \Sigma \frac{dT}{dq'}\delta q\right]_{t_0}^{t_1},$$

where H is the reciprocal function of L, by Art. 378.

The integral $\int_{t_0}^{t_1} L\, dt$ has been called by Sir W. R. Hamilton the *principal function*, and is usually represented by the letter S.

If the geometrical equations do not contain the time explicitly, we have $H = T - U$. In this case the equation of vis viva will hold, and if h be the constant of vis viva we have

$$\delta \int_{t_0}^{t_1} L\, dt = -h(\delta t_1 - \delta t_0) + \left[\Sigma \frac{dT}{dq'}\delta q\right]_{t_0}^{t_1}.$$

392. Other functions may be used instead of S. Let us put

$$V = S + [Ht]_{t_0}^{t_1},$$

$$\therefore \delta V = \delta S + [H\delta t + t\delta H]_{t_0}^{t_1}$$

$$= \left[t\delta H + \Sigma \frac{dT}{dq'}\delta q\right]_{t_0}^{t_1}.$$

The function V is called the *characteristic function.*

If the geometrical equations do not contain the time explicitly, we have $H = h$, where h is a constant which may be used to represent the whole energy of the system. In this case

$$V = S + h(t_1 - t_0)$$

$$= \int_{t_0}^{t_1} (T + U)\,dt + \int_{t_0}^{t_1} (T - U)\,dt$$

$$= 2\int_{t_0}^{t_1} T dt.$$

The function V therefore expresses the whole accumulation of the vis viva, i.e. the *action* of the system in passing from its position at the time t_0 to its position at the time t_1.

393. In the proof of these theorems we have supposed that all the forces are conservative. If in addition to the impressed forces there are any reactions, such as rolling friction, which cannot be taken account of by reducing the number of independent co-ordinates, we must use Lagrange's equation in the form

$$\frac{d}{dt}\frac{dL}{dq'} - \frac{dL}{dq} = P,$$

where, as explained in Art. 385, $P\delta q$ is the virtual moment of these reactions corresponding to a displacement δq. In this case the quantity under the integral sign will not vanish unless the variations are such that

$$\Sigma P(\delta q - q'\delta t) = 0.$$

Now q being the value of any co-ordinate in the actual motion at the time t, $q + \delta q$ is its value in a neighbouring motion at the time $t + \delta t$. But $q'\delta t$ is the change of q in the time δt, hence $q + \delta q - q'\delta t$ is the value of the co-ordinate in the neighbouring motion at the time t. The neighbouring motions must therefore be such that the virtual moments of the reactions corresponding to a displacement of the system from any position in the actual motion into its position in a neighbouring motion at the same time is zero. With this restriction on the variations, the two equations which express the variations of S and V will still be true.

394. The two fundamental equations, giving the values of δS and δV, will be found to lead to many important theorems which we shall now proceed to consider.

Let us call the positions of the system at the times t_0 and t_1 the initial and terminal positions, and let us suppose these fixed, so

that the actual motion and all its neighbouring motions are to have the same initial and terminal positions. In this case δq vanishes at each limit, and the two fundamental equations take the form*

$$\delta \int_{t_0}^{t_1} L\, dt = -h\,(\delta t_1 - \delta t_0),$$

$$2\delta \int_{t_0}^{t_1} T\, dt = (t_1 - t_0)\,\delta h,$$

* We may easily establish these theorems without the use of Lagrange's theorems. Let (x, y, z) be the rectangular co-ordinates of any particle and let m be the mass of this particle. Let X, Y, Z be the components of the impressed accelerating forces on it. Then

$$L = \frac{1}{2}\Sigma m\,(x'^2 + y'^2 + z'^2) + U,$$

and by the fundamental theorem in the Calculus of Variations

$$\delta \int_{t_0}^{t_1} Ldt = [L\delta t]_{t_0}^{t_1} + \int_{t_0}^{t_1} \Sigma \left(\frac{dL}{dx} - \frac{d}{dt}\frac{dL}{dx'}\right)(\delta x - x'\delta t)\, dt + \left[\Sigma \frac{dL}{dx'}(\delta x - x'\delta t)\right]_{t_0}^{t_1}.$$

If we substitute for L and remember that T is a homogeneous function of x', y', z', this becomes

$$\delta \int_{t_0}^{t_1} Ldt = [(U - T)\,\delta t + \Sigma m x'\delta x]_{t_0}^{t_1} + \int_{t_0}^{t_1} \Sigma m\,(X - x'')\,(\delta x - x'\delta t) dt.$$

If we consider the positions of the system at the times t_0 and t_1 to be given, δx is zero in the part taken between limits.

If the time of transit be given it is unnecessary to vary the time. Putting $\delta t = 0$, the part under the integral sign vanishes by the principle of virtual velocities. The part outside the integral sign is also zero and therefore $\delta \int_{t_0}^{t_1} Ldt = 0$.

If the time be varied, $\delta x - x'\delta t$ is the projection on the axis of x of the displacement of the particle m from its position in the actual motion at time t to its position in a neighbouring motion at the same time. Hence the part under the integral sign vanishes as before by the principle of virtual velocities. Let us suppose that the geometrical conditions do not contain the time explicitly, then $T - U = h$ and $L = 2T - h$. The equation then becomes

$$2\delta \int_{t_0}^{t_1} Tdt - [\delta(ht)]_{t_0}^{t_1} = [-h\delta t]_{t_0}^{t_1}.$$

If h be given we have $\delta \int_{t_0}^{t_1} Tdt = 0$.

From the general value of the variation in Cartesian co-ordinates we can also deduce the values of δS and δV given in the text. For the term $\Sigma m x'$ is clearly the virtual moment of the momenta, and this by Art. 368 is $\frac{dT}{dq'}\delta q$. The method followed in the text seems however to be preferable.

Lagrange has given a general view of his transformation from Cartesian co-ordinates which seems worthy of notice. Let L be any function of x, x', &c., y, y', &c. and of t, and let the variables x, y, &c. be transformed into others

where it has been supposed that the geometrical equations do not contain the time explicitly.

If the time of transit of the system from its initial to its terminal position be also given, we have $\delta t_1 = \delta t_0$, and therefore $\delta \int_{t_0}^{t_1} L\, dt = 0$.

Hence $\int_{t_0}^{t_1} L\, dt$ is either a maximum or a minimum. It cannot be the former, since by causing the bodies to take circuitous paths we may make it as large as we please. It is therefore a minimum.

If the constant h be given, or which is the same thing (since the terminal position is given) if the energy of the system be given, we have $\delta h = 0$, and therefore $\delta \int_{t_0}^{t_1} T\, dt = 0$. We may now infer the two following theorems.

Let any two positions of a dynamical system be given, the actual motion is such that $\int T\, dt$ is less than if the system were constrained, without violating any geometrical conditions, to move in some other manner from the one position to the other with the same energy; these other motions being such that, throughout, T is the same function of the co-ordinates and their differential coefficients.

This is called the principle of Least Action.

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q_1, q_2 &c. by writing for x, y, &c. any functions of q_1, q_2, &c. and of t. The function L is thus expressed in two ways, and by comparing the two values of $\delta \int_{t_0}^{t_1} L\, dt$ given by the Calculus of Variations, we see that the integral of

$$\Sigma \left\{ \left(\frac{dL}{dx'} - \&c. \right) \delta x - \left(\frac{dL}{dq'} - \&c. \right) \delta q \right\}$$

may be completely found. Hence this expression must be a perfect differential with regard to t, quite independently of the operation δ. But this cannot be unless it vanishes, because it contains only the variations δx, δq, &c. and not the differential coefficients of these variations. We have therefore the general equation of transformation

$$\Sigma \left(\frac{dL}{dx} - \frac{d}{dt} \frac{dL}{dx'} + \&c. \right) \delta x = \Sigma \left(\frac{dL}{dq} - \frac{d}{dt} \frac{dL}{dq'} + \&c. \right) \delta q,$$

where the Σ implies summation for all the variables x, y, &c. or q_1, q_2, &c.

If x, y, &c. be Cartesian co-ordinates the left-hand side of this equality vanishes by virtual velocities. Hence $\Sigma \left(\frac{dL}{dq} - \&c. \right) \delta q$ must also vanish. The q's being all independent, we are led to Lagrange's equations.

In the same way if the system moves in the varied motion, not with the same energy, but in the same time, from the one given position to the other, then $\int L\,dt$ is a minimum.

395. Maupertuis conceived that he could establish *à priori* by theological arguments, that all mechanical changes must take place in the world so as to occasion the least possible quantity of *action*. In asserting this, it was proposed to measure the action by the product of velocity and space; and this measure being adopted, mathematicians, though they did not generally assent to Maupertuis' reasonings, found that his principle expressed a remarkable and useful truth, which might be established on known mechanical grounds. Whewell's *History of the Inductive Sciences*, Vol. II. p. 119.

396. Conversely, from either of these theorems we may deduce the motion of any system, by making $\int_{t_0}^{t_1} L\,dt$ or $\int_{t_0}^{t_1} T\,dt$ a minimum according to the rules of the Calculus of Variations*. That this

* Lagrange's equations are the ordinary equations supplied by the Calculus of Variations when we make $\int L dt$ a minimum under known conditions. Sir W. Hamilton put these equations under a form (see Art. 381) which is very useful in Dynamics. It is an interesting question to determine what is the corresponding transformation when L is a function of differential coefficients higher than the first. This was considered by Ostrogradsky in a *Mémoire sur les equations differentielles relative au problème des Isopérimètres*, published in the *Memoirs of the Academy of Sciences* at St Petersburgh in 1850. The Memoir is rather difficult on account of the immense length of the algebraical transformations. The following short account may therefore prove useful.

Let L be a function of t and of m variables, of which q is any one, and let it be a function of the first n differential coefficients of q with regard to t.

Let Q_k stand for the partial differential coefficient of L with regard to $\frac{d^kq}{dt^k}$, and let

$$\overline{Q}_k = Q_k - Q'_{k+1} + Q''_{k+2} - \ldots\ldots,$$

where, as usual, accents denote differential coefficients with regard to t, and let k accents be denoted by (k). The relations between these variables are, therefore,

$$\left.\begin{aligned} \overline{Q}_0 &= \frac{dL}{dq} - \overline{Q}'_1 \\ \overline{Q}_1 &= \frac{dL}{dq'} - \overline{Q}'_2 \\ &\text{and so on up to} \\ \overline{Q}_{n-1} &= \frac{dL}{dq^{(n-1)}} - \overline{Q}'_n \\ &\text{and the last is} \\ \overline{Q}_n &= \frac{dL}{dq^{(n)}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

By the principles of the Calculus of Variations, the minimum is given by the typical equation $\overline{Q}_0 = 0$.

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process will really lead to the equations of motion may be seen by simply reversing our steps. Thus granting that $\delta\int L\,dt=0$ under the known conditions, we have

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When L contains no differential coefficient above the first, Sir W. Hamilton eliminated the m first differential coefficients typified by q' by introducing m new variables typified by $Q_1=\dfrac{dL}{dq'}$. Let us in the same way eliminate the highest differential coefficients typified by $q^{(n)}$ and introduce instead the m new variables typified by Q_n. Let

$$H=L-\Sigma\,(\bar{Q}_1q'+\bar{Q}_2q''+\ldots+\bar{Q}_nq^{(n)}),$$

where the Σ refers to summation for all the q's. Let $q^{(n)}$ be found from the equation $\bar{Q}_n=\dfrac{dL}{dq^{(n)}}$ and let its value be substituted in this expression for H so that H is now a function of t, q, $q'\ldots q^{(n-1)}$, $\bar{Q}_1$, $\bar{Q}_2\ldots\bar{Q}_n$. Since L was originally a function of t, q, $q'\ldots q^{(n)}$ it is now a function of t, q, $q'\ldots q^{(n-1)}$ and $\bar{Q}_n$.

We have by differentiation

$$\frac{dH}{d\bar{Q}_{k+1}}=-q^{(k+1)}=-\frac{d}{dt}q^{(k)}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

provided $k+1$ is not n. In that case

$$\frac{dH}{d\bar{Q}_n}=\frac{dL}{dq^{(n)}}\frac{dq^{(n)}}{d\bar{Q}_n}-q^{(n)}-\bar{Q}_n\frac{dq^{(n)}}{d\bar{Q}_n},$$

but the first and third of these terms destroy each other, so that the theorem (2) is also true when $k+1=n$. Also

$$\frac{dH}{dq^{(k)}}=\frac{dL}{dq^{(k)}}+\frac{dL}{dq^{(n)}}\frac{dq^{(n)}}{dq^{(k)}}-\bar{Q}_k-\bar{Q}_n\frac{dq^{(n)}}{dq^{(k)}}.$$

Here the second and fourth terms destroy each other. The first and third, by (1), become $\bar{Q}'_{k+1}$ or $\dfrac{d}{dt}\bar{Q}_{k+1}$. Thus all the equations may be written in the typical Hamiltonian form

$$\left.\begin{aligned}\frac{dH}{d\bar{Q}_{k+1}}&=-\frac{d}{dt}q^{(k)}\\ \frac{dH}{dq^{(k)}}&=\frac{d}{dt}\bar{Q}_{k+1}\end{aligned}\right\},$$

which are true for all values of k from $k=0$ to $k=n-1$. Thus there are $2n$ equations corresponding to each q.

We may show in the same way as in Art. 382, that the total differential coefficient of H with regard to t is equal to its partial differential coefficient. So that when L, and therefore H, are not explicit functions of t, we have as one integral $H=h$, where h is a constant. Writing this at length it becomes

$$L=\Sigma\,(\bar{Q}_1q'+\bar{Q}_2q''+\ldots)+h,$$

which is the integral continually used in the Calculus of Variations. We see that this integral corresponds to the equation of Vis Viva in Dynamics.

$$\int_{t_0}^{t_1} \Sigma \left(\frac{dL}{dq} - \frac{d}{dt} \frac{dL}{dq'} \right) \delta q dt = 0$$

for all variations. The δq's being all arbitrary and independent, each coefficient under the integral sign must vanish separately, and this leads to the typical Lagrange's equation.

Ex. 1. There is another method of deducing Lagrange's equations from the principle of Least Action which is worthy of notice. We are to make $\int_{t_0}^{t_1} T dt$ a minimum, subject to the condition $T - U = h$. By Lagrange's rule in the Calculus of Variations we are to make

$$\delta \int \{T + \lambda (T - U - h)\} dt = 0,$$

without regard to the given condition, and afterwards make λ such a function of t that the given condition is satisfied. This will be found an excellent exercise in the Calculus of Variations.

The solution may be indicated as follows. Putting $W = T + \lambda (T - U)$ we have with the same notation as before

$$\delta \int W dt = [W \delta t] + \Sigma \int \left(\frac{dW}{dq} - \frac{d}{dt} \frac{dW}{dq'} \right) (\delta q - q' \delta t) dt + \left[\Sigma \frac{dW}{dq'} (\delta q - q' \delta t) \right],$$

and this must be equal to $h \delta \int \lambda dt$. The integrals are to be taken between the limits, which are omitted for the sake of brevity.

First, let us consider the part outside the integral sign. The initial and final positions being given $\delta q = 0$, and we have

$$W \delta t - \Sigma \frac{dW}{dq'} q' \delta t = h \delta \int \lambda dt = h \lambda \delta t.$$

This equation is satisfied by $\delta t = 0$, but since the time of transit is not to be the same in the actual and varied motions, this factor must be rejected. Also T is a homogeneous function of the q's, hence $\Sigma \frac{dT}{dq'} q' = 2T$. Substituting for W its value and using this equation we find $(1 + \lambda) T + \lambda U + h\lambda = 0$. But λ is such that $T - U = h$, hence $(1 + 2\lambda) T = 0$ and $\therefore \lambda = -\frac{1}{2}$.

Next, let us consider the part under the integral sign. By the rule in the Calculus of Variations this gives at once the typical equation

$$\frac{dW}{dq} - \frac{d}{dt} \frac{dW}{dq'} = 0.$$

Substituting for W we have the typical Lagrange's equation.

Ex. 2. If we add to the conditions given in the principle of Least Action, the condition that the time of transit is to be always the same, show that the minimum does not in general lead to Lagrange's equations. Following the notation of the last Article, show that the minimum for a given time is determined by $\lambda = -\frac{1}{2} + \frac{A}{T}$,

where A is an arbitrary constant to be chosen so that the constant of vis viva has its given value, while the absolute minimum is determined by $\lambda = -\frac{1}{2}$.

397. When the geometrical equations do not contain the time explicitly the symbol H or h may be used to express the energy of the system. If we represent the energy by E, Sir W. R. Hamilton's fundamental equation may be written

$$2\delta\int_0^t Tdt = t\delta E + \left[\Sigma \frac{dT}{dq'}\delta q\right]_0^t.$$

This equation has been applied to the motion of a system of bodies oscillating in such a manner that the motion repeats itself in all respects at some constant interval. Let this interval be i. Suppose that some disturbance is given to the system by the addition of a quantity of energy δE. Let the system be such that the motion still recurs after a constant interval, and let this interval be now $i+\delta i$. The symbols of variation in Hamilton's equation may be used to imply a change from one kind of motion to the other. If the time t be taken equal to the period i of complete recurrence, the initial and terminal states of motion are the same and therefore the last term vanishes when taken between the limits. The equation reduces to $2\delta\int_0^i Tdt = i\delta E$. Let T_m be the mean vis viva of the system during a period of complete recurrence of the motion, then $\int_0^i Tdt = iT_m$. We therefore have $\frac{\delta E}{T_m} = 2\frac{\delta(iT_m)}{iT_m}$.

This equation may be put into another form. Let P_m be the mean potential energy of the system during a period of complete recurrence; then we have

$$\delta P_m + \delta T_m = \delta E,$$
$$\delta P_m - \delta T_m = 2T_m\frac{\delta i}{i},$$

which serve to determine the change in the mean potential and kinetic energies when any additional energy δE is added to the system.

These or equivalent equations have been applied by Bolzman, Clausius and Szily to the Dynamical Theory of Heat. The papers of the two latter are in various numbers of the *Philosophical Magazine* extending from 1870 to the present time. The second of the equations above written may be called Clausius' equation.

398. Ex. 1. If the period of complete recurrence of a dynamical system be not altered by the addition of energy, prove that this additional energy is equally distributed into potential and kinetic energy.

Ex. 2. A quantity of energy dE is communicated to a system whose mean semi-vis-viva during a period of complete recurrence is T_m. This is repeated continually, so that at last the mean vis viva and the period of complete recurrence are the same as at first. Prove that $\int\frac{dE}{T_m} = 0$.

This example is due to M. Szily, and is important in the Dynamical Theory of Heat.

On the Solution of the General Equations of Motion.

399. Sir W. R. Hamilton has applied his fundamental theorem expressing the variation of the Principal and Characteristic functions to obtain a new method of solving dynamical problems.

Let $(a_1, a_1', a_2, a_2', \&c.)$ be the values of $(q_1, q_1', q_2, q_2', \&c.)$ when $t = t_0$, and let T_0 be the same function of $(a_1, a_1', \&c.)$ that T is of $(q_1, q_1', \&c.)$. We have then when t is written for the upper limit

$$\delta S = \Sigma \frac{dT}{dq'} \delta q - \Sigma \frac{dT_0}{da'} \delta a - H\delta t + H_0 \delta t_0,$$

$$\delta V = \Sigma \frac{dT}{dq'} \delta q - \Sigma \frac{dT_0}{da'} \delta a + t\delta H - t_0 \delta H_0.$$

It is clear that both S and V may be regarded as functions of the time and the initial conditions of the system of bodies, i.e. we may regard either of these quantities as a function of t, a_1, a_2, &c., a_1', a_2', &c. Also the co-ordinates q_1, q_2, &c. are functions of t and the same initial conditions. Though these functions are in general unknown, yet we can conceive the initial velocities a_1', a_2', &c. eliminated, so that S and V are now functions of t, and a_1, a_2, &c., q_1, q_2, &c. the co-ordinates of the system at the times t_0 and t.

Let S be thus expressed, then, by the equation for δS, we have the typical equations

$$\frac{dS}{dq} = \frac{dT}{dq'}, \quad \frac{dS}{da} = -\frac{dT_0}{da'}.$$

Since T is not a function of q'', the first of these equations contains no differential coefficient of a co-ordinate higher than the first. This equation, therefore, represents typically all the first integrals of the equations of motion.

Since T_0 contains only the initial co-ordinates and the initial velocities, the second equation has no differential coefficient of any co-ordinate in it. This equation, therefore, represents typically all the second integrals of the motion.

Besides these we have the two equations

$$\frac{dS}{dt} = -H, \quad \frac{dS}{dt_0} = H_0,$$

where, if the geometrical equations do not contain the time explicitly, we may put h for H, h being a constant. In this case

the integrals may be used to connect the constant of vis viva with the constants $(a, a', \&c.)$.

Comparing Art. 394 with these results we see that S is such a function, that all the equations of motion and their integrals are included in the statement that δS is a known function of the variation of the limits. If we keep the limits fixed, we get Lagrange's equations; if we vary the limits we get the integrals.

400. In just the same way, if we regard q_1', q_2', &c. as functions of t, the initial co-ordinates and their initial velocities, we may eliminate t also by means of the equation

$$H = -U - T + \Sigma \frac{dT}{dq'} q',$$

which reduces to $H = T - U$ when the geometrical equations do not contain the time explicitly.

Let us suppose V to be expressed in this manner as a function of the initial co-ordinates, the co-ordinates at the time t, and of H. Then, by the equation for δV,

$$\frac{dV}{dq} = \frac{dT}{dq'}, \quad \frac{dV}{da} = -\frac{dT_0}{da'}, \quad \frac{dV}{dH} = t.$$

Supposing V to be known, the first of these equations gives in a typical form all the first integrals of the equations of motion. The second supplies as many equations as there are co-ordinates $(q_1, q_2, \&c.)$. When the geometrical equations do not contain the time explicitly these do not contain t, but they all contain h. One of them, therefore, reduces to the relation between this constant and the constants $(a, a', \&c.)$. The equation $\dfrac{dV}{dh} = t$ will give another second integral of the equations of motion containing the time.

401. Ex. If $Q = \int_{t_0}^{t} (\Sigma q p' + H)\, dt$, where $p = \dfrac{dT}{dq'}$, prove that $\delta Q = [H\delta t + \Sigma q \delta p]_{t_0}^{t}$. Thence show that if Q be expressed as a function of the initial and terminal components of momentum, viz. $(b_1, b_2, \&c.)$ and $(p_1, p_2, \&c.)$ and of the time, then $\dfrac{dQ}{dp} = q$, $\dfrac{dQ}{db} = -a$, $\dfrac{dQ}{dt} = H$. This result is due to Sir W. R. Hamilton.

402. Ex. 1. A homogeneous sphere of unit mass rolls down a perfectly rough fixed inclined plane. If the position of the sphere is defined by the distance q of the point of contact from a fixed point on the inclined plane, show that

$$S = \frac{7}{10}\frac{(q-a)^2}{t} + \frac{1}{2}(q+a)gt - \frac{5}{168}g^2t^3,$$

where g is the resolved part of gravity down the plane and $t_0 = 0$.

Thence obtain by substitution the Hamiltonian first and second integrals of the equation of motion.

We easily find, as in Art. 133, that $q=a+a't+\frac{5}{14}gt^2$. Also $T=\frac{7}{10}q'^2$, $U=gq$. To find S, we substitute in $S=\int_0^t(T+U)\,dt$. After integration we must eliminate a' by means of the equation for q.

Ex. 2. Taking the same circumstances of motion as in the last example, show that $V=\frac{2}{3g}\sqrt{\frac{14}{5}}\{(gq+h)^{\frac{3}{2}}-(ga+h)^{\frac{3}{2}}\}$. Thence also deduce the Hamiltonian first and second integrals.

Ex. 3. Show how to deduce the equation of vis viva, from the Hamiltonian integrals.

We have V a function of q_1, q_2, &c. and H. Hence $\frac{dV}{dt}=\Sigma\frac{dV}{dq}q'+\frac{dV}{dH}\frac{dH}{dt}$, which becomes by Hamilton's integrals $2T=\Sigma\frac{dT}{dq}q'+t\frac{dH}{dt}$. When T is a homogeneous quadratic function of (q_1', q_2', &c.) this gives $\frac{dH}{dt}=0$, or $H=$constant. The equation of vis viva may also be deduced from Hamilton's principal function.

Ex. 4. When the geometrical equations do not contain the time explicitly, show that no two of the Hamiltonian integrals can be the same and no one can be deduced from two others.

If it were possible that two should be the same, the ratio of $\frac{dT}{dq_1'}$ to $\frac{dT}{dq_2'}$, must be some constant m. Integrating this partial differential equation we find T to be a homogeneous quadratic function of $q_1'-mq_2'$, q_3', &c. It would, therefore, be possible to set the system in motion, with values of q_1' and q_2' which are not zero, and yet so that the system is without vis viva.

403. By the preceding reasoning all the integrals of a dynamical system of equations can be expressed in terms of the differential coefficients of a single function. But the method supplies no means of discovering this function *à priori*. We shall now show that this function must always satisfy a certain differential equation, so that the solution of all dynamical problems may be reduced to the integration of this one equation.

Let us, for the sake of brevity, suppose that the geometrical equations do not contain the time explicitly. We have then $H=T-U$. If we follow the process indicated in Art. 378, we put $\frac{dT}{dq_1'}=p_1$, $\frac{dT}{dq_2'}=p_2$, &c. and eliminate q_1', q_2', &c. Let the reciprocal function of H thus found be

$$H=F(q_1,\,p_1,\,q_2,\,p_2,\,\&c.).$$

But $p_1 = \frac{dS}{dq_1}$, $p_2 = \frac{dS}{dq_2}$, &c. and $H = -\frac{dS}{dt}$. Hence S must satisfy the equation

$$\frac{dS}{dt} + F\left(q_1, \frac{dS}{dq_1}, q_2, \frac{dS}{dq_2}, \&c.\right) = 0.$$

In just the same way, $p_1 = \frac{dV}{dq_1}$, $p_2 = \frac{dV}{dq_2}$, &c. and the equation of vis viva gives $H = h$. Hence V must satisfy the equation

$$F\left(q_1, \frac{dV}{dq_1}, q_2, \frac{dV}{dq_2}, \&c.\right) = h.$$

If we consider the initial value of T, we shall have another equation of a similar form with a_1, a_2, &c. written for q_1, q_2, &c., and t_0 for t. It is necessary that the functions should satisfy both these equations.

Ex. Taking the same circumstance of motion as in Ex. 1 of Art. 402, show that the differential equation to find V is $\frac{5}{14}\left(\frac{dV}{dq}\right)^2 - gq = h$. Integrate this equation and thence find the motion.

404. When there are several independent variables, the equation to find V is of the form

$$\frac{1}{2} B_{11} \left(\frac{dV}{dq_1}\right)^2 + B_{12} \frac{dV}{dq_1} \frac{dV}{dq_2} + \&c. = U + h \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $(B_{11}, B_{12}, \&c.)$ are functions of q_1, q_2, &c. only. The left-hand side of this equation, by Ex. 2 of Art. 384, may be written in the form of a determinant. We have only to replace u, v, &c. by their values $\frac{dV}{dq_1}$, $\frac{dV}{dq_2}$, &c.

We thus have, in general, a partial differential equation to find V, and Sir W. Hamilton gave no rule to determine which integral is to be taken. This rule has been supplied by Jacobi in the following proposition.

Suppose a solution to have been found containing $n-1$ *constants* besides* h, *and the constant which may be introduced by simple addition to the function* V. *These need not be the initial values of* $q_1, q_2 \ldots q_n$, *but may be any constants whatever. Let them be denoted by* $a_1, a_2 \ldots a_{n-1}$, *so that*

$$V = f(q_1, q_2 \ldots q_n, \quad a_1, a_2 \ldots a_{n-1}) + a_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Then the integrals of the dynamical equations will be

$$\frac{df}{da_1} = \beta_1, \;\&c., \; \frac{df}{da_{n-1}} = \beta_{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{df}{dh} = t + \epsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

* An integral of a partial differential equation has been called by Lagrange "complete," when it contains as many arbitrary constants as there are independent variables. It is implied that the constants enter in such a manner into the integral that they cannot by any algebraic process be reduced to a smaller number. For instance, if two of the constants enter in the form $a_1 + a_2$, they amount on the whole to only one.

where $\beta_1, \beta_2 \dots \beta_{n-1}$ and ϵ are n new arbitrary constants. And the first integrals of the equations may be written in the form

$$\frac{df}{dq_1}=\frac{dT}{dq_1'},\ \frac{df}{dq_2}=\frac{dT}{dq_2'},\ \&c.=\&c. \qquad (5).$$

Let the expression for the semi-vis-viva be

$$T=\frac{1}{2}A_{11}q_1'^2+A_{12}q_1'q_2'+\&c. \qquad (6),$$

where the coefficients A_{11}, A_{12}, &c. are functions of q_1, q_2, &c. only.

Let Q_1, $Q_2 \dots Q_n$ be such functions of q_1, $q_2 \dots q_n$ and the constants, that they may satisfy *identically* the n equations

$$\left.\begin{aligned}\frac{df}{dq_1}&=A_{11}Q_1+A_{12}Q_2+\dots\\ \frac{df}{dq_2}&=A_{21}Q_1+A_{22}Q_2+\dots\\ \&c.&=\&c.\end{aligned}\right\} \qquad (7).$$

Then from the mode in which the differential equation to find V has been formed, in Art. 403, we know Q_1, Q_2 will also satisfy *identically* the equation

$$U+h=\frac{1}{2}A_{11}Q_1^2+A_{12}Q_1Q_2+ \qquad (8).$$

Firstly, we shall prove that $Q_1=q_1'$, $Q_2=q_2'$, &c., it will then follow that the equations (5) are satisfied. Differentiating equations (3) and (4), we have

$$\left.\begin{aligned}\frac{d^2f}{da_1\,dq_1}\frac{dq_1}{dt}+\frac{d^2f}{da_1\,dq_2}\frac{dq_2}{dt}+\dots&=0\\ \&c.\qquad\qquad &=0\\ \frac{d^2f}{dh\,dq_1}\frac{dq_1}{dt}+\frac{d^2f}{dh\,dq_2}\frac{dq_2}{dt}+\dots&=1\end{aligned}\right\} \qquad (9).$$

These are the equations to find $\frac{dq_1}{dt}$, $\frac{dq_2}{dt}$, &c.

But differentiating (7) with regard to a_1, we have

$$\left.\begin{aligned}\frac{d^2f}{da_1\,dq_1}&=A_{11}\frac{dQ_1}{da_1}+A_{12}\frac{dQ_2}{da_1}+\dots\\ \frac{d^2f}{da_1\,dq_2}&=A_{21}\frac{dQ_1}{da_1}+A_{22}\frac{dQ_2}{da_1}+\dots\\ \&c.&=\&c.\end{aligned}\right\} \qquad (10),$$

because A_{11}, A_{12}, &c. are not functions of the constants. Multiplying these equations by Q_1, $Q_2 \dots$, and adding, we get

$$\frac{d^2f}{da_1\,dq_1}Q_1+\frac{d^2f}{da_1\,dq_2}Q_2+\dots=\frac{d}{da_1}\left\{\frac{1}{2}A_{11}Q_1^2+A_{12}Q_1Q_2+\dots\right\}.$$

Since the equation (8) is an identical equation the quantity in brackets on the right-hand side does not contain a_1, being equal to $U+h$. Hence the expression on the left-hand side vanishes. Thus we have an equation connecting Q_1, $Q_2 \dots$ exactly similar to the first of equations (9). Similarly by differentiating equations (7) with respect to $a_2 \dots h$ successively, we shall have equations similar to the second, &c. and last of equations (9). We have therefore exactly the same equations to find Q_1, $Q_2 \dots$ and q_1', $q_2' \dots$. Hence $Q_1=q_1'$, $Q_2=q_2'$, &c.

Secondly, we shall prove that (3) and (4) satisfy the equations of motion. Let us consider the equation of Lagrange*,

$$\frac{d}{dt}\frac{dT}{dq_1'}-\frac{dT}{dq_1}=\frac{dU}{dq_1}.$$

When $q_1', q_2' \ldots q_n'$ have been expressed in terms of $q_1, q_2 \ldots q_n$ and the constant by means of equations (5), we have *identically*

$$U+h=\frac{1}{2}A_{11}q_1'^2+A_{12}q_1'q_2'+\ldots$$

Therefore, differentiating partially,

$$\frac{dU}{dq_1}=\frac{1}{2}\frac{dA_{11}}{dq_1}q_1'^2+\frac{dA_{12}}{dq_1}q_1'q_2'+\ldots+\left\{A_{11}\frac{dq_1'}{dq_1}+A_{12}\frac{dq_2'}{dq_1}+\ldots\right\}q_1'$$
$$+\left\{A_{12}\frac{dq_1'}{dq_1}+A_{22}\frac{dq_2'}{dq_1}+\ldots\right\}q_2'+\&c.$$

But differentiating (5) written at length, with regard to q_1, we have

$$A_{11}\frac{dq_1'}{dq_1}+A_{12}\frac{dq_2'}{dq_1}+\ldots=\frac{d^2f}{dq_1^2}-q_1'\frac{dA_{11}}{dq_1}-q_2'\frac{dA_{12}}{dq_1}-\ldots$$

$$A_{21}\frac{dq_1'}{dq_1}+A_{22}\frac{dq_2'}{dq_1}+\ldots=\frac{d^2f}{dq_1\,dq_2}-q_1'\frac{dA_{21}}{dq_1}-q_2'\frac{dA_{22}}{dq_1}-\ldots$$

$$\&c.=\&c.$$

Hence, substituting,

$$\frac{dU}{dq_1}=\frac{d^2f}{dq_1^2}q_1'+\frac{d^2f}{dq_1\,dq_2}q_2'+\ldots-\frac{1}{2}\frac{dA_{11}}{dq_1}q_1'^2-\frac{dA_{12}}{dq_1}q_1'q_2'-\ldots$$

* We may also show that the Jacobian integrals satisfy the Hamiltonian form of the equations of motion. The peculiar relation of the differential equation to the Hamiltonian function H adapts it to this process. If we substitute the value of V given by (2) in the differential equation (1), the result is an identical equation. Differentiating this identity with regard to each of the n constants and replacing $\frac{dV}{dq}$ by p, we get n equations of the form $\frac{dH}{dp_1}\frac{d^2f}{dq_1\,da}+\frac{dH}{dp_2}\frac{d^2f}{dq_2\,da}+\ldots=0$ to find $\frac{dH}{dp_1}$, $\frac{dH}{dp_2}$, &c. These are the same as the equations (9) in the text, hence $\frac{dH}{dp}=q'$. Again, differentiating H partially with regard to q_1, we have

$$\frac{dH}{dq_1}+\frac{dH}{dp_1}\frac{d^2f}{dq_1^2}+\frac{dH}{dp_2}\frac{d^2f}{dq_1\,dq_2}+\ldots=0.$$

But all the terms of this equation except the first are together equal to the total differential coefficient $\frac{dp_1}{dt}$. Hence $\frac{dH}{dq_1}=-\frac{dp_1}{dt}$. The investigations of Hamilton and Jacobi apply to a system of free particles mutually attracting each other referred to Cartesian co-ordinates. In the text the reasoning has been applied to a system of bodies referred to any co-ordinates.

Next let us consider the expression for T: we see that the partial differential coefficient

$$\frac{dT}{dq_1}=\frac{1}{2}\frac{dA_{11}}{dq_1}q_1'^2+\frac{dA_{12}}{dq_1}q_1'q_2'+\dots$$

is the same as the latter part of the expression for $\frac{dU}{dq_1}$.

Also $\frac{dT}{dq_1'}=\frac{df}{dq_1}$, therefore taking the total differential coefficient, we have

$$\frac{d}{dt}\frac{dT}{dq_1'}=\frac{d^2f}{dq_1^2}q_1'+\frac{d^2f}{dq_1dq_2}q_2'+\dots,$$

which is the same as the first part of the expression for $\frac{dU}{dq_1}$. Hence the differential equation of motion is satisfied.

We have also, since T is homogeneous,

$$2T=\frac{dT}{dq_1'}q_1'+\frac{dT}{dq_2'}q_2'+\dots=\frac{df}{dq_1}q_1'+\frac{df}{dq_2}q_2'+\dots=\frac{df}{dt},$$

where the differential coefficient is total. This shows that the function f represents the whole accumulated "action" in the time t. See Art. 392.

405. Jacobi has extended his theorem to the case in which the geometrical equations do contain the time explicitly. But for this we have no space. It is no part of the plan of this book to enter on Theoretical Dynamics. We cannot therefore do more than allude to Professor Donkin's theorem that a knowledge of half the integrals of the Hamiltonian system will in certain cases lead to a determination of the rest.

In Boole's *Differential Equations* it is shown that when the Hamiltonian equations are four in number, and one integral besides Vis Viva is known, both the remaining integrals can be found by integrating an exact differential equation. Miscellaneous Exercises, No. 15.

Variation of the Elements.

406. Let the integrals of a dynamical problem be

$$\left.\begin{aligned} c_1&=f_1(p_1, q_1, p_2, q_2, \dots t)\\ c_2&=f_2(p_1, q_1, p_2, q_2, \dots t)\\ \&c.&=\&c. \end{aligned}\right\}\dots\dots\dots\dots(1),$$

where $p, q, \dots$ are some variables which determine the position and motion of the system, and which are such that the equations of motion may be written in the forms

$$p'=-\frac{dH}{dq},\quad q'=\frac{dH}{dp}\dots\dots\dots\dots(2),$$

in the manner explained in Art. 381. Let the equations of motion of a second dynamical problem be

$$p'=-\frac{dH}{dq}-\frac{dK}{dq},\quad q'=\frac{dH}{dp}+\frac{dK}{dp}\dots\dots\dots\dots(3),$$

where K is some function of $p, q, \ldots t$. If we consider $c_1, c_2, \ldots$ the constants of the solution of the first problem to be functions of p, q, and t, we may suppose the solution of the second problem to be represented by integrals of the same form (1) as those of the first problem. It is therefore our object to discover what functions $c_1, c_2, \ldots$ are of p, q, and t. The function K is called "the disturbing function," and is usually small as compared with H.

Since the equations (1) are the integrals of the differential equations (2), we shall obtain identical expressions by substituting from (1) in (2). Hence differentiating (1), and substituting for p' and q' their values given by (2), we get

$$\left.\begin{aligned} 0 &= -\frac{dc_1}{dp}\frac{dH}{dq} + \frac{dc_1}{dq}\frac{dH}{dp} + \ldots + \frac{dc_1}{dt} \\ 0 &= \&c. \end{aligned}\right\} \qquad (4).$$

But when $c_1, c_2, \ldots$ are considered as variables, the equations (1) are the integrals of the differential equations (3). Hence repeating the same process, we have

$$\left.\begin{aligned} \frac{dc_1}{dt} &= -\frac{dc_1}{dp}\frac{dH}{dq} + \frac{dc_1}{dq}\frac{dH}{dp} + \ldots + \frac{dc_1}{dt} \\ &\quad -\frac{dc_1}{dp}\frac{dK}{dq} + \frac{dc_1}{dq}\frac{dK}{dq} + \ldots\ldots \\ \frac{dc_2}{dt} &= \&c. \end{aligned}\right\},$$

where the differential coefficients on the left-hand side are total, and those on the right-hand side partial.

Hence, using the identities (4), we get

$$\frac{dc_1}{dt} = -\frac{dc_1}{dp}\frac{dK}{dq} + \frac{dc_1}{dq}\frac{dK}{dp} + \ldots \qquad (5),$$

with similar expressions for $\frac{dc_2}{dt}$, &c.

If K be given as a function of p, q, &c. and t, we have $\frac{dc_1}{dt}$, &c. expressed as functions of p, q, &c. and t. Joining these equations to those marked (1) we find $c_1, c_2 \ldots$ as functions of t. If K be given as a function of $c_1, c_2, \ldots$ and t we may continue thus,

$$\frac{dK}{dp} = \frac{dK}{dc_1}\frac{dc_1}{dp} + \frac{dK}{dc_2}\frac{dc_2}{dp} + \ldots, \qquad \frac{dK}{dq} = \frac{dK}{dc_1}\frac{dc_1}{dq} + \frac{dK}{dc_2}\frac{dc_2}{dq} + \ldots$$

Substituting in the expression for $\frac{dc_1}{dt}$, we get

$$\frac{dc_1}{dt} = \Sigma\left[\frac{dc_1}{dq}\frac{dc_2}{dp} - \frac{dc_1}{dp}\frac{dc_2}{dq}\right]\frac{dK}{dc_2} + \Sigma\left[\frac{dc_1}{dq}\frac{dc_3}{dp} - \frac{dc_1}{dp}\frac{dc_3}{dq}\right]\frac{dK}{dc_3} + \ldots,$$

where the Σ means summation for all values of p, q, viz. p_1, q_1, p_2, q_2, &c.

Since by hypothesis $c_1, c_2, \ldots$ are supposed expressed as functions of p_1, q_1, &c. and t, these coefficients may be found by simple differentiation. It will, of course, be more convenient to express them in terms of c_1, c_2, &c. and t by substituting for p_1, q_1, &c. their values given by the integrals (1).

407. On effecting this substitution it will be found that t disappears from the expressions. This may be proved as follows. Let A be any coefficient, so that

$A=\Sigma\left[\frac{dc_1}{dq}\frac{dc_2}{dp}-\frac{dc_1}{dp}\frac{dc_2}{dq}\right]$, we have to prove that A being regarded as a function of p_1, q_1, &c. and t, the total differential coefficient $\frac{d.A}{dt}$ is zero. Now

$$\frac{d.A}{dt}=\frac{dA}{dt}+\frac{dA}{dp}p'+\frac{dA}{dq}q'+\ldots$$

The letters p_1, q_1, &c. enter into the expression for A only through c_1 and c_2. Let us consider only the part of $\frac{d.A}{dt}$ due to the variation of c_1, then the part due to the variation of c_2 may be found by interchanging c_1 and c_2, and changing the sign of the whole. The complete value of $\frac{d.A}{dt}$ is the sum of these two parts.

The part of $\frac{d.A}{dt}$ due to the variation of c_1 is

$$\Sigma\left[\frac{dc_2}{dp}\left\{\frac{d}{dq}\frac{dc_1}{dt}-\frac{d^2c_1}{dp\,dq}\frac{dH}{dq}+\frac{d^2c_1}{dq^2}\frac{dH}{dp}+\ldots\right\}-\frac{dc_2}{dq}\left\{\frac{d}{dp}\frac{dc_1}{dt}-\frac{d^2c_1}{dp^2}\frac{dA}{dq}+\frac{d^2c_1}{dp\,dq}\frac{dH}{dp}+\ldots\right\}\right].$$

If we substitute for $\frac{dc_1}{dt}$ its value given by the identity (4), we get

$$\Sigma\left[\frac{dc_2}{dp}\left\{\frac{dc_1}{dp}\frac{d^2H}{dq^2}-\frac{dc_1}{dq}\frac{d^2H}{dp\,dq}\right\}-\frac{dc_2}{dq}\left\{\frac{dc_1}{dp}\frac{d^2H}{dp\,dq}-\frac{dc_1}{dq}\frac{d^2H}{dp^2}\right\}\right].$$

If we now interchange c_1 and c_2 we get the same result. Hence when the two parts of $\frac{d.A}{dt}$ are added together, the signs being opposite, the sum is zero.

408. Let the expression $\Sigma\left[\frac{dc_1}{dq}\frac{dc_2}{dp}-\frac{dc_1}{dp}\frac{dc_2}{dq}\right]$, where the Σ means summation for all the values of p, q, be represented shortly by (c_1, c_2). Then in any dynamical problem if K be the disturbing function, the variations of the parameters c_1, c_2 ... are given by $\frac{dc_1}{dt}=(c_1, c_2)\frac{dK}{dc_2}+(c_1, c_3)\frac{dK}{dc_3}+\ldots$, where all the coefficients are functions of the parameters only and not of t.

This equation may be greatly simplified by a proper choice of the constants c_1, c_2,... In the *Mécanique Analytique* of Lagrange, it is shown that if the constants chosen be the *initial* values of p_1, p_2,... and q_1, q_2,..., viz. α, β, γ,... and λ, μ, ν,... respectively, then the equations become

$$\left.\begin{aligned}\frac{d\alpha}{dt}&=-\frac{dK}{d\lambda}, & \frac{d\beta}{dt}&=-\frac{dK}{d\mu}, \text{ \&c.}\\ \frac{d\lambda}{dt}&=\frac{dK}{d\alpha}, & \frac{d\mu}{dt}&=\frac{dK}{d\beta}, \text{ \&c.}\end{aligned}\right\}.$$

It is assumed in the demonstration that K is a function of q_1, q_2,... only. This simplification has been extended by Sir W. Hamilton and Jacobi to other cases, but for this we must refer the reader to books which treat on theoretical dynamics.

409. It follows from the investigation in Art. 407, that if two integrals of a dynamical problem be found, viz. $c_1=\alpha$, $c_2=\beta$, where c_1 and c_2 stand for some functions of p_1, q_1, p_2, q_2, ... and t, and α and β are constants, then (c_1, c_2) is also constant. So that $(c_1, c_2)=\gamma$, where γ is a constant, is either a third integral of the equations of motion or an identity. If it is an integral it may be either a new integral or one derivable from the two c_1 and c_2 already found.

EXAMPLES*.

1. A screw of Archimedes is capable of turning freely about its axis, which is fixed in a vertical position: a heavy particle is placed at the top of the tube and runs down through it; determine the whole angular velocity communicated to the screw.

Result. Let n be the ratio of the mass of the screw to that of the particle, α = the angle the tangent to the screw makes with the horizon, h the height descended by the particle. Then the angular velocity generated is

$$= \sqrt{\frac{2gh \cos^2 \alpha}{a^2 (n+1)(n+\sin^2 \alpha)}}.$$

2. A fine circular tube, carrying within it a heavy particle, is set revolving about a vertical diameter. Show that the difference of the squares of the absolute velocities of the particle at any two given points of the tube equidistant from the axis is the same for all initial velocities of the particle and tube.

3. A circular wire ring, carrying a small bead, lies on a smooth horizontal table; an elastic thread the natural length of which is less than the diameter of the ring, has one end attached to the bead and the other to a point in the wire; the bead is placed initially so that the thread coincides very nearly with a diameter of the ring; find the vis viva of the system when the string has contracted to its original length.

4. A straight tube of given length is capable of turning freely about one extremity in a horizontal plane, two equal particles are placed at different points within it at rest, an angular velocity is given to the system, determine the velocity of each particle on leaving the tube.

5. A smooth circular tube of mass M has placed within it two equal particles of mass m, which are connected by an elastic string whose natural length is $\frac{1}{4}$ of the circumference. The string is stretched until the particles are in contact and the tube is placed flat on a smooth horizontal table and left to itself. Show that when the string arrives at its natural length, the actual energy of the two particles is to the work done in stretching the string as

$$2(M^2+Mm+m^2) : (M+2m)(2M+m).$$

6. An endless flexible and inextensible chain in which the mass for unit length is μ through one continuous half and μ' through the other half is stretched over two equal perfectly rough uniform circular discs (radius a, mass M) which can turn freely about their centres at a distance b in the same vertical line. Prove that the time of a small oscillation of the chain under the action of gravity is

$$2\pi \sqrt{\frac{M+(\pi a+b)(\mu+\mu')}{2(\mu-\mu')g}}.$$

7. Two particles of masses m, m' are connected by an elastic string of length a. The former is placed in a smooth straight groove and the latter is projected in a

* These examples are taken from the Examination Papers which have been set in the University and in the Colleges.

direction perpendicular to the groove with a velocity V. Prove that the particle m will oscillate through a space $\dfrac{2am'}{m+m'}$, and if m be large compared with m' the time of oscillation is nearly $\dfrac{2\pi a}{V}\left(1-\dfrac{m'}{4m}\right)$.

8. A rough plane rotates with uniform angular velocity n about a horizontal axis which is parallel to it but not in it. A heavy sphere of radius a being placed on the plane when in a horizontal position, rolls down it under the action of gravity. If the centre of the sphere be originally in the plane containing the moving axis and perpendicular to the moving plane, and if x be its distance from this plane at a subsequent time t before the sphere leaves the plane, then

$$x=\frac{1}{24\sqrt{35}}\left(\frac{35g}{n^2}-84a-60c\right)\left(e^{\sqrt{\frac{5}{7}}nt}-e^{-\sqrt{\frac{5}{7}}nt}\right)-\frac{5}{12}\frac{g}{n^2}\sin nt,$$

c being the distance from the axis to the plane measured upwards.

9. The extremities of a uniform heavy beam of length $2a$ slide on a smooth wire in the form of the curve whose equation is $r=a(1-\cos\theta)$ the prime radius being vertical and the vertex of the curve downwards. Prove that if the beam be placed in a vertical position and displaced with a velocity just sufficient to bring it into a horizontal position $\tan\theta=\frac{1}{2}\left\{e^{\sqrt{\frac{3g}{2a}}t}-e^{-\sqrt{\frac{3g}{2a}}t}\right\}$, where θ is the angle through which the rod has turned after a time t.

10. A rigid body whose radius of gyration about G the centre of gravity is k, is attached to a fixed point C by a string fastened to a point A on its surface. CA $(=b)$ and AG $(=a)$ are initially in one line, and to G is given a velocity V at right angles to that line. No impressed forces are supposed to act, and the string is attached so as always to remain in one right line. If θ be the angle between AG and AC at time t, show that $\left(\dfrac{d\theta}{dt}\right)^2=\dfrac{V^2}{b^2}\,\dfrac{k^2-4ab\sin^2\dfrac{\theta}{2}}{k^2+a^2\sin^2\theta}$, and if the amplitude of θ, i.e. $2\sin^{-1}\dfrac{k}{2\sqrt{ab}}$ be very small, the period is $\dfrac{2\pi bk}{V\sqrt{a(a+b)}}$.

11. A fine weightless string having a particle at one extremity is partially coiled round a hoop which is placed on a smooth horizontal plane, and is capable of motion about a fixed vertical axis through its centre. If the hoop be initially at rest and the particle be projected in a direction perpendicular to the length of the string, and if s be the portion of the string unwound at any time t, then

$$s^2-b^2=\frac{\mu}{m+\mu}V^2t^2+2Vat,$$

where b is the initial value of s, m and μ the masses of the hoop and particle, a the radius of the hoop and V the velocity of projection.

12. A square formed of four similar uniform rods jointed freely at their extremities is laid upon a smooth horizontal table, one of its angular points being fixed: if angular velocities ω, ω' in the plane of the table be communicated to the two sides containing this angle, show that the greatest value of the angle $(2a)$ between them is given by the equation $\cos 2a=-\dfrac{5}{6}\dfrac{(\omega-\omega')^2}{\omega^2+\omega'^2}$.

13. Two particles of masses m, m' lying on a smooth horizontal table are connected by an inelastic string extended to its full length and passing through a small ring on the table. The particles are at distances a, a' from the ring and are projected with velocities v, v' at right angles to the string. Prove that if $mv^2a^2 = m'v'^2a'^2$ their second apsidal distances from the ring will be a', a respectively.

14. If a uniform thin rod PQ move in consequence of a primitive impulse between two smooth curves in the same plane, prove that the square of the angular velocity varies inversely as the difference between the sum of the squares of the normals OP, OQ to the curves at the extremities of the rods, and $\frac{5}{12}$ of the square of the whole length of the rod.

15. A small bead can slide freely along an equiangular spiral of equal mass and angle α which can turn freely about its pole as a fixed point. A centre of repulsive force F is situated in the pole and acts on the particle. If the system start from rest when the particle is at a distance a, show that the angular velocity of the spiral when the particle is at a distance k from the pole is $\sqrt{\dfrac{\int_a^k F dr}{mk^2(1+2\cot^2\alpha)}}$ where mk^2 is the moment of inertia of the spiral about its pole.

16. The extremities of a uniform beam of length $2a$, slide on two slender rods without inertia, the plane of the rods being vertical, their point of intersection fixed and the rods inclined at angles $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ to the horizon. The system is set rotating about the vertical line through the point of intersection of the rods with an angular velocity ω, prove that if θ be the inclination of the beam to the vertical at the time t and α the initial value of θ

$$4\left(\frac{d\theta}{dt}\right)^2 + \frac{(3\cos^2\alpha + \sin^2\alpha)^2}{3\cos^2\theta + \sin^2\theta}\,\omega^2 = (3\cos^2\alpha + \sin^2\alpha)\,\omega^2 + \frac{6g}{a}(\sin\alpha - \sin\theta).$$

17. A perfectly rough sphere of radius a is placed close to the intersection of the highest generating lines of two fixed equal horizontal cylinders of radius c the axes being inclined at an angle 2α to each other, and is allowed to roll down between them. Prove that the vertical velocity of its centre in any position will be $\sin\alpha\cos\phi\left\{\dfrac{10g(a+c)(1-\sin\phi)}{7-5\cos^2\phi\cos^2\alpha}\right\}^{\frac{1}{2}}$, where ϕ is the inclination to the horizon of the radius to the point of contact.

18. Let a complete integral of the equation $\dfrac{d^2x}{dt^2} = \dfrac{dT}{dx}$ in which T is a function of x be $x = X$, X being a known function of a and b two arbitrary constants and t. Then the solution of $\dfrac{d^2x}{dt^2} = \dfrac{dT}{dx} + \dfrac{dR}{dx}$, R being a function of x may also be represented by $x = X$ provided a and b are variable quantities determined by the equations $\dfrac{da}{dt} = k\dfrac{dR}{db}$, $\dfrac{db}{dt} = -k\dfrac{dR}{da}$, where k is a function of a and b which does not contain the time explicitly.

CHAPTER VIII.

ON SMALL OSCILLATIONS.

Oscillations with one degree of freedom.

410. WHEN a system of bodies admits of only one independent motion and is making small oscillations about some mean position, or some mean state of motion, it is in general our object to reduce the equation of motion to the form

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = c,$$

where x is some small quantity which determines the position of the system at the time t. This reduction is effected by neglecting the squares of the small quantity x.

411. It will generally happen that a, b, c are all constants, and in this case we can completely determine the oscillation. By putting $x = \frac{c}{b} + \xi e^{-\frac{at}{2}}$, we reduce the equation to the well known form

$$\frac{d^2\xi}{dt^2} + \left(b - \frac{a^2}{4}\right)\xi = 0.$$

When $b - \frac{a^2}{4}$ is positive, we therefore have

$$x = \frac{c}{b} + Ae^{-\frac{at}{2}} \sin\left\{\sqrt{b - \frac{a^2}{4}}\,t + B\right\},$$

where A and B are two undetermined constants which depend on the initial conditions of the motion. The physical interpretation of this equation is not difficult. It represents an oscillatory motion. If we write for t, $t + \dfrac{2\pi}{\sqrt{b - \frac{a^2}{4}}}$, we have exactly the same expression for x with A_1 written for A, where $A_1 = Ae^{-\frac{\pi a}{\sqrt{b-\frac{a^2}{4}}}}$; we

therefore infer that the time of a complete oscillation is $\dfrac{2\pi}{\sqrt{b-\dfrac{a^2}{4}}}$.

The central position about which the system oscillates is determined by $x=\dfrac{c}{b}$. To find the times at which the system comes momentarily to rest we put $\dfrac{dx}{dt}=0$. This gives

$$\tan\left(\sqrt{b-\frac{a^2}{4}}\,t+B\right)=\frac{\sqrt{4b-a^2}}{a}.$$

The extent of the oscillations on each side of the central position may be found by substituting the values of t given by this equation in the expression for $x-\dfrac{c}{b}$. Since these must occur at a constant interval equal to $\dfrac{\pi}{\sqrt{b-\dfrac{a^2}{4}}}$ we see that the extent of the oscillation continually decreases, and that the successive arcs on each side of the position of equilibrium form a geometrical progression whose common ratio is $e^{-\frac{a\pi}{\sqrt{4b-a^2}}}$.

If $b-\dfrac{a^2}{4}$ is negative, the sine must be replaced by its exponential value, and the integral becomes

$$x=\frac{c}{b}+A'e^{\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}-b}\right)t}+B'e^{\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}-b}\right)t},$$

where A' and B' are two undetermined constants. The motion is now no longer oscillatory. If a and b are both positive, or if the initial conditions are such that the coefficient of the exponential which has a positive index is zero, x will ultimately become equal to $\dfrac{c}{b}$ and the system will ultimately continually approach the position determined by this value of x.

If $b-\dfrac{a^2}{4}=0$, the integral takes a different form and we have

$$x=\frac{c}{b}+(A''t+B'')\,e^{-\frac{at}{2}},$$

where A'' and B'' are two undetermined constants. If a is positive, the system will ultimately continually approach the position determined by $x=\dfrac{c}{b}$.

When the value of x as given by these equations becomes large, the terms depending on x^2 which have been neglected in forming the equation may also become great. It is possible that these terms may alter the whole character of the motion. In such cases the equilibrium, or the undisturbed motion of the system as the case may be, is called unstable, and these equations can represent only the nature of the motion with which the system *begins* to move from its undisturbed state.

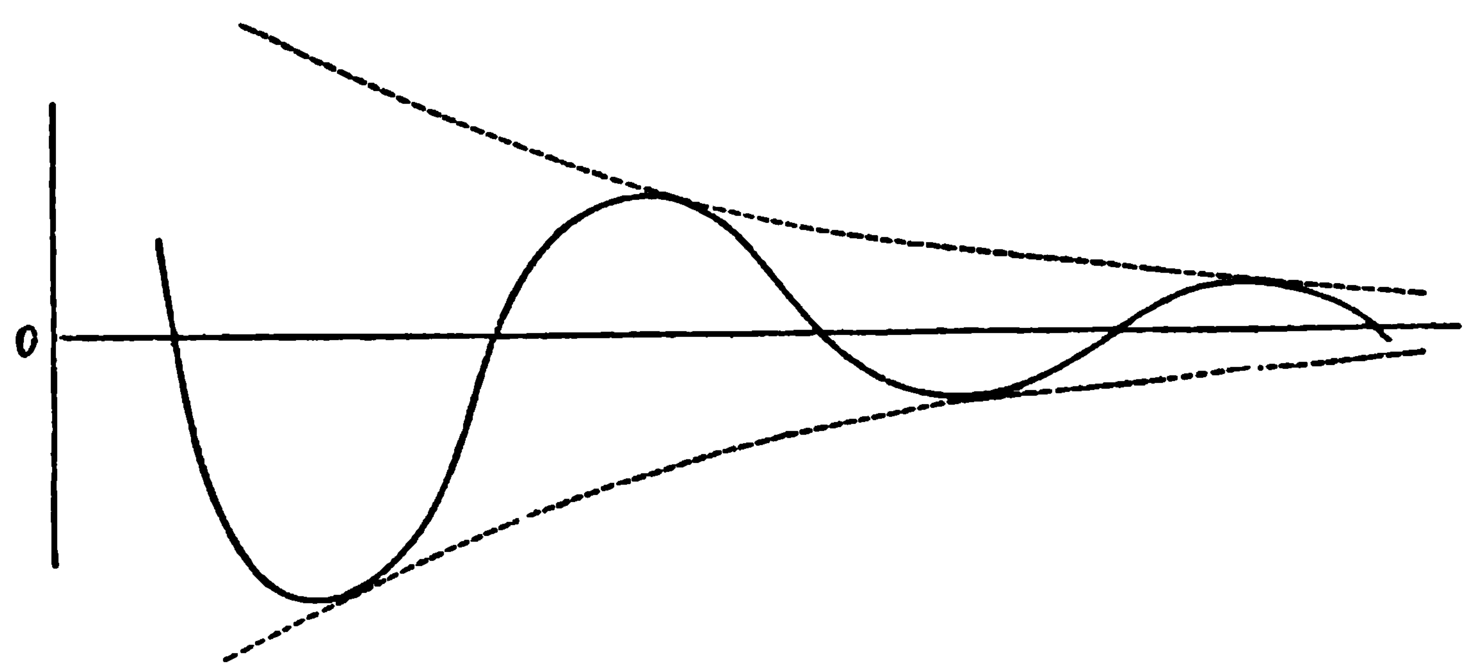
Ex. Show that the complete solution of $\frac{d^2x}{dt^2}+a\frac{dx}{dt}+bx=f(t)$ is

$$x=e^{-\frac{1}{2}at}\left\{x_0'\frac{\sin b't}{b'}+x_0\left(\cos b't+\frac{a}{2b'}\sin b't\right)\right\}+\frac{1}{b'}\int_0^t e^{-\frac{a}{2}(t-t')}\sin b'(t-t')f(t')\,dt',$$

where $b'^2=b-\frac{a^2}{4}$ and x_0, x'_0 are the values of x and $\frac{dx}{dt}$ when $t=0$.

[Math. Tripos, 1876.]

412. It will be often found advantageous to trace the motion of the system by a figure. Let equal increments of the abscissa of a point P represent on any scale equal increments of the time, and let the ordinate represent the deviation of the co-ordinate x from its mean value. Then the curve traced out by the representative point P will exhibit to the eye the whole motion of the system. In the case in which a and $b-\frac{a^2}{4}$ are both positive the curve takes the form



The dotted lines correspond to the ordinate $\pm Ae^{-\frac{at}{2}}$. The representative point P oscillates between these, and its path alternately touches each of them. In just the same way we may trace the representative curve for other values of a and b.

The most important case in dynamics is when $a=0$. The motion is then given by

$$x - \frac{c}{b} = A \sin(\sqrt{b}\,t + B).$$

The representative curve is then the curve of sines. In this case the oscillation is usually called *harmonic*.

413. Ex. 1. A system oscillates about a mean position, and its deviation is measured by x. If x_0 and x_0' be the initial values of x and $\frac{dx}{dt}$, show the system will never deviate from its mean position by so much as $\left\{\frac{x_0'^2 + ax_0x_0' + bx_0^2}{b - \frac{a^2}{4}}\right\}^{\frac{1}{2}}$ if $4b$ is greater than a^2.

Ex. 2. A system oscillates about a position of equilibrium. It is required to find by observations on its motion the numerical values of a, b, c.

Any three determinations of the co-ordinate x at three different times will generally supply sufficient equations to find a, b, c, but some measurements can be made more easily than others. For example, the values of x when the system comes momentarily to rest can be conveniently observed, because the system is then moving slowly and a measurement at a time slightly wrong will cause an error only of the second order, while the values of t at such times cannot be conveniently observed, because, owing to the slowness of the motion, it is difficult to determine the precise moment at which $\frac{dx}{dt}$ vanishes.

If three successive values of x thus found be x_1, x_2, x_3, the ratio of the two successive arcs $x_2 - x_1$ and $x_3 - x_2$ is a known function of a and b and one equation can thus be formed to find the constants. If the position of equilibrium is unknown, we may form a second equation from the fact that the three arcs $x_1 - \frac{c}{b}$, $x_2 - \frac{c}{b}$, $x_3 - \frac{c}{b}$ also form a geometrical progression. In this way we find $\frac{c}{b}$ which is the value of x corresponding to the position of equilibrium.

The position of equilibrium being known, the interval between two successive passages of the system through it is also a known function of a and b, and thus a third equation may be formed.

Ex. 3. A body performs rectilinear vibrations in a medium whose resistance is proportional to the velocity, under the action of an attractive force tending towards a fixed centre and proportional to the distance therefrom. If the observed period of vibration is T and the co-ordinates of the extremities of three consecutive semi-vibrations are p, q, r; prove that the co-ordinate of the position of equilibrium and the time of vibration if there were no resistance are respectively

$$\frac{pr - q^2}{p + r - 2q} \text{ and } T\left\{1 + \frac{1}{\pi^2}\left(\log\frac{p-q}{r-q}\right)^2\right\}^{-\frac{1}{2}}.$$

[Math. Tripos, 1870.]

414. When the coefficients are functions of the time, the equation can be integrated only by some artifice suited to the particular case under consideration. Let the equation be

$$\frac{d^2x}{dt^2}+p\frac{dx}{dt}+qx=r,$$

then a few useful methods of solution will be indicated in the following examples.

Ex. 1. If $q-\frac{p^2}{4}-\frac{1}{2}\frac{dp}{dt}$ is a positive constant, viz. n^2, prove that the successive oscillations of the system will be performed in the same time, though the extent of the oscillations may follow any law.

This may be proved by clearing the equation of the second term in the usual way, *i.e.* put $x=\xi e^{-\frac{1}{2}\int p\,dt}$.

Ex. 2. If $r=0$ and $\frac{p}{\sqrt{q}}-\frac{d}{dt}\frac{1}{\sqrt{q}}=a$, where a is a constant, prove that

$$x=e^{-\frac{a}{2}\int_0^t\sqrt{q}\,dt}A\sin\left\{\sqrt{1-\frac{a^2}{4}}\int_0^t\sqrt{q}\,dt+B\right\}.$$

Thence show that if $\int_0^t\sqrt{q}\,dt$ does not become infinite, the time of oscillation is independent of the arc of oscillation but the successive oscillations are not performed in the same time.

This may be proved by writing $t=\psi(\lambda)$, and then so choosing the form of ψ that the coefficient of x in the differential equation becomes unity or some constant.

Ex. 3. A system oscillates about a position of equilibrium and its motion is determined by the equation $\frac{d^2x}{dt^2}+qx=0$, where q is a known function of t, which during the time under consideration always lies between β^2 and β'^2, the latter being the greater. If the system be started with an initial co-ordinate x_0 and an initial velocity x_0' in a direction away from the position of equilibrium, show that the system will begin to return before x becomes so great as $\sqrt{x_0^2+\frac{x_0'^2}{\beta^2}}$. If $\pm m$, $\mp m'$ be two successive maximum values of x, prove that m' cannot be so great as $\frac{\beta'}{\beta}m$, and that the time from one maximum to the next lies between $\frac{\pi}{\beta}$ and $\frac{\pi}{\beta'}$.

415. When the arc of oscillation is not small, the equation cannot always be reduced to the linear form, and no general rule can be given for its solution. In many cases it is important to ascertain if the motion of the system is tautochronous. Various methods of determining this will be shown in the following examples.

Ex. 1. Show that if the equation of motion be

$$\frac{d^2x}{dt^2}=\left(\text{a homogeneous function of } \frac{dx}{dt} \text{ and } x \text{ of the first degree}\right),$$

then, in whatever position the system is placed at rest, the time of arriving at the position determined by $x=0$ is the same.

Let the homogeneous function be written $xf\left(\frac{1}{x}\frac{dx}{dt}\right)$. Let x and ξ be the coordinates of two systems starting from rest in two different positions, and let $x=a$, $\xi=\kappa a$ initially. It is easy to see that the differential equation of one system is changed into that of the other by writing $\xi=\kappa x$. If therefore the motion of one system is given by $x=\phi(t, A, B)$, that of the other is given by $\xi=\kappa\phi(t, A', B')$. To determine the arbitrary constants, A, B and A', B', we have exactly the same conditions, viz. when $t=0$, $\phi=a$ and $\frac{d\phi}{dt}=0$. Since only one motion can follow from the same initial conditions we have $A'=A$, and $B'=B$. Hence throughout the motion $\xi=\kappa x$ and therefore x and ξ vanish together. It follows that the motions of the two systems are perfectly similar.

This result may also be obtained by integrating the differential equation. If we put $\frac{1}{x}\frac{dx}{dt}=p$, we find $x=A\phi(t+B)$. When $t=0$, $\frac{dx}{dt}=0$, and therefore $\phi'(B)=0$. Thus B is known and x vanishes when $\phi(t+B)=0$ whatever be the value of A.

Ex. 2. If the equation of motion of the system be

$$\frac{d^2x}{dt^2}=-\left(\frac{dx}{dt}\right)^2\frac{f'(x)}{f(x)}+\left(\text{a homogeneous function of } \frac{dx}{dt} \text{ and } f(x) \text{ of the first degree}\right),$$

where $f(x)$ is any function of x, show that in whatever position the system is placed the time of arriving at the position determined by $x=0$ is the same.

This is Lagrange's general expression for a force which makes a tautochronous motion. The formula was given by him in the Berlin Memoirs for 1765 and 1770. Another very complicated demonstration was given in the same volume by D'Alembert, which required variations as well as differentiations. Lagrange seems to have believed that his expression for a tautochronous force was both necessary and sufficient. But it has been pointed out by M. Fontaine and M. Bertrand that though sufficient it is not necessary. At the same time the latter reduced the demonstration to a few simple principles. A more general expression than Lagrange's has been lately given by Brioschi.

In practice it will be more convenient to apply Bertrand's method than Lagrange's rule. Suppose the equation of motion to be $\frac{d^2x}{dt^2}=F\left(x, \frac{dx}{dt}\right)$. Put $x=\phi(y)$ and if possible so choose the form of ϕ, that $\frac{d^2y}{dt^2}$ becomes a homogeneous function of y and $\frac{dy}{dt}$ of the first degree. If this can be done, the motion is, by Ex. 1, tautochronous.

Ex. 3. If the motion of any system is tautochronous according to Lagrange's formula in vacuo, it will also be tautochronous in a resisting medium, if the effect of the resistance is to add on to the differential equation of motion a term proportional to the velocity. This theorem is due to Lagrange.

Ex. 4. A particle, acted on by a repulsive force varying as the distance and tending from a fixed point, is constrained to move along a rough curve in a medium resisting as the velocity, find the curve that the motion may be tautochronous by Lagrange's rule.

Let v be the velocity, s the arc to be described, r the radius vector of the particle, p the perpendicular on the tangent, ρ the radius of curvature. Let ar be

the repulsive force, b the coefficient of friction. Then omitting the resistance by Ex. 3, the equations of motion are

$$\left.\begin{aligned}\frac{d^2s}{dt^2}&=-a\sqrt{r^2-p^2}+bR\\ \frac{v^2}{\rho}&=-ap+R\end{aligned}\right\}.$$

Eliminating the pressure R, we have

$$\frac{d^2s}{dt^2}=b\frac{v^2}{\rho}+abp-a\sqrt{r^2-p^2}.$$

By Lagrange's rule, the motion is tautochronous if, when $f(s)=abp-a\sqrt{r^2-p^2}$, we find $\frac{b}{\rho}=-\frac{f'(s)}{f(s)}$. This will be found to give $\rho=(1+b^2)p$, which is an epicycloid.

First Method of forming the Equations of Motion.

416. When the system under consideration is a single body, there is a simple method of forming the equation of motion which is sometimes of great use.

First, let the motion be in two dimensions.

It has been shown in Art. 175, that if we neglect the squares of small quantities we may take moments about the instantaneous centre as a fixed centre. Usually the unknown reactions will be such that their lines of action will pass through this point, their moments will then be zero, and thus we shall have an equation containing only known quantities.

Since the body is supposed to be turning about the instantaneous centre as a point fixed for the moment, the direction of motion of any point of the body is perpendicular to the straight line joining it to the centre. Conversely when the directions of motion of two points of the body are known, the position of the instantaneous centre can be found. For if we draw perpendiculars at these points to their directions of motion, these perpendiculars must meet in the instantaneous centre of rotation.

The equation will, in general, reduce to the form

$$Mk^2\frac{d^2\theta}{dt^2}=\left(\begin{matrix}\text{moment of impressed forces about}\\ \text{the instantaneous centre}\end{matrix}\right),$$

where θ is the angle some straight line fixed in the body makes with a fixed line in space. In this formula Mk^2 is the moment of inertia of the body about the instantaneous centre, and since the left-hand side of the equation contains the small factor $\frac{d^2\theta}{dt^2}$ we may here suppose the instantaneous centre to have

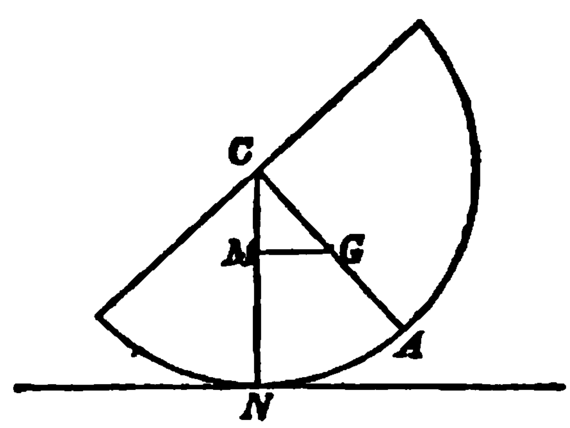
its mean or undisturbed position. On the right-hand side there is no small factor, and we must therefore be careful either to take the moment of the forces about the instantaneous centre in its disturbed position, or to include the moment of any unknown reaction which passes through the instantaneous centre.

Ex. If a body with only one independent motion can be in equilibrium in the same position under two different systems of forces, and if L_1, L_2 are the lengths of the simple equivalent pendulums for these systems acting separately, then the length L of the equivalent pendulum when they act together is given by

$$\frac{1}{L}=\frac{1}{L_1}+\frac{1}{L_2}.$$

417. Ex. *A homogeneous hemisphere performs small oscillations on a perfectly rough horizontal plane: find the motion.*

Let C be the centre, G the centre of gravity of the hemisphere, N the point of contact with the rough plane. Let the radius$=a$, $CG=c$, $\theta=\angle NCG$.



Here the point N is the centre of instantaneous rotation, because the plane being perfectly rough, sufficient friction will be called into play to keep N at rest. Hence taking moments about N

$$(k^2+GN^2)\frac{d^2\theta}{dt^2}=-gc.\sin\theta.$$

Since we can put $GN=a-c$ in the small terms, this reduces to

$$\{k^2+(a-c)^2\}\frac{d^2\theta}{dt^2}+gc.\theta=0.$$

Therefore the time of a small oscillation is $=2\pi\sqrt{\dfrac{k^2+(a-c)^2}{cg}}$.

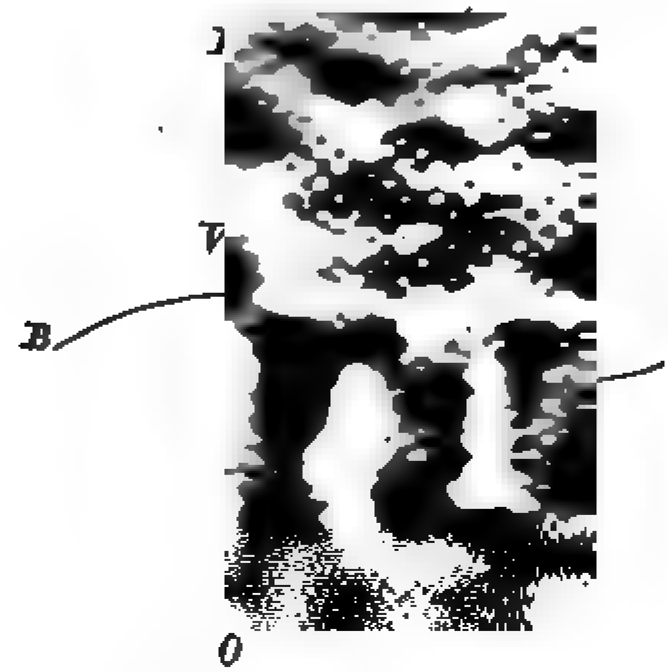
It is clear that $k^2+c^2=(\text{rad.})^2$ of gyration about $C=\frac{2}{5}a^2$ and $c=\frac{3}{8}a$.

If the plane had been smooth, M would have been the instantaneous axis, GM being the perpendicular on CN. For the motion of N is in a horizontal direction, because the sphere remains in contact with the plane, and the motion of G is vertical by Art. 79. Hence the two perpendiculars GM, NM meet in the instantaneous axis. By reasoning similar to the above the time will be found to be $2\pi\sqrt{\dfrac{k^2}{cg}}$.

418. *A cylindrical surface of any form rests in stable equilibrium on another perfectly rough cylindrical surface, the axes*

of the cylinders being parallel. A small disturbance being given to the upper surface, find the time of a small oscillation.

Let BAP, $B'A'P$ be the sections of the cylinders perpendicular to their axes. Let OA, CA' be normals at those points A, A'



which before disturbance were in contact, and let α be the angle AO makes with the vertical. Let OPC be the common normal at the time t. Let G be the centre of gravity of the moving body, then before disturbance $A'G$ was vertical. Let $A'G = r$.

Now we have only to determine the time of oscillation when the motion decreases without limit. Hence the arcs AP, $A'P$ will be ultimately zero, and therefore C and O may be taken as the centres of curvature of AP, $A'P$. Let $\rho = OA$, $\rho' = CA'$, and let the angles AOP, $A'CP$ be denoted by ϕ, ϕ' respectively.

Let θ be the angle turned round by the body in moving from the position of equilibrium into the position $B'A'P$. Then since before disturbance, $A'C$ and AO were in the same straight line, we have $\theta = \angle CDE = \phi + \phi'$, where CA' meets OAE in D. Also since one body rolls on the other, the arc $AP = \text{arc} A'P$, $\therefore \rho\phi = \rho'\phi'$, $\therefore \phi = \dfrac{\rho'}{\rho + \rho'}\theta$.

Again, in order to take moments about P, we require the horizontal distance of G from P; this may be found by projecting the broken line $PA' + A'G$ on the horizontal. The projection of $PA' = PA' \cos(\alpha + \theta) = \rho\phi\cos\alpha$ when we neglect the squares of small quantities. The projection of $A'G$ is $r\theta$. Thus the horizontal distance required is $\left(\dfrac{\rho\rho'}{\rho + \rho'}\cos\alpha - r\right)\theta$.

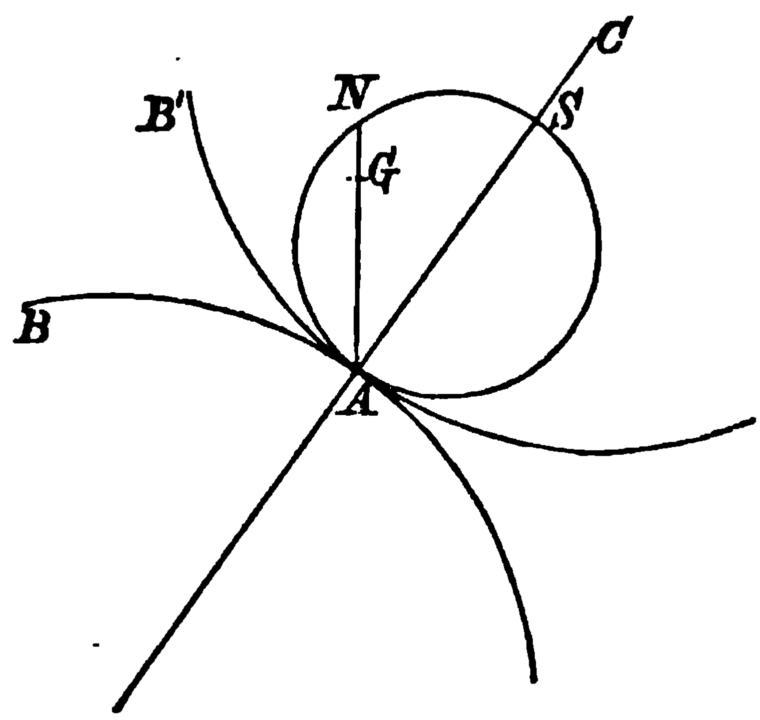
If k be the radius of gyration about the centre of gravity, the equation of motion is

$$(k^2+GA^2)\frac{d^2\theta}{dt^2}=-g\theta\left(\frac{\rho\rho'}{\rho+\rho'}\cos\alpha-r\right).$$

If L be the length of the simple equivalent pendulum, we have

$$\frac{k^2+r^2}{L}=\frac{\rho\rho'}{\rho+\rho'}\cos\alpha-r.$$

Along the common normal at the point of contact A of the two cylindrical surfaces measure a length $AS=s$ where $\frac{1}{s}=\frac{1}{\rho}+\frac{1}{\rho'}$, and describe a circle on AS as diameter. Let AG, produced if necessary, cut this circle in N. Then $GN=s\cos\alpha-r$,



the positive direction being from N towards A. The length L of the simple equivalent pendulum is given by the formula

$$\frac{k^2+GA^2}{L}=GN.$$

It is clear from this formula, if G* lie without the circle and above the tangent at A, L is negative and the equilibrium is

* Let R be the radius of curvature of the path traced out by G as the one cylinder rolls on the other, then we know that $R=-\frac{AG^2}{NG}$, so that all points without the circle described on AS as diameter are describing curves whose concavity is turned towards A, while those within the circle are describing curves whose convexity is turned towards A. It is then clear that the equilibrium is stable, unstable, or neutral, according as the centre of gravity lies within, without, or on the circumference of the circle.

unstable, if within L is positive and equilibrium is stable. This circle is called the *circle of stability*.

419. It may be noticed that the preceding problem is perfectly general and may be used in all cases in which the locus of the instantaneous axis is known. Thus ρ' is the radius of curvature of the locus in the body, ρ that of the locus in space, and α the inclination of its path to the horizon.

If dx be the horizontal displacement of the instantaneous centre produced by a rotation $d\theta$ of the body, then the equation to find the length of the simple equivalent pendulum of a body oscillating under gravity may be written

$$\frac{k^2+r^2}{L}=\frac{dx}{d\theta}-r.$$

This follows at once from the reasoning in Art. 418. It may also be easily seen that the diameter of the circle of stability is equal to the ratio of the velocity in space of the instantaneous axis to the angular velocity of the body.

Ex. 1. A homogeneous sphere makes small oscillations inside a fixed sphere so that its centre moves in a vertical plane. If the roughness be sufficient to prevent all sliding, prove that the length of the equivalent pendulum is seven-fifths of the difference of the radii. If the spheres were smooth the length of the equivalent pendulum would be equal to the difference of the radii.

Ex. 2. A homogeneous hemisphere being placed on a rough fixed plane, which is inclined to the horizon at an angle $\sin^{-1}\dfrac{1}{2\sqrt{2}}$, makes small oscillations in a vertical plane. Shew that, if a is the radius of the hemisphere, the length of the equivalent pendulum is $\left(\dfrac{46}{5}-\dfrac{\sqrt{56}}{4}\right)a$.

420. If the body be acted on by any force which passes through the centre of gravity, the results must be slightly modified. Just as before the force in equilibrium must act along the straight line joining the centre of gravity G to the instantaneous centre A. When the body is displaced the force will cut its former line of action in some point F, which we shall assume to be known. Let $AF=f$, taking f positive when G and F are on opposite sides of the locus of the instantaneous centre. Then it may be shown by similar reasoning, that the length L of the simple equivalent pendulum under this force, supposed constant and equal to gravity, is given by

$$\frac{k^2+r^2}{L}=\frac{\rho\rho'}{\rho+\rho'}\cos\alpha-\frac{fr}{f+r},$$

where α is the angle the direction of the force makes with the normal to the path of the instantaneous centre.

If we measure along the line AG a length AG' so that $\dfrac{1}{AG'}=\dfrac{1}{AG}+\dfrac{1}{AF}$, then the expression for L takes the form

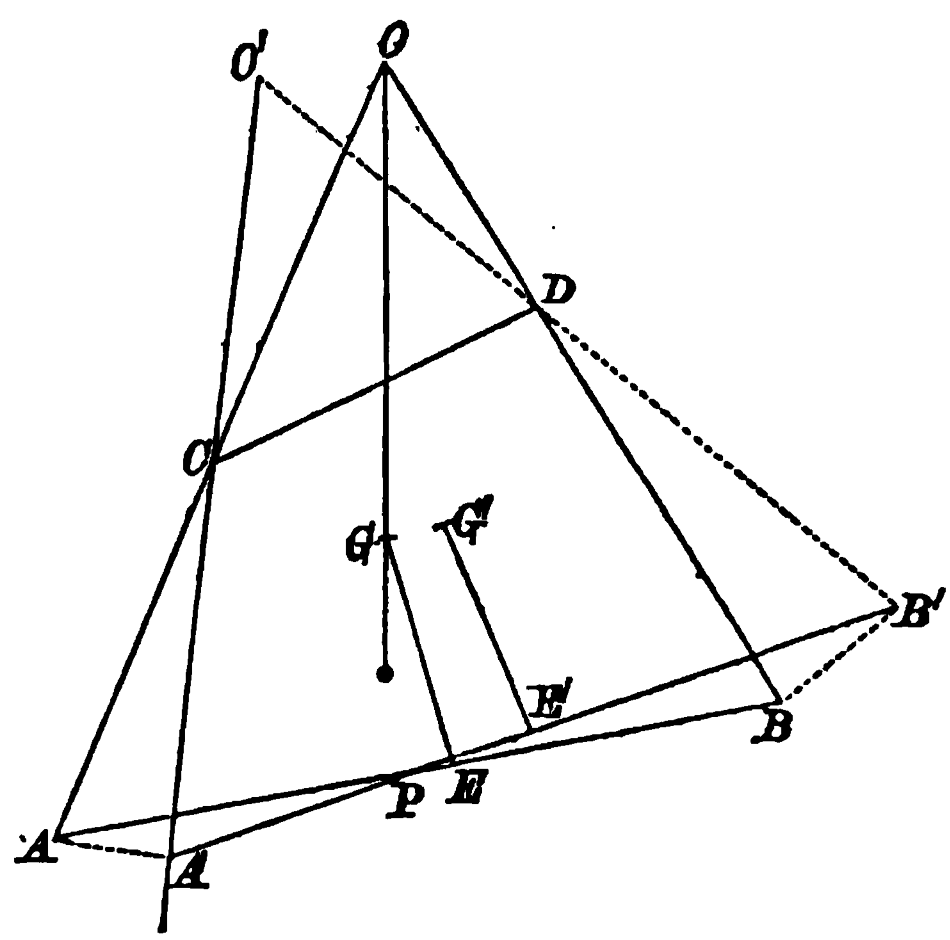
$$\frac{k^2+r^2}{L}=G'N.$$

The equilibrium is therefore stable or unstable according as G' lies within or without the circle of stability.

421. ***Two points A, B of a body are constrained to describe given curves, and the body is in equilibrium under the action of gravity. A small disturbance being given, find the time of an oscillation.***

Let C, D be the centres of curvature of the given curves at the two points A, B. Let AC, BD meet in O. Let G be the centre of gravity of the body, GE a perpendicular on AB. Then in the position of equilibrium OG is vertical. Let i, j be the angles CA, BD make with the vertical, and let α be the angle AOB. Let A', B'...denote the positions into which A, B...have been moved when the body has been turned through an angle θ. Let $ACA'=\phi$, $BDB'=\phi'$. Since the body may be brought from the position AB into the position $A'B'$ by turning it about O through an angle θ, we have $\dfrac{CA\,.\,\phi}{OA}=\dfrac{BD\,.\,\phi'}{OB}=\theta$. Also GG' is ultimately perpendicular to OG, and we have $GG'=OG\,.\,\theta$. Also let x, y be the projections of OO' on the horizontal and vertical through O. Then by projections

$$x\cos j+y\sin j=\text{distance of } O' \text{ from } OD=OD\,.\,\phi',$$



$$x\cos i-y\sin i=\text{distance of } O' \text{ from } OC=OC\,.\,\phi;$$

$$\therefore\ x=\frac{OD\,.\sin i\,.\,\phi'+OC\,.\sin j\,.\,\phi}{\sin\alpha}.$$

Now taking moments about O' as the centre of instantaneous rotation, we have

$$(k^2 + OG^2)\frac{d^2\theta}{dt^2} = -g\,.\,(GG' + x)$$

$$= -g\theta\left(OG + \frac{OD\,.\,OB}{BD}\,.\,\frac{\sin i}{\sin a} + \frac{OC\,.\,OA}{CA}\,\frac{\sin j}{\sin a}\right),$$

where k is the radius of gyration about the centre of gravity.

Hence if L be the length of the simple equivalent pendulum, we have

$$\frac{k^2 + OG^2}{L} = OG + \frac{OD\,.\,OB}{BD}\,.\,\frac{\sin i}{\sin a} + \frac{OC\,.\,OA}{AC}\,.\,\frac{\sin j}{\sin a}.$$

COR. If the given curves, on which the points A, B are constrained to move, be straight lines, the centres of curvature C and D are at infinity. In this case, we may put $\frac{OD}{BD} = -1$, $\frac{OC}{AC} = -1$, and the expression becomes

$$\frac{k^2 + OG^2}{L} = OG - OB\,.\,\frac{\sin i}{\sin a} - OA\,.\,\frac{\sin j}{\sin a}.$$

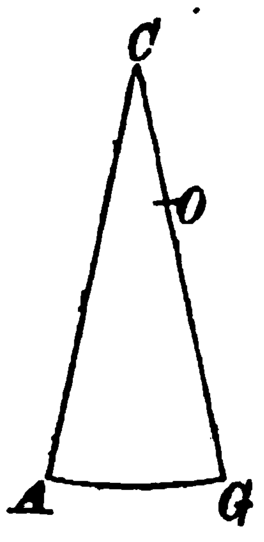
If OA and OB be at right angles, this takes the simple form

$$\frac{k^2 + OG^2}{L} = OG - 2OF,$$

where F is the projection on OG of the middle point of AB.

422. *A body oscillates about a position of equilibrium under the action of gravity, the radius of curvature of the path of the centre of gravity being known, find the time of oscillation.*

Let A be the position of the centre of gravity of the body when it is in its position of equilibrium, G the position of the centre of gravity at the time t. Then since in equilibrium the altitude of the centre of gravity is a maximum or minimum, the tangent at A to the curve AG is horizontal. Let the normal GC to the



curve at G meet the normal at A in C. Then when the oscillation becomes indefinitely small C is the centre of curvature of the curve at A. Let $AG = s$, the angle $ACG = \psi$, and let R be the radius of curvature of the curve at A.

Let θ be the angle turned round by the body in moving from the position of equilibrium into the position in which the centre of gravity is at G; then $\frac{d\theta}{dt}$ is the angular velocity of the body. Since G is moving along the tangent at G, the

centre of instantaneous rotation lies in the normal GC, at such a point O, that $OG\frac{d\theta}{dt}=$ vel. of $G=\frac{ds}{dt}$, $\therefore GO=\frac{ds}{d\theta}$.

Let Mk^2 be the moment of inertia of the body about its centre of gravity, then taking moments about O, we have

$$(k^2+OG^2)\frac{d^2\theta}{dt^2}=-g\,.\,OG\sin\psi.$$

Now ultimately when the angle θ is indefinitely small $\frac{\psi}{\theta}=\frac{d\psi}{d\theta}=\frac{OG}{R}$; $\therefore$ the equation of motion becomes

$$(k^2+OG^2)\frac{d^2\theta}{dt^2}=-g\frac{OG^2}{R}\,.\,\theta.$$

Hence if L be the length of the simple equivalent pendulum we have

$$L=\left(1+\frac{k^2}{OG^2}\right)R.$$

423. When the system of bodies in motion admits of only one independent motion, the time of a small oscillation may frequently be deduced from the equation of Vis Viva. This equation will be one of the second order of small quantities, and in forming the equation it will be necessary to take into account small quantities of that order. This will sometimes involve rather troublesome considerations. On the other hand the equation will be free from all the unknown reactions, and we may thus frequently save much elimination.

The method of proceeding will be made clear by the following example, by which a comparison may be made with the method of the last article.

The motion of a body in space of two dimensions is given by the co-ordinates x, y *of its centre of gravity, and the angle θ which any fixed line in the body makes with a line fixed in space. The body being in equilibrium under the action of gravity it is required to find the time of a small oscillation.*

Since the body is capable of only one independent motion, we may express (x, y) as functions of θ, thus

$$x=F(\theta),\quad y=f(\theta).$$

Let Mk^2 be the moment of inertia of the body about its centre of gravity, then the equation of Vis Viva becomes

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+k^2\left(\frac{d\theta}{dt}\right)^2=C-2gy,$$

where C is an arbitrary constant.

Let α be the value of θ when the body is in the position of equilibrium, and suppose that at the time t, $\theta=\alpha+\phi$. Then, by M'Laurin's theorem,

$$y=y_0+y_0'\phi+y_0''\frac{\phi^2}{2}+\dots,$$

where y_0', y_0'' are the values of $\frac{dy}{d\theta}$, $\frac{d^2y}{d\theta^2}$ when $\theta=\alpha$. But in the position of equilibrium y is a maximum or minimum; $\therefore y_0'=0$. Hence the equation of Vis Viva becomes

$$(x_0'^2+k^2)\left(\frac{d\phi}{dt}\right)^2=C-gy_0''\phi^2,$$

where x_0' is the value of $\frac{dx}{d\theta}$ when $\theta=\alpha$; differentiating we get

$$(x_0'^2+k^2)\frac{d^2\phi}{dt^2}=-gy_0''\phi.$$

If L be the length of the simple equivalent pendulum, we have

$$L=\frac{k^2+\left(\frac{dx}{d\theta}\right)^2}{\frac{d^2y}{d\theta^2}},$$

where for θ we are to write its value α after the differentiations have been effected. It is not difficult to see that the geometrical meaning of this result is the same as that given in the last article.

This analytical result was given by Mr Holditch, in the eighth volume of the *Cambridge Transactions*. It is a convenient formula to use when the motion of the oscillating body is known with reference to its centre of gravity.

424. When a body moves in space with one independent motion there is not in general an instantaneous axis. It has, however, been proved in Art. 186 that the motion may always be reduced to a rotation about some central axis and a translation along that axis.

Let I be the moment of inertia of the body about the instantaneous central axis, Ω the angular velocity about it, V the velocity of translation along it, M the mass of the body, then by the principle of vis viva $\frac{1}{2}I\Omega^2+\frac{1}{2}MV^2=U+C$, where U is the force-function, and C some constant. Differentiating we get

$$I\frac{d\Omega}{dt}+\frac{1}{2}\Omega\frac{dI}{dt}+M\frac{V}{\Omega}\frac{dV}{dt}=\frac{dU}{\Omega dt}.$$

Let L be the moment of the impressed forces about the instantaneous central axis, then $L=\frac{dU}{\Omega dt}$ by Art. 326.

Let p be the pitch of the screw-motion of the body, then $V=p\Omega$. The equation of motion therefore becomes

$$(I+Mp^2)\frac{d\Omega}{dt}+\frac{1}{2}\Omega\frac{dI}{dt}+Mp\Omega\frac{dp}{dt}=L.$$

If the body be performing small oscillations about a position of equilibrium, we may reject the second and third terms, and the equation becomes

$$(I+Mp^2)\frac{d\Omega}{dt}=L.$$

If there be an instantaneous axis $p=0$, and we see that we may take moments about the instantaneous axis exactly as if it were fixed in space and in the body.

425. Ex. *A heavy body oscillates in three dimensions, with one degree of freedom, on a fixed rough surface of any form in such a manner that there is no rotation about the common normal. Find the motion.*

(1) Let O be the point of contact, Oz the common normal, Oy a tangent to the arc of rolling determined by the geometrical conditions of the question, OI the instantaneous axis. Then OI, Oy are conjugate diameters in the relative indicatrix.

The relative indicatrix is a conic having its centre at O and lying in the common tangent plane at O, such that the difference of the curvatures of the normal sections through any radius vector OR varies as $\frac{1}{OR^2}$.

(2) Let ρ, ρ' be the radii of curvature of the normal sections through Oy, taken positively when the curvatures are in opposite directions, and let $\frac{1}{s}=\frac{1}{\rho}+\frac{1}{\rho'}$. Then s may be called the radius of relative curvature.

Measure a length $s \,.\, \sin^2 yOI$ along the common normal Oz, and describe a cylinder on it as diameter, the axis being parallel to OI. If the centre of gravity of the body be inside, the equilibrium is stable; if outside and above the plane of xy, unstable. This cylinder may therefore be called the *cylinder of stability*.

(3) Let G be the centre of gravity, and let OG produced cut the cylinder of stability in V; then if K be the radius of gyration about OI, the length L of the simple equivalent pendulum is given by $\frac{K^2}{L}=GV \,.\, \sin^2 GOI$. This equation may also be written in the form $\frac{K^2}{L}=s\cos Goz \,.\, \sin^2 yOI - OG \,.\, \sin^2 GOI$.

This result may be obtained by taking moments about the instantaneous axis. Let O' be the point of contact, G' the position of the centre of gravity at the time t and let $O'I'$ be the instantaneous axis. In the small terms we may consider these as coincident with O, G and OI respectively. If θ be the angle turned round the instantaneous axis, it may be shown that the arc OO' rolled over is $\theta s \sin yOI$. Let this be called σ. To find the moment of the weight we resolve gravity parallel and perpendicular to $O'I'$. The former may be neglected, the latter is $g \sin GOI'$. Let this force act parallel to some line KO. The moment required is the product of resolved gravity into the difference of the projections of OO' and OG' on a plane through $O'I'$ perpendicular to KO. The projection of the former is $\sigma \sin yOI \cos KOz$. The projection of the latter is $\theta \,.\, OG \,.\, \sin GOI$. The result then follows by the same reasoning as in Art. 418.

(4) The motion of the upper body is the same as if the fixed surface were plane and the curvature of the upper body at the point of contact altered so that the relative indicatrix remain the same as before. This supplies an easy method of finding the oscillations in any particular case.

426. Ex. A heavy cone of any form oscillates on a fixed rough conical surface, the vertices being coincident. Let O be the common vertex, OI the line of contact in the position of equilibrium, G the centre of gravity. Let K be the radius of gyration about OI, z = inclination of OI to the vertical measured in the direction opposite to gravity. Let $OG=h$, and the angle $GOI=r$. Let n be the inclination of the vertical plane GOI to the normal plane to the two cones along OI. Let ρ, ρ'

be the semi-angles of the two right circular osculating cones of contact along OI taken positively when the curvatures are in opposite directions. Then the length L of the simple equivalent pendulum is given by

$$\frac{K^2}{hL} = \sin(z-r)\cos n\frac{\sin\rho\sin\rho'}{\sin(\rho+\rho')} - \sin r\sin z.$$

If the upper body be a right cone of semi-angle ρ, and if it be on the top of any conical surface, the preceding expression takes the form

$$\frac{K^2}{hL} = \frac{\sin(z+\rho')\sin^2\rho}{\sin(\rho+\rho')}.$$

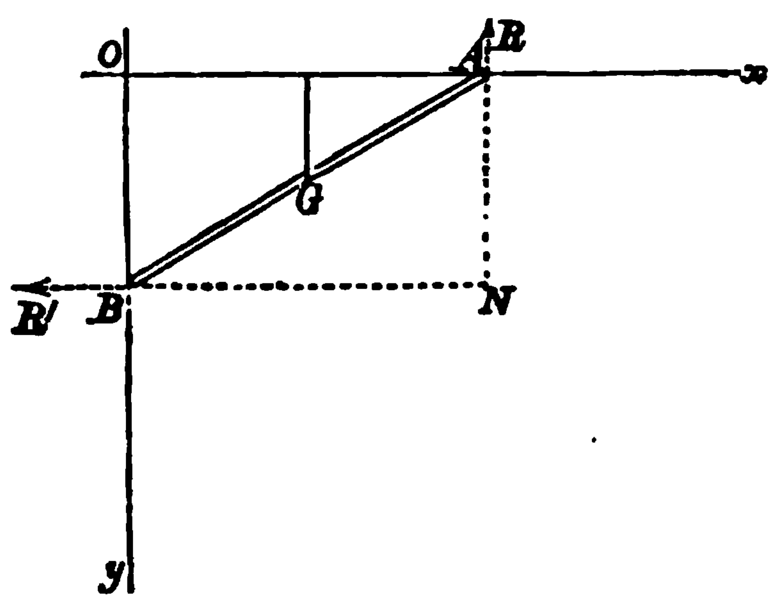
Second Method of forming the Equations of Motion.

427. Let the general equations of motion of all the bodies be formed. If the position about which the system oscillates be known, some of the quantities involved will be small. The squares and higher powers of these may be neglected, and all the equations will become linear. If the unknown reactions be then eliminated the resulting equations may be easily solved.

If the position about which the system oscillates be unknown, it is not necessary to solve the statical problem first. We may by one process determine the positions of rest, ascertain whether they are stable or not, and find the time of oscillation. The method of proceeding will be best explained by an example.

428. Ex. *The ends of a uniform heavy rod* AB *of length* 2l *are constrained to move, the one along a horizontal line* Ox, *and the other along a vertical line* Oy. *If the whole system turn round* Oy *with a uniform angular velocity* ω, *it is required to find the positions of equilibrium and the time of a small oscillation.*

Let x, y be the co-ordinates of G the middle point of the rod, θ the angle OAB which the rod makes with Ox. Let R, R' be the reactions at A and B resolved in the plane xOy. Let the mass of a unit of length be taken as the unit of mass.



The accelerations of any element dr of the rod whose coordinates are (ξ, η) are $\frac{d^2\xi}{dt^2} - \omega^2\xi$ parallel to Ox, $\frac{1}{\xi}\frac{d}{dt}(\xi^2\omega)$ perpendicular to the plane xOy and $\frac{d^2\eta}{dt^2}$ parallel to Oy.

As it will not be necessary to take moments about Ox, Oy, or to resolve perpendicular to the plane xOy, the second acceleration will not be required. The resultants of the effective forces $\frac{d^2\xi}{dt^2}dr$ and $\frac{d^2\eta}{dt^2}dr$, taken throughout the body, are $2l\frac{d^2x}{dt^2}$ and $2l\frac{d^2y}{dt^2}$ acting at G, and a couple $2lk^2\frac{d^2\theta}{dt^2}$ tending to turn the body round G. The resultants of the effective forces $\omega^2\xi dr$ taken throughout the body is a single force acting at $G = \int_{-l}^{+l}\omega^2(x + r\cos\theta)\,dr = \omega^2 x \,.\, 2l$, and a couple* round $G = \int_{-l}^{+l}\omega^2(x + r\cos\theta)\,r\sin\theta\,dr = \omega^2 \,.\, 2l \,.\, \frac{l^2}{3}\sin\theta\cos\theta$, the distance r being measured from G towards A.

Then we have, by resolving along Ox, Oy, and by taking moments about G, the *dynamical equations*

$$\left.\begin{aligned} 2l \,.\, \frac{d^2x}{dt^2} &= -R' + \omega^2 x \,.\, 2l \\ 2l \,.\, \frac{d^2y}{dt^2} &= -R + g \,.\, 2l \\ 2l \,.\, k^2 \,.\, \frac{d^2\theta}{dt^2} &= Rx - R'y - \omega^2 \,.\, 2l \,.\, \frac{l^2}{3}\sin\theta\cos\theta \end{aligned}\right\}\text{............(1).}$$

We have also the *geometrical equations*

$$x = l\cos\theta,\ \ y = l\sin\theta \text{......................(2).}$$

Eliminating R, R', from the equations (1), we get

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} + k^2\frac{d^2\theta}{dt^2} = gx - \omega^2 xy - \omega^2\frac{l^2}{3}\sin\theta\cos\theta \text{.......(3).}$$

* If a body in one plane be turning about an axis in its own plane with an angular velocity ω, a general expression can be found for the resultants of the centrifugal forces on all the elements of the body. Take the centre of gravity G as origin and the axis of y parallel to the fixed axis. Let c be the distance of G from the axis of rotation. Then all the centrifugal forces are equivalent to a single resultant force at G

$$= \int\omega^2(c + x)\,dm = \omega^2 \,.\, Mc, \text{ since } \bar{x} = 0,$$

and to a single resultant couple

$$= \int\omega^2(c + x)\,y\,dm, = \omega^2\int xy\,dm, \text{ since } \bar{y} = 0.$$

To find the position of rest. We observe that if the rod were placed at rest in that position it would always remain there, and that therefore $\frac{d^2x}{dt^2}=0$, $\frac{d^2y}{dt^2}=0$, $\frac{d^2\theta}{dt^2}=0$. This gives

$$gx-\omega^2xy-\omega^2\frac{l^2}{3}\sin\theta\cos\theta=0 \quad\quad (4).$$

Joining this with equations (2), we get $\theta=\frac{\pi}{2}$, or $\sin\theta=\frac{3g}{4\omega^2 l}$, and thus the positions of equilibrium are found. Let any one of these positions be represented by $\theta=\alpha$, $x=a$, $y=b$.

To find the motion of oscillation. Let $x=a+x'$, $y=b+y'$, $\theta=\alpha+\theta'$, where x', y', θ' are all small quantities, then we must substitute these values in equation (3). On the left-hand side since $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$, are all small, we have simply to write a, b, α, for x, y, θ. On the right-hand side the substitution should be made by Taylor's Theorem, thus

$$f(a+x',\ b+y',\ \alpha+\theta')=\frac{df}{da}x'+\frac{df}{db}y'+\frac{df}{d\alpha}\theta'.$$

We know that the first term $f(a, b, \alpha)$ will be zero, because this was the very equation (4) from which a, b, α were found. We therefore get

$$a\frac{d^2y'}{dt^2}-b\frac{d^2x'}{dt^2}+k^2\frac{d^2\theta'}{dt^2}=(g-\omega^2b)\,x'-\omega^2ay'-\omega^2\frac{l^2}{3}\cos 2\alpha\,.\,\theta'.$$

But by putting $\theta=\alpha+\theta'$ in equations (2), we get by Taylor's Theorem $x'=-l\sin\alpha\,.\,\theta'$, $y'=l\cos\alpha\,.\,\theta'$.

Hence the equation to determine the motion is

$$(l^2+k^2)\frac{d^2\theta'}{dt^2}+(gl\sin\alpha+\tfrac{4}{3}\omega^2l^2\cos 2\alpha)\,\theta'=0.$$

Now, if $gl\sin\alpha+\frac{4}{3}\omega^2l^2\cos 2\alpha=n$ be positive when either of the two values of α is substituted, that position of equilibrium is *stable*, and the time of a small oscillation is $2\pi\sqrt{\frac{l^2+k^2}{n}}$.

If n be negative the equilibrium is *unstable*, and there can be no oscillation.

If $\omega^2>\frac{3g}{4l}$ there are two positions of equilibrium of the rod. It will be found by substitution that the position in which the rod is inclined to the vertical is stable, and the other position unstable.

If $\omega^2 < \frac{3g}{4l}$ the only position in which the rod can rest is vertical, and this position is stable.

If $n = 0$, the body is in a position of neutral equilibrium. To determine the small oscillations we must retain terms of an order higher than the first. By a known transformation we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left(l^2\frac{d\theta}{dt}\right).$$

Hence the left-hand side of equation (3) becomes $(l^2 + k^2)\frac{d^2\theta}{dt^2}$. The right-hand side becomes by Taylor's Theorem

$$\frac{d^2}{d\alpha^2}\left(gl\cos\alpha - \frac{2}{3}\omega^2 l^2 \sin 2\alpha\right)\frac{\theta'^2}{1\,.\,2} + \&c.$$

When $n = 0$, we have $\alpha = \frac{\pi}{2}$ and $\omega^2 = \frac{3g}{4l}$. Making the necessary substitutions the equation of motion becomes

$$(l^2 + k^2)\frac{d^2\theta'}{dt^2} = -\frac{gl}{2}\theta'^3.$$

Since the lowest power of θ' on the right-hand side is odd and its coefficient negative, the equilibrium is stable for a displacement on either side of the position of equilibrium. Let a be the initial value of θ', then the time T of reaching the position of equilibrium is

$$T = \sqrt{\frac{4(l^2 + k^2)}{gl}}\int_0^a \frac{d\theta'}{\sqrt{a^4 - \theta'^4}},$$

put $\theta' = a\phi$, then

$$T = \sqrt{\frac{4(l^2 + k^2)}{gl}}\,.\int_0^1 \frac{d\phi}{\sqrt{1 - \phi^4}}\,.\frac{1}{a}.$$

Hence the time of reaching the position of equilibrium varies inversely as the arc. When the initial displacement is indefinitely small, the time becomes infinite.

This definite integral may be otherwise expressed in terms of the Gamma function. It may be easily shown that $\int_0^1 \frac{d\phi}{\sqrt{1-\phi^4}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{2\pi}}$.

429. This problem might have been easily solved by the first method. For if the two perpendiculars to Ox, Oy at A and B meet in N, N is the instantaneous axis. Taking moments about N, we have the equation

$$(l^2+k^2)\frac{d^2\theta}{dt^2}=gl\cos\theta-\int_{-l}^{+l}\omega^2(l+r)^2\sin\theta\cos\theta\frac{dr}{2l}$$

$$=gl\cos\theta-\frac{4l^2}{3}\omega^2\sin\theta\cos\theta$$

$$=f(\theta).$$

Then the position of equilibrium can be found from the equation $f(\alpha)=0$ and the time of oscillation from the equation

$$(l^2+k^2)\frac{d^2\theta'}{dt^2}=\frac{df(\alpha)}{d\alpha}\theta'.$$

430. Ex. 1. If the mass of the rod AB is M show that the magnitude of the couple which constrains the system to turn round Oy with uniform angular velocity is $M\frac{4l^2}{3}\omega\frac{d\theta}{dt}\sin 2\theta.$

Would the magnitude of this couple be altered if Ox or Oy had any mass?

Ex. 2. The upper extremity of a uniform beam of length $2l$ is constrained to slide on a smooth horizontal rod without inertia, and the lower along a smooth vertical rod through the upper extremity of which the horizontal rod passes: the system rotates freely about the vertical rod, prove that if α be the inclination of the beam to the vertical when in a position of relative equilibrium, the angular velocity of the system will be $\left(\frac{3g}{4l\cos\alpha}\right)^{\frac{1}{2}}$, and if the beam be slightly displaced from this position show that it will make a small oscillation in the time

$$\frac{4\pi}{\left\{\frac{3g}{l}(\sec\alpha+3\cos\alpha)\right\}^{\frac{1}{2}}}.\quad\text{[Coll. Exam.].}$$

In the example in the text the system is constrained to turn round the vertical with uniform angular velocity, but in this example the system rotates freely. The angular velocity about the vertical is therefore not constant, and its small variations must be found by the principle of angular momentum.

Oscillations with two or more Degrees of Freedom.

431. When the position of a system of bodies depends on several independent co-ordinates, the equations to determine the motion become rather complicated. In order to separate the difficulties of analysis from those of dynamics, we shall consider the case in which the system depends on two independent co-ordinates, though the remarks about to be made will be for the most part quite general, and will apply, no matter how many co-ordinates the system may have. In the sequel we shall consider Lagrange's general method of forming the equations when the system has n co-ordinates.

432. The equations of motion of a dynamical system performing small oscillations with two independent motions are of the form

$$A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx+F\frac{d^2y}{dt^2}+G\frac{dy}{dt}+Hy=0,$$

$$A'\frac{d^2x}{dt^2}+B'\frac{dx}{dt}+C'x+F'\frac{d^2y}{dt^2}+G'\frac{dy}{dt}+H'y=0.$$

To solve these, we eliminate either x or y; if D stand for $\dfrac{d}{dt}$, we have

$$\begin{vmatrix} AD^2+BD+C, & FD^2+GD+H \\ A'D^2+B'D+C', & F'D^2+G'D+H' \end{vmatrix} x=0,$$

with a similar equation for y. If $\overline{AB}$ stand for the determinant $\begin{vmatrix} A, & B \\ A', & B' \end{vmatrix}$ this biquadratic becomes, when x is omitted,

$$\overline{AF}D^4+(\overline{AG}+\overline{BF})D^3+(\overline{AH}+\overline{BG}+\overline{CF})D^2+(\overline{BH}+\overline{CG})D+\overline{CH}=0.$$

If the roots of this biquadratic be m_1, m_2, m_3, m_4, we have by the theory of Linear Differential Equations

$$x=M_1e^{m_1t}+M_2e^{m_2t}+M_3e^{m_3t}+M_4e^{m_4t},$$

where M_1, M_2, M_3, M_4 are arbitrary constants. Similarly we have

$$y=M_1'e^{m_1t}+M_2'e^{m_2t}+M_3'e^{m_3t}+M_4'e^{m_4t}.$$

The M''s are not independent of the M's, for by substituting in either differential equation and taking any M and M' as typical of all,

$$(Am^2+Bm+C)M=-(Fm^2+Gm+H)M'.$$

There are therefore just four arbitrary constants, and these are to be determined by the initial values of x, y, $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$.

433. If the position of the system depends on three independent co-ordinates x, y, z, we shall have three equations of motion similar to the two at the beginning of this article. These may be solved in the same way. In this case we obtain a subsidiary equation of the sixth degree to determine the exponentials which occur in the variables. The relations between the coefficients of corresponding exponentials can be found by substitution in any two of the equations of motion.

In certain cases it may be more convenient to choose x or y to be itself a differential coefficient of a co-ordinate. In this case the biquadratic or sextic equation will reduce to a cubic or quintic.

434. It appears from this summary that the character of the motion depends on the forms of the roots* of this biquadratic.

* If the general character of the motion is required it will be necessary to analyse the biquadratic. Rules by which this is made to depend on a cubic equation are given in most of the books on the theory of equations, but as the final results are not stated, it will be useful to give here a short analysis for reference.

Let the biquadratic be

$$ax^4+4bx^3+6cx^2+4dx+e=0,$$

so that the invariants are $I=ae-4bd+3c^2$ and $J=ace+2bcd-ad^2-eb^2-c^3$. This last may also be written in the form of a determinant. It will generally be found convenient to clear the equation of the second term. Let the equation so transformed be

$$\xi^4-\frac{6H}{a^2}\xi^2+\frac{4G}{a^3}\xi+\frac{F}{a^4}=0,$$

where $H=b^2-ac$ and $G=2b^3-3abc+a^2d$. By using the invariants or by actual transformation, it is easy to see that $Ia^2=F+3H^2$ and $a^3J=4H^3-G^2-a^2IH$.

Let Δ be the discriminant, *i.e.* $\Delta=I^3-27J^2$, then it is proved in all books on the theory of equations that if Δ is negative and not zero the biquadratic has two real and two imaginary roots. If Δ is positive and not zero, the roots are either all real or all imaginary.

Usually we can distinguish whether the roots are all real or all imaginary by ascertaining if the biquadratic has or has not a real root, thus if a and e have opposite signs one root is, and therefore all the roots are, real. In any case we may use the following criterion. Let $Ka^4=9H^2-F=12H^2-Ia^2$. Then if $\alpha, \beta, \gamma, \delta$ be the roots of the transformed equation it is easy to prove

$$\left.\begin{aligned} \frac{3H}{a^2}&=\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{4} \\ K&=\left(\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{4}\right)^2-\alpha\beta\gamma\delta \end{aligned}\right\}.$$

If all the roots are real H must be finite and positive. Since the arithmetic mean of four positive quantities is greater than their geometrical mean, it is clear that K is also positive, and can vanish only when all the squares of the roots are equal. If all the roots are imaginary, let them be $p\pm p'\sqrt{-1},\ -p\pm q'\sqrt{-1}$. We then have

$$\left.\begin{aligned} \frac{3H}{a^2}&=\frac{2p^2-(p'^2+q'^2)}{2} \\ K&=\left(\frac{p'^2-q'^2}{2}\right)^2-2p^2(p'^2+q'^2) \end{aligned}\right\}.$$

If H is positive or zero, it is easy to see that K must be negative. If therefore H and K are both positive, the four roots are real, if either is negative or zero, the four roots are imaginary.

If the discriminant Δ is zero, but I and J not zero, it is known that the biquadratic has two roots equal. If two of the roots are real and equal and the other

If any one of the roots is real and positive, x and y will ultimately become large, unless the initial conditions are such that the term depending on this root disappears from the values of x and y. If the roots are all real and negative, the motion will gradually disappear and the system will come to rest at the end of an infinite time.

If two of the roots are imaginary, we have a pair of imaginary exponentials with imaginary coefficients, which can be rationalized into a sine and a cosine. This rationalization will be however unnecessary if, as usually happens, only the character of the oscillations is required. Suppose the roots to be $a \pm p\sqrt{(-1)}$, we have

$$x = e^{at}(N_1 \cos pt + N_2 \sin pt) + \&c.,$$

where N_1, N_2 are arbitrary constants. There will be a similar expression for y with N' written for N. Thus the period of the oscillation is $\frac{2\pi}{p}$. The oscillation will ultimately become very large or vanish away, according as a is positive or negative. If $a = 0$, the oscillations will continue throughout of the same magnitude.

If it be required to find not merely the character of the motion, but also the motion resulting from given initial conditions, it will be necessary to determine the relations between the arbitrary constants which enter into the expression for x and y. This may be effected very easily in the following manner. Let $D^2 + fD + g$ be the factor which equated to zero gives the imaginary roots, then f and g are known in terms of a and p. Let us now substitute $-fD - g$ for D^2 in the two first equations of Art. 432. They reduce to equations of the form

$$\left.\begin{array}{l}\left(B_1 \dfrac{d}{dt} + C_1\right) x + \left(G_1 \dfrac{d}{dt} + H_1\right) y = 0 \\ \left(B_1' \dfrac{d}{dt} + C_1'\right) x + \left(G_1' \dfrac{d}{dt} + H_1'\right) y = 0\end{array}\right\},$$

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two imaginary, we see by putting q' zero that if H is positive or zero, K must be negative. Hence if H and K are both positive all the roots are real, if H or K is negative or zero, two roots are real and two imaginary. If G is zero, there are then two pairs of equal roots. In this case K is zero, and these roots are all real if H is positive, all imaginary if H is negative.

Lastly if Δ is zero and also both I and J zero. The biquadratic has three roots equal, and therefore all the roots are real. If $H = 0$ also, the four roots are all equal and real.

where B_1, C_1, &c. are some constant coefficients. Eliminating $\frac{dy}{dt}$ from these equations, we have an equation of the form

$$y = \left(K\frac{d}{dt} + L\right) x,$$

where K and L are constants, so that when the two terms of x, which depend on this factor, are known, the corresponding terms of y can be found immediately. If there be another pair of imaginary roots, we obtain by a similar process a similar equation with different constants for K and L, to find the corresponding terms in y.

If two of the roots are equal, say $m_1 = m_2$, then, by the theory of Linear Equations, we know that

$$x = (N_1 + N_2 t)\, e^{m_1 t} + M_3 e^{m_3 t} + M_4 e^{m_4 t},$$

where N_1 and N_2, &c. are arbitrary constants. If three roots are equal, there will be a term with t^2 and so on. The expression for y will of course contain similar terms. Let it be

$$y = (N_1' + N_2' t)\, e^{m_1 t} + M_3'\, e^{m_3 t} + M_4'\, e^{m_4 t}.$$

The terms containing t as a factor will at first increase with t, and if m_1 is positive or zero will become very great, but if m_1 is negative, they will ultimately vanish. The motion will, in the latter case, be stable if the initial increase of the terms is not such that the values of x and y become large, *i.e.* if the system is not at first so much disturbed that the motion cannot be considered as a small oscillation.

In some cases the relations between the constants in the expressions for x and y are such that the coefficients of both the terms containing the factor t vanish*. When this occurs the four

* To prove this let us find the relations between the constants. Substituting the values of x and y in the two first equations of Art 432, we find

$$(Am_1^2 + Bm_1 + C)\, N_2 = -(Fm_1^2 + Gm_1 + H)\, N_2',$$

$$(Am_1^2 + Bm_1 + C)\, N_1 + (2Am_1 + B)\, N_2 = -(Fm_1^2 + Gm_1 + H)\, N_1' - (2Fm_1 + G)\, N_2',$$

with two similar equations which may be obtained from these by accenting the letters A, B, C, F, G, H. If then

$$\left.\begin{aligned} Am_1^2 + Bm_1 + C &= 0 \\ A'm_1^2 + B'm_1 + C' &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} Fm_1^2 + Gm_1 + H &= 0 \\ F'm_1^2 + G'm_1 + H' &= 0 \end{aligned}\right\},$$

while the two expressions

$$(2Am_1 + B)\,(2F'm_1 + G') \text{ and } (2A'm_1 + B')\,(2Fm_1 + G)$$

are unequal, we have N_2, N_2' both zero, and N_1, N_1' both arbitrary. If the two expressions just written down were equal also, it may be shown that the biquadratic to find D would have three equal roots.

arbitrary constants will be N_1, N_1', M_3 and M_4. In such cases the motion is stable for all initial conditions.

435. The most important case is that in which there are no real exponentials in the values of x and y. If $\overline{AG}+\overline{BF}$ and $\overline{BH}+\overline{CG}$ both vanish, there will be no odd powers in the subsidiary biquadratic. The biquadratic may now be regarded as quadratic in D^2. If its roots are real and negative, let them be $-p^2$ and $-q^2$. The expression for x will then take the form

$$x = N_1 \sin(pt+\nu_1) + N_2 \sin(qt+\nu_2),$$

where N_1, N_2, ν_1, ν_2 are arbitrary constants. The corresponding terms in y may be found by the rule just given. Eliminating $\dfrac{dy}{dt}$ between the two given equations of motion, let the result be

$$A'\frac{d^2x}{dt^2} + B'\frac{dx}{dt} + C'x + F'\frac{d^2y}{dt^2} + H'y = 0.$$

Then writing $-p^2$ for $\dfrac{d^2}{dt^2}$, we have

$$\begin{aligned} y &= -\frac{C'-A'p^2}{H'-F'p^2}x - \frac{B'}{H'-F'p^2}\frac{dx}{dt} \\ &= -\frac{C'-A'p^2}{H'-F'p^2}N_1 \sin(pt+\nu_1) - \frac{B'p}{H'-F'p^2}N_1 \cos(pt+\nu_1) \\ &\quad -\frac{C'-A'q^2}{H'-F'q^2}N_2 \sin(qt+\nu_2) - \frac{B'q}{H'-F'q^2}N_2 \cos(qt+\nu_2). \end{aligned}$$

436. In many cases it will be found impracticable to solve the biquadratic on which the character of the motion depends. If however we only wish to ascertain whether the position of equilibrium, or the steady motion about which the system is in oscillation, is stable or unstable, we may proceed without solving the biquadratic.

With the reservations in the case of equal roots mentioned in Art. 434, the necessary and sufficient conditions for stability are, that the real roots and the real parts of the imaginary roots should be all negative. It is proposed here to investigate a method of easy application to decide whether the roots are of this character.

Let the biquadratic be written in the form

$$\phi(D) = aD^4 + bD^3 + cD^2 + dD + e = 0.$$

Let us form that symmetrical function of the roots which is the

product of the sums of the roots taken two and two. If this be called $\frac{X}{a^3}$, we find*

$$X = bcd - ad^2 - eb^2$$
$$= \frac{1}{2} \begin{vmatrix} 2a & b & c \\ b & 0 & d \\ c & d & 2e \end{vmatrix}.$$

It will be convenient to consider first the case in which X is finite.

Suppose we know the roots to be imaginary, say $\alpha \pm p\sqrt{-1}$, and $\beta \pm q\sqrt{-1}$.

Then $$\frac{X}{a^3} = 4\alpha\beta\,\{(\alpha+\beta)^2 + (p+q)^2\}\,\{(\alpha+\beta)^2 + (p-q)^2\}.$$

Thus, $\alpha\beta$ always takes the sign of $\frac{X}{a}$, and $\alpha+\beta$ always takes the sign of $-\frac{b}{a}$. Thus the signs of both α and β can be determined; and if a, b, X have the same sign, the real parts of the roots are all negative.

Suppose, next, that two of the roots are real and two imaginary. Writing $q'\sqrt{-1}$ for q, so that the roots are $\alpha \pm p\sqrt{-1}$ and $\beta \pm q'$, we find

$$\frac{X}{a^3} = 4\alpha\beta\,\{[(\alpha+\beta)^2 + p^2 - q'^2]^2 + 4p^2q'^2\}.$$

* This value of X may be found in several ways more or less elementary. If we substitute $D = E \pm Z$ in the given biquadratic and equate to zero the even and odd powers of Z, we have

$$\left.\begin{aligned} aZ^4 + (6aE^2 + 3bE + c)\,Z^2 + AE^4 + bE^3 + cE^2 + dE + e = 0 \\ (4aE + b)\,Z^3 + (4aE^3 + 3bE^2 + 2cE + d)\,Z = 0 \end{aligned}\right\}.$$

Rejecting the root $Z = 0$ and eliminating Z we have

$$64a^3E^6 + \ldots\ldots + bcd - ad^2 - eb^2 = 0,$$

where only the first and last terms of the equation are retained, the others not being required for our present purpose. Since $x = E \pm Z$ it is clear that each value of E is the arithmetic mean of two values of x. We have an equation of the sixth degree to find E because there are six ways of combining the four roots of the biquadratic two and two. The product of the roots of the equation in E may be deduced in the usual manner from the first and last terms, and thence the value of X is seen to be that given in the text.

If we eliminated E we should obtain an equation in Z whose roots are the arithmetic means of the differences of the roots of the given equation taken two and two.

Just as before, $\alpha\beta$ takes the sign of $\frac{X}{a}$, and $\alpha+\beta$ takes the sign of $-\frac{b}{a}$. Also, $\beta^2-q'^2$ takes the sign of the last term $\frac{e}{a}$ of the biquadratic. This determines whether β is numerically greater or less than q'. If, then, a, b, e, and X have the same sign, the real roots and the real parts of the imaginary roots are all negative.

Lastly, suppose the roots to be all real. Then, if all the coefficients are positive, we know, by Descartes' rule, that the roots must be all negative, and the coefficients cannot be all positive unless all the roots are negative. In this case, since X is the product of the sums of the roots taken two and two, it is clear that $\frac{X}{a}$ will be positive.

Whatever the nature of the roots may be, yet if the real roots and the real parts of the imaginary roots are negative, the biquadratic must be the product of quadratic factors all whose terms are positive. It is therefore necessary for stability that every coefficient of the biquadratic should have the same sign. It is also clear that no coefficient of the equation can be zero unless either some real root is zero or two of the imaginary roots are equal and opposite.

Summing up the several results which have just been proved, we conclude that *if X is finite, the necessary and sufficient conditions that the real roots and the real parts of the imaginary roots should be negatine or zero are that every coefficient of the biquadratic and also X should have the same sign.*

The case in which $X=0$ does not present any difficulty. It follows from the definition of X, that if X vanishes two of the roots must be equal with opposite signs and conversely if two roots are equal with opposite signs X must vanish. Writing $-D$ for D in the biquadratic and subtracting the result thus obtained from the original equation we find $bD^3+dD=0$. The equal and opposite roots are therefore given by $D=\pm\sqrt{-\frac{d}{b}}$. If b and d have opposite signs these roots are real, one being positive and one negative. If b and d have the same sign, they are a pair of imaginary roots with the real parts zero.

The sum of the other two roots is equal to $-\frac{b}{a}$ and their product is $\frac{be}{ad}$. We therefore conclude that *if $X=0$, the real roots and the real parts of the negative roots will be negative or zero if every coefficient of the biquadratic is finite and has the same sign.*

If either a or e vanishes, the biquadratic reduces to a cubic. Putting e zero, we have

$$\frac{X}{a^3 d} = bc - ad.$$

If the coefficients have all the same sign it is easy to see that it is necessary for stability that $bc - ad$ should be positive or zero.

If a and e be not zero and one of the two b, d vanish, the other must vanish also, for otherwise X could not have the same sign as a. In this case X vanishes, and the biquadratic reduces to the quadratic

$$aD^4 + cD^2 + e = 0.$$

As this equation admits of an easy solution, no difficulty can arise in practice from this case. It is necessary for stability that the roots of the quadratic should be real and negative. The conditions for this are, *firstly* the coefficients a, c, e must all have the same sign, *secondly* that $c^2 > 4ae$.

437. If the equation on which the motion of the system depends is of the fifth degree, we may proceed in the same way. Let the equation be

$$\phi(D) = aD^5 + bD^4 + cD^3 + dD^2 + eD + f = 0,$$

and let us suppose the coefficient a to be positive.

Form, as before, the product of the sums of the roots taken two and two. If this be $\frac{X}{a^4}$, we find

$$X = \begin{vmatrix} bc - ad & be - af \\ be - af & de - cf \end{vmatrix}.$$

Let us consider first the case in which X is finite.

Suppose that there are four imaginary roots $\alpha \pm p\sqrt{-1}$, $\beta \pm q\sqrt{-1}$, and one real root γ. Then γ has the sign opposite to f, and $\alpha\beta$ takes the sign of X, while $2(\alpha+\beta) + \gamma = -\frac{b}{a}$. If then f be positive, γ is negative. If b be positive and $\phi\left(-\frac{b}{a}\right)$ negative, the root γ is numerically less than $\frac{b}{a}$, so that $\alpha + \beta$ is negative. If therefore a, b, f, X, and $-\phi\left(-\frac{b}{a}\right)$ be all positive, α, β, γ will be all negative.

Suppose that there are two imaginary roots $\alpha \pm p\sqrt{-1}$, and three real roots β, γ, δ. Then, if all the coefficients are positive, β, γ, δ are negative, and X takes the sign opposite to α; so that, if X be also positive, α, β, γ, δ will be all negative.

Suppose all the roots to be real; then, if all the coefficients be positive, the roots will be all negative, and not otherwise; and it is also clear that X, being the product of ten negative quantities, will be positive.

In both these cases, if the real roots and the real parts of the imaginary roots be negative, it is clear that $\phi\left(-\frac{b}{a}\right)$ must have the sign opposite to a.

Conversely, if all the real roots and the real parts of the imaginary roots be negative, the factors of the equation, and therefore the equation itself, must have all the coefficients of the same sign.

We therefore conclude that *it is necessary and sufficient for stability of equilibrium that every coefficient of the equation, $-\phi\left(-\frac{b}{a}\right)$, and also X, should be positive.*

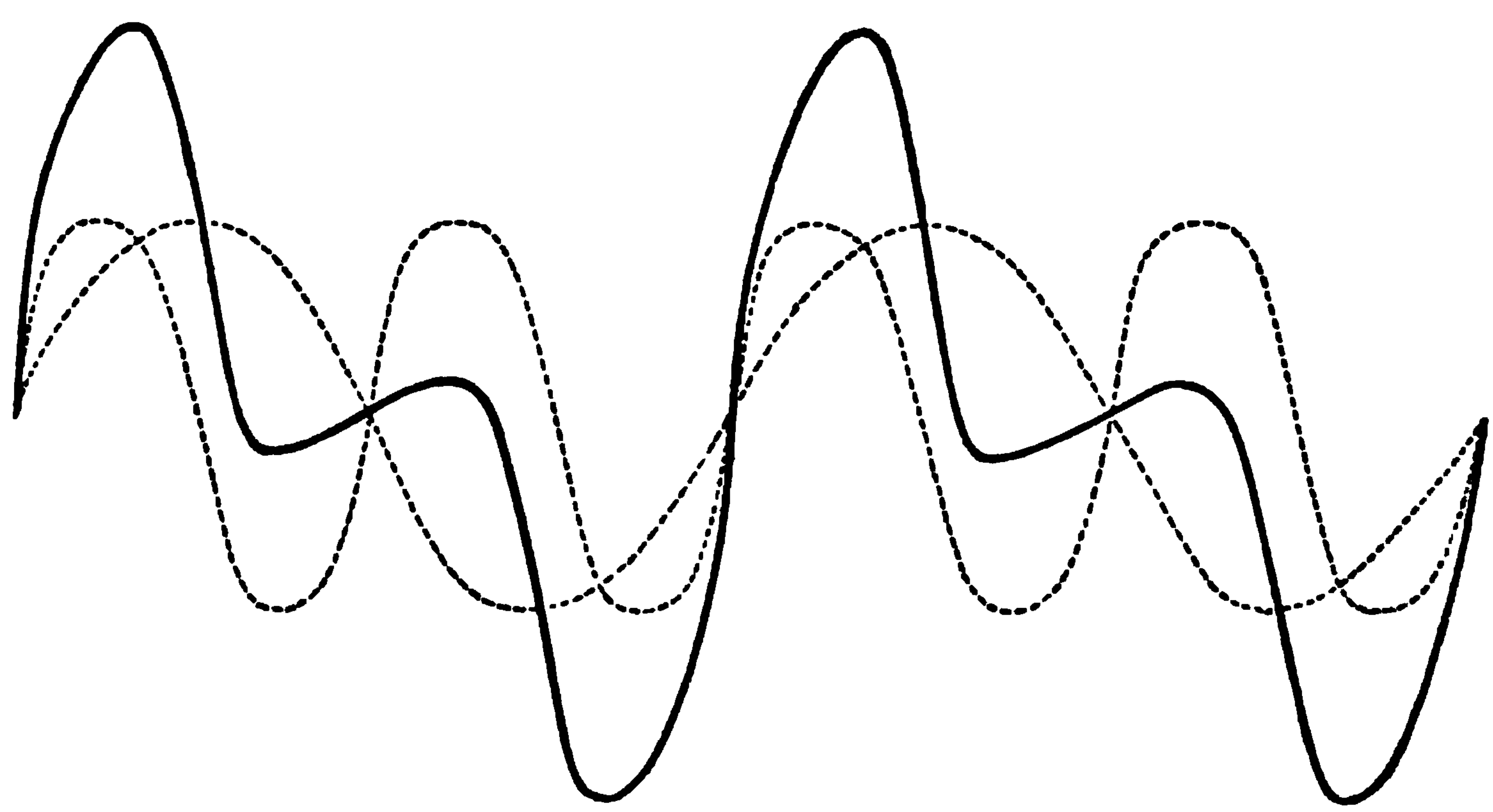
The case in which X is zero may be treated in the same manner as in the biquadratic.

As it is very seldom that equations beyond the fourth or fifth degrees present themselves in dynamics, it is unnecessary to consider any other cases in detail. A more general method of proceeding will be indicated in a note.

438. It will be often found advantageous to trace the more complicated cases of motion by the help of a figure. There are various methods of effecting this, some being more suited to illustrate one kind of motion, others to illustrate another. We might, for instance, follow the method indicated in Art. 412. Let the abscissa of a point P represent on any scale the time elapsed since some epoch, and let the ordinate represent the value of x. In the same way the curve traced out by another point Q will represent the changes of y. Suppose, for example, we wished to trace the motion represented by

$$x = N \sin pt + N \sin 2pt,$$

the coefficients being equal in magnitude. There will be no difficulty in tracing the two curves $x_1 = N \sin pt$ and $x_2 = N \sin 2pt$. Let these be the two dotted lines. We obtain the required curve by adding the ordinates corresponding to each abscissa. Let this be the continuous line.



In the figure the axis of the abscissæ is not drawn. It clearly joins the two extreme points on the right and left hand sides.

We see from a simple inspection of the figure, that the motion consists of a violent oscillation to each side of the mean position

followed by a very slight one, and so on alternately. This figure resembles that used in Astronomy to trace the changes of magnitude of the equation of time throughout the year.

439. Ex. 1. Show that the motion represented by $x = N\sin pt + N\sin 3pt$ consists of two large oscillations to one side of the mean position followed by two equally large ones to the other side and so on continually.

Ex. 2. Trace the motion represented by $x = N\sin 2pt + N\sin 3pt$, and point out the difference between the two parts of the large oscillation.

440. Let us trace the motion represented by $x = N_1 \sin(pt + \nu_1) + N_2 \sin(qt + \nu_2)$, where N_1 and N_2 are both positive, firstly when p and q are nearly equal, and secondly when p is small compared with q.

In the first case, consider any time at which $pt + \nu_1$ and $qt + \nu_2$ differ from each other by an even multiple of π. At this instant the two trigonometrical terms have the same sign, and, since p and q are nearly equal, they will increase and decrease together for several oscillations, how many will depend on the nearness of p and q to each other. The value of x will therefore vary between the limits $\pm(N_1 + N_2)$. Next consider any time at which $pt + \nu_1$ and $qt + \nu_2$ differ by an odd multiple of π. The two trigonometrical terms have opposite signs and will continue to have opposite signs for several oscillations. The value of x will therefore vary between the limits $\pm(N_1 - N_2)$. We see that the motion of that part of the dynamical system which depends on the co-ordinate x undergoes a periodic change of character. At one time, this part of the system is oscillating with an arc $N_1 + N_2$, after an interval equal to $\dfrac{\pi}{p-q}$, the arc of oscillation is $N_1 - N_2$. If N_1 and N_2 are nearly equal, this last arc may be so small, that the motion is invisible to the eye. Thus there will be alternate periods of comparative activity and rest. This alternation is sometimes called *beats*. Usually the two co-ordinates x and y will be so related that the period of comparative rest in one will coincide with the period of comparative activity in the other. When this is the case *there will be an alternate transference of energy from one part of the system to another and back again.*

441. Ex. Show that, if p and q be unequal, x may be written in the form

$$x = N\sin\frac{1}{2}(pt + \nu_1 + qt + \nu_2 + \delta),$$

where
$$N^2 = N_1^2 + N_2^2 + 2N_1N_2\cos(pt + \nu_1 - qt - \nu_2),$$

$$\tan\frac{\delta}{2} = \frac{N_1 - N_2}{N_1 + N_2}\tan\frac{1}{2}(pt + \nu_1 - qt - \nu_2).$$

Thence show that when p and q are nearly equal, the oscillation will appear to the eye to be harmonic, but the arc of oscillation will slowly vary between the limits $N_1 \pm N_2$.

442. Next, let p be small compared with q. In this case $qt + \nu_2$ increases by 2π while $pt + \nu_1$ alters only by $\dfrac{p}{q}2\pi$, so that the second trigonometrical term goes through all its changes while the first is only very slightly altered. The system will therefore appear to oscillate about a mean position determined by the instan-

taneous value of the first trigonometrical term. Thus *the oscillations will appear to be harmonic to the eye, while the apparent mean position will travel first to one side and then to the other of the real mean.*

443. Ex. Investigate the following geometrical construction to represent the motion

$$\pm x = N_1 \sin pt + N_2 \sin qt.$$

Let q be greater than p in the standard case and let x have a sign such that N_1 is positive. Describe a circle with centre O and radius equal to $\frac{q-p}{q}N_1$. Let another circle with centre C and radius equal to $\frac{p}{q}N_1$ touch the first circle externally at a point A. Measure CP equal to N_2 in the direction OC, so that if N_2 is negative CP must be measured in the opposite direction. If the second circle be now made to roll on the first, the point P traces out an epitrochoid. If C' and P' be corresponding positions of the centre of the moving circle and the generating point, then the distance of P' from the fixed straight line OA is the value of x, while the angle $C'OA$ is equal to pt.

Apply this to trace the motion when p and q are nearly equal.

The third or Lagrange's method of forming the equations of motion.

444. Let a system of bodies be in equilibrium under any conservative forces. When disturbed into any other position let U be the force function, $2T$ the vis viva. Let the position of the system be defined by n co-ordinates θ, ϕ, &c., which are such that they vanish in the position of equilibrium. Then if the system oscillate about the position of equilibrium, θ, ϕ, &c. will be small throughout the whole motion. As before, let accents denote differential coefficients with regard to t.

Let us suppose that the geometrical equations do not contain the time explicitly, then by Art. 367 T may be expressed as a homogeneous function of θ', ϕ', &c., of the form

$$2T = A_{11}\theta'^2 + 2A_{12}\theta'\phi' + A_{22}\phi'^2 + \&c. \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Here the coefficients A_{11}, &c. are all functions of θ, ϕ, &c., and we may suppose them to be expanded in a series of some powers of these co-ordinates. If the oscillations of the system are so small that we may reject all powers of the small quantities θ, ϕ, &c. except the lowest which occur, we may reject all except the constant terms of these series. We shall therefore regard the coefficients A_{11}, &c. as constants.

In the same way we may expand U in a series of powers of θ, ϕ, &c. In this series the terms containing the first powers will vanish, because by the principle of virtual velocities

$$dU = \Sigma m\,(Xdx + Ydy + Zdz)$$

vanishes in the position of equilibrium. Hence we may put

$$2U = 2U_0 + a_{11}\theta^2 + 2a_{12}\theta\phi + \&c. \qquad \text{.......................} \quad (2),$$

where U_0 is a constant, which is easily seen to be the value of U in the position of equilibrium. It is necessary for the success of Lagrange's method that both these expansions should be possible.

We have now to substitute these values of T and U in the n Lagrange's equations

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta} \qquad \text{........................} (3),$$

with similar equations for ϕ, ψ. Since the expression for T does not contain θ, ϕ, &c., we have

$$\frac{dT}{d\theta} = 0,\ \frac{dT}{d\phi} = 0,\ \&c.$$

The n equations (3) therefore become

$$\left.\begin{array}{c} A_{11}\theta'' + A_{12}\phi'' + \ldots = a_{11}\theta + a_{12}\phi + \ldots \\ A_{12}\theta'' + A_{22}\phi'' + \ldots = a_{12}\theta + a_{22}\phi + \ldots \\ \&c. = \&c. \end{array}\right\} \text{.........} \quad (4).$$

These are Lagrange's equations to determine the small oscillations of any system about a position of equilibrium, under any conservative forces, provided the geometrical equations do not contain the time explicitly, and are not functions of the differential coefficients of the co-ordinates.

These equations may be obtained in a variety of ways. In many cases it is more convenient to use the process of taking moments and resolving. The advantage of using Lagrange's method is that the whole motion is made to depend on one function only, viz. $T + U$.

445. We shall now proceed to the solution of the equations. We notice that these equations do not contain any differential coefficients of the first order. This will be the case when a dynamical system oscillates about a position of equilibrium under conservative forces. This peculiarity greatly simplifies the solution. Instead of using exponentials, as in Art. 432, which (when we want anything more than the periods) have afterwards to be rationalized, we may now conveniently introduce the trigonometrical expressions at once. Let us then put

$$\left.\begin{array}{c} \theta = L_1 \sin(p_1 t + \alpha_1) + L_2 \sin(p_2 t + \alpha_2) + \&c. \\ \phi = M_1 \sin(p_1 t + \alpha_1) + M_2 \sin(p_2 t + \alpha_2) + \&c. \\ \&c. = \&c. \end{array}\right\} \text{......} (5),$$

which may be written in the typical form

$$\theta = L \sin (pt + \alpha),\ \phi = M \sin (pt + \alpha),\ \&c.$$

If we substitute in equations (4) we have

$$\left.\begin{array}{l}(A_{11} p^2 + a_{11}) L + (A_{12} p^2 + a_{12}) M + \&c. = 0 \\ (A_{12} p^2 + a_{12}) L + (A_{22} p^2 + a_{22}) M + \&c. = 0 \\ \quad \&c. \qquad\qquad\qquad \&c. \qquad\qquad = 0\end{array}\right\} \ \ldots\ldots\ldots (6).$$

Eliminating L, M, &c., we have a determinantal equation

$$\begin{vmatrix} A_{11} p^2 + a_{11}, & A_{12} p^2 + a_{12}, & \&c. \\ A_{12} p^2 + a_{12}, & A_{22} p^2 + a_{22}, & \&c. \\ \&c. & \&c. & \&c. \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (7),$$

which, it will be observed, is symmetrical about the leading diagonal. This equation is of the n^{th} degree to find p^2. It will be presently shown that its roots are real. Taking any root positive or negative, the equations (6) determine the ratios of M, N, &c. to L; and we notice that these ratios will also be all real. If all the roots are positive, the equations (5) will give the whole motion, with $2n$ arbitrary constants, viz. L_1, $L_2 \ldots L_n$; α_1, $\alpha_2 \ldots \alpha_n$. These have to be determined by the initial values of θ, ϕ, &c., θ', ϕ', &c. If any root be negative, the corresponding sine will resume its exponential form, the coefficient being rationalized by giving the coefficient L an imaginary form.

That the determinant should contain no odd powers of p is just what we should have expected *a priori*. In our preliminary assumption (5) each sine is really the sum of two exponentials with indices of opposite signs. The equation therefore of Art. 432 to determine p should here give pairs of equal roots of opposite signs.

The equation (7) may be written down without difficulty as soon as the values of T and U have been expanded in powers of θ', &c., θ, &c., respectively. In finding the times of oscillation of a system about a position of equilibrium, it is not necessary to go through all the intermediate steps; we may, if we please, write down at once the determinantal equation. The rule will be as follows. *Omitting the accents in the expression for* T, *and the constant term in* U, *equate to zero the discriminant of* $p^2T + U$. *The roots of the equation thus formed are the values of* p. If we require the motion as well as the periods, we shall require equations (6). But these may be also very simply found in the following manner. *Omitting accents as before and taking any of the values of* p *just found, form the equations**

* These equations are given by Lagrange.

$$\frac{d}{d\theta}(p^2T+U)=0,\ \frac{d}{d\phi}(p^2T+U)=0,\ \&c. \quad\quad (8).$$

The θ, ϕ, &c. in these equations may be replaced by the coefficients required in the equations (5).

If we solve these equations we see that the ratios of L, M, &c. are equal to the ratios of the minors of the constituents of any one line in the determinant (7).

Ex. 1. A rod AB whose length is $2a$ and mass m is suspended from a fixed point O by a string OA the length of which is l. The rod oscillates under gravity in a vertical plane, find the periods of the small oscillations.

Let θ, ϕ be the angles the string and the rod make with the vertical. Proceeding as in Art. 136, we find that when powers of θ and ϕ higher than the second are neglected,

$$T=\frac{1}{2}m\{l^2\theta'^2+2al\theta'\phi'+(k^2+a^2)\phi'^2\},$$

$$U=U_0-\frac{1}{2}mg(l\theta^2+a\phi^2).$$

Forming the discriminant of p^2T+U and dividing out the common factor m, we have

$$\begin{vmatrix} p^2l^2-gl & alp^2 \\ alp^2 & p^2(k^2+a^2)-ag \end{vmatrix}=0.$$

This quadratic gives two values of p^2. If these be p_1^2 and p_2^2, we have

$$\left.\begin{aligned}\theta&=L_1\sin(p_1t+\alpha_1)+L_2\sin(p_2t+\alpha_2),\\ \phi&=-L_1\frac{p_1^2l^2-gl}{alp_1^2}\sin(p_1t+\alpha_1)-L_2\frac{p_2^2l^2-gl}{alp_2^2}\sin(p_2t+\alpha_2)\end{aligned}\right\}.$$

Ex. 2. Show that when the determinant (7) is zero, the ratios of the minors of the constituents of any one line are equal to the ratios of the corresponding minors of the constituents of any other line.

Ex. 3. If (T_1, U_1), (T_2, U_2), &c. be the values of T and $U-U_0$ when $(L_1, M_1, \&c.)$, $(L_2, M_2, \&c.)$ are substituted for $(\theta', \phi', \&c.)$ or $(\theta, \phi, \&c.)$, prove that $T_1p_1^2+U_1=0$, $T_2p_2^2+U_2=0$, &c.

This follows from equations (8) by Euler's theorem on homogeneous functions.

446. In order to determine the values of p^2, it will often be necessary to expand the determinant. This may be done by the use of Taylor's theorem. Let Δ be the discriminant of T and let Π represent the operation

$$\Pi=a_{11}\frac{d}{dA_{11}}+a_{12}\frac{d}{dA_{12}}+a_{22}\frac{d}{dA_{22}}+\&c.,$$

then the determinant when expanded becomes

$$\Delta p^{2n}+\Pi(\Delta)p^{2n-2}+\Pi^2(\Delta)p^{2n-4}+\ldots=0.$$

If Δ' be the discriminant of U and Π' the operation Π when the great and small letters are interchanged, we may write the equation in the form

$$\Delta'+\Pi'(\Delta')p^2+\Pi'^2(\Delta')p^4+\ldots=0.$$

When there are only three independent co-ordinates, we may adopt the notation used in the chapter on Invariants in Dr Salmon's *Conics*.

Ex. 1. If a system be in a position of equilibrium, show that the equilibrium will be stable if $-\Pi(\Delta)$, $\Pi^2(\Delta)$, $-\Pi^3(\Delta)$, &c. be all positive.

Firstly, we may show that Δ is necessarily positive, and secondly that these are then the conditions that the roots of the equation (7) are all real.

Ex. 2. If S_k be the sum of the products of each kth minor of the discriminant Δ' into the conjugate minor of Δ, prove that S_k is the coefficient of p^{2k}.

Ex. 3. The same dynamical system can oscillate about the same position of equilibrium under two different sets of forces. If $\rho_1, \rho_2 \ldots$ and $\sigma_1, \sigma_2 \ldots$ be the periods of the oscillations when the two sets act separately, $R_1, R_2 \ldots$ the periods when they act together, prove that $\Sigma \frac{1}{\rho^2} + \Sigma \frac{1}{\sigma^2} = \Sigma \frac{1}{R^2}$.

This follows from the fact that $\Pi(\Delta)$ contains A_{11} &c. only in their first powers.

Ex. 4. Two different systems of bodies when acted on by the same set of forces oscillate in periods $\rho_1, \rho_2 \ldots$ and $\sigma_1, \sigma_2 \ldots$. If $R_1, R_2 \ldots$ be the periods when they are both set in oscillation by the same set of forces, prove that $\Sigma\rho^2 + \Sigma\sigma^2 = \Sigma R^2$.

Ex. 5. Prove that the equation giving the periods of the oscillations may be expressed as a determinant of $2n$ rows and columns by using Sir W. Hamilton's equations given in Art. 381.

447. If we refer the motion of the system to any other co-ordinates ξ, η, ζ, &c. which vanish in the position of equilibrium, it is clear that when θ, ϕ, ψ, &c. are expressed in terms of ξ, &c. and the squares of small quantities neglected, we shall have equations of the form

$$\left.\begin{aligned} \theta &= \lambda_1\xi + \lambda_2\eta + \&c. \\ \phi &= \mu_1\xi + \mu_2\eta + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \ldots\ldots (9).$$

Now θ, ϕ, &c. being expanded in a series of sines as in equations (5) it is clear that ξ, η, &c. will be expanded in a series of the *same* sines but with different coefficients. Hence the values of p^2 found from the determinantal equation will be the same whatever co-ordinates the system is referred to. The ratio of the coefficients of the several powers of p are therefore invariable.

If μ be the determinant of transformation, we know that Δ becomes $\mu^2\Delta$. Hence all the other coefficients will be altered in the same ratio. The quantities Δ, $\Pi(\Delta)$, $\Pi^2(\Delta)$, &c. are therefore called the *invariants* of the dynamical system.

448. *To show that the values of p^2 are all real**.

Since T is essentially a positive quantity for all values of θ', ϕ', &c. the coefficients of θ'^2, ϕ'^2, &c., viz. A_{11}, A_{22}, &c., must be all positive. Let us collect together the terms containing θ'^2, θ', and complete the square by adding and subtracting the proper quadratic function of ϕ', ψ', &c. We have

* This theorem seems to have been first discovered by Sir W. Thomson.

$$2T = \xi'^2 + B_{22}\phi'^2 + 2B_{23}\phi'\psi' + \dots,$$

where $$\xi = \sqrt{A_{11}}\,\theta + \frac{A_{12}}{\sqrt{A_{11}}}\phi + \frac{A_{13}}{\sqrt{A_{11}}}\psi + \dots;$$

and since A_{11} is positive, this transformation is real. In the same way B_{22} must be positive, and we may repeat the process. We thus have

$$2T = \xi'^2 + \eta'^2 + \&c.,$$

where $$\eta = \sqrt{B_{22}}\,\phi + \frac{B_{23}}{\sqrt{B_{22}}}\psi + \dots,$$

and it is clear that this process may be repeated continually.

We may take ξ, η, &c. as co-ordinates of the system because they are independent of each other and vanish in the position of equilibrium. We thus have

$$\left.\begin{aligned} 2T &= \xi'^2 + \eta'^2 + \dots \\ 2(U - U_0) &= f_{11}\xi^2 + 2f_{12}\xi\eta + \dots \end{aligned}\right\},$$

where f_{11}, f_{12}, &c. are all real constants. The determinantal equation now takes the form

$$\begin{vmatrix} p^2 + f_{11}, & f_{12}, & \&c. \\ f_{12}, & p^2 + f_{22}, & \&c. \\ \&c. & \&c. & \&c. \end{vmatrix} = 0.$$

When there are only three co-ordinates, this is the discriminating cubic used in Solid Geometry, and we know that its roots are all real. When there are more than three co-ordinates, it is proved in Dr Salmon's *Higher Algebra*, Lesson VI., that the roots are all real.

449. To explain what is meant by the principal co-ordinates of a dynamical system.

When we have two homogeneous quadratic functions of any number of variables, one of which is essentially positive for all values of the variables, it is known that by a real linear transformation of the variables we may clear both expressions of the terms containing the products of the variables, and also make the coefficients of the squares in the positive function each equal to unity. If the co-ordinates θ, ϕ, &c. be changed into ξ, η, &c. by the equations (9) of Art. 447, we observe that θ', ϕ', &c. will be changed into ξ', η', &c. by the same transformation. Also the vis viva is essentially positive. Hence we infer that by a proper choice of new co-ordinates, we may express the vis viva and force function in the form

$$\left.\begin{aligned} 2T &= \xi'^2 + \eta'^2 + \zeta'^2 + \dots \\ 2(U - U_0) &= b_{11}\xi^2 + b_{22}\eta^2 + \dots \end{aligned}\right\}.$$

These new co-ordinates ξ, η, &c. are called the *principal co-ordinates* of the dynamical system. A great variety of other names have been given to these co-ordinates; such as *Harmonic*, *simple* and *normal* co-ordinates.

450. When a dynamical system is referred to principal co-ordinates, Lagrange's equations take the form

$$\frac{d^2\xi}{dt^2} - b_{11}\xi = 0, \quad \frac{d^2\eta}{dt^2} - b_{22}\eta = 0, \text{ \&c.,}$$

so that the whole motion is given by

$$\xi = E\sin(p_1 t + \alpha_1), \quad \eta = F\sin(p_2 t + \alpha_2), \text{ \&c.,}$$

where E, F, &c., α_1, α_2, &c. are arbitrary constants to be determined by the initial conditions and $p_1^2 = -b_{11}$, $p^2 = -b_{22}$, &c.

When the initial conditions are such that all the principal co-ordinates are zero except one, the system is said to be performing a *principal* or *harmonic* oscillation.

451. The physical peculiarities of a principal oscillation are:

1. The motion recurs at a constant interval, i.e. after this interval the system occupies the same position as before, and is moving in exactly the same way.

2. The system passes through the position of equilibrium, twice in each complete oscillation. For taking ξ as the variable co-ordinate, we see that ξ vanishes twice while $p_1 t$ increases by 2π.

3. The velocity of every particle of the system becomes zero at the same instant, and this occurs twice in every complete oscillation. For $\frac{d\xi}{dt}$ vanishes twice while $p_1 t$ increases by 2π. These may be called the *extreme positions* of the oscillation.

4. The system being referred to any co-ordinates, θ, ϕ, ψ, &c., which are all variable, the ratios of these co-ordinates to each other are constant throughout the motion*. For referring to the equations (9) of Art. 447, we see that when η, ζ are all zero, and only ξ is variable,

$$\frac{\theta}{\lambda_1} = \frac{\phi}{\mu_1} = \dots = \xi.$$

* This property is mentioned by Lagrange, who on several occasions uses principal co-ordinates though not the name.

452. When some of the roots of the equation to find p^2 are equal, we know by the theory of linear differential equations that either terms of the form $(At+B)\sin pt$ enter into the values of θ, ϕ, &c., or else there must be an indeterminateness in the coefficients L, M, &c. given by equations (8). Referring the system to principal co-ordinates we see that the first alternative is in general excluded. If two values of p^2 were equal, say $b_{11} = b_{22}$, the trigonometrical expressions for ξ and η have equal periods, but terms which contain t as a factor do not make their appearance. The physical peculiarity of this case is that the system has more than one set of principal, or harmonic oscillations. For it is clear that, without introducing any terms containing the products of the co-ordinates into the expressions for T or U, we may change ξ, η into any other co-ordinates ξ_1, η_1, which make $\xi^2+\eta^2=\xi_1^2+\eta_1^2$, the other co-ordinates ζ, &c. remaining unchanged. For example we may put $\xi = \xi_1 \cos\alpha - \eta_1 \sin\alpha$ and $\eta = \xi_1 \sin\alpha + \eta_1 \cos\alpha$, where α has any value we please. These new quantities ξ_1, η_1, ζ, &c. will evidently be principal co-ordinates, according to the definition of Art. 449.

One important exception must however be noticed, viz. when one or more of the values of p are zero. If, for example, $b_{11}=0$ we have $\xi = At + B$, where A and B are two undetermined constants. The physical peculiarity of this case is that the position of equilibrium from which the system is disturbed is not solitary. To show this, we remark that the equations giving the position of equilibrium are $\dfrac{dU}{d\xi}=0$, $\dfrac{dU}{d\eta}=0$, &c., where U has the value given at the end of Art. 449. These in general require that ξ, η, &c. should all vanish, but if $b_{11}=0$ they are satisfied whatever ξ may be, provided η, ζ, &c. are zero. These values of ξ must however be very small, because the cubes of ξ, η, &c. have been rejected. It follows therefore that there are other positions of equilibrium in the immediate neighbourhood of the given position. Unless the initial conditions of disturbance are such as to make the terms of the form $At+B$ zero, it may be necessary to examine the terms of the higher order to obtain an approximation to the motion.

453. The motion being referred to any co-ordinates θ, ϕ... it may be required to find the principal oscillation. This may be done by finding λ, μ, &c. in equations (9) Art. 447, by the analytical process of clearing the two quadratic expressions of the terms containing the products, in the manner explained in Art. 449. We may also proceed thus, Let the system be performing the principal oscillation whose period is $\dfrac{2\pi}{p_1}$. Then in the equations (5), L_2, M_2, &c., L_3, M_3, &c. are all zero, hence θ, ϕ, ψ, &c. are in

the ratio L_1, M_1, &c. But these ratios are given by (6) or (8), in the form

$$\frac{d}{d\theta}(p_1^2 T + U) = 0, \quad \frac{d}{d\phi}(p_1^2 T + U) = 0, \text{ \&c.} \ldots\ldots\ldots\ldots (8),$$

where the accents in T have been omitted. These equations give the relations between θ, ϕ, &c., when the system is performing a principal oscillation.

454. When the dynamical system has but three co-ordinates, we may obtain a geometrical interpretation of this process. If we regard θ, ϕ, ψ as the Cartesian co-ordinates of a point P, it is clear that the position of P at any instant will give the position of the system. Omitting the accents in T and the constant term in U, the equations $T = \alpha$, $U = -\beta$, where α and β are any constants, represent two quadric surfaces which have their centres at the origin. These have a common set of conjugate diameters which may be found by the following process. Let θ, ϕ, ψ be the co-ordinates of any point on one of the three conjugates. Then, since the diametral planes of this point in the two quadrics are parallel, we have

$$\mu\frac{dT}{d\theta} = \frac{dU}{d\theta}, \quad \mu\frac{dT}{d\phi} = \frac{dU}{d\phi}, \quad \mu\frac{dT}{d\psi} = \frac{dU}{d\psi}.$$

Comparing these with equations (8), we see that when the system is performing a principal oscillation, the representative point P oscillates on one of the common conjugate diameters of the two quadrics.

By Euler's theorem on homogeneous functions we clearly have $\mu T = U$. Applying the same reasoning to equations (8) we see that $\mu = -p^2$. Let the diameter considered cut the quadrics T and U in the points D and D' and let O be the origin. Putting the point P at D we have $T = \alpha$ and since U is homogeneous $U = -\left(\frac{OD}{OD'}\right)^2 \beta$. Hence $p^2 = -\mu = \left(\frac{OD}{OD'}\right)^2 \frac{\beta}{\alpha}$. The period of the oscillation corresponding to the diameter ODD' is therefore $2\pi \cdot \frac{OD'}{OD}\sqrt{\frac{\alpha}{\beta}}$.

Since θ, ϕ, ψ contain only a single trigonometrical term (Art. 450) when the system is performing a principal oscillation, we see that the representative point P moves with an acceleration tending to the origin and varying as the distance therefrom.

455. As an example of this geometrical analogy let us consider the following problem. *A rigid body, free to move about a fixed point O, is under the action of any forces and makes small oscillations about a position of equilibrium; find the principal oscillations.*

Let OA, OB, OC be the positions of the principal axes in the position of equilibrium, OA', OB', OC' their positions at the time t. The position of the body may be defined by the angles between (1) the planes AOC, AOC', (2) the planes BOC, BOC', (3) the planes COA, COA'. Let these be called θ, ϕ, ψ respectively. Then θ, ϕ, ψ are angular displacements of the body about OA, OB, OC. Taking these as the axes of co-ordinates in the geometrical analogy; a small displacement of P from the origin to a point θ, ϕ, ψ represents a rotation of the body about the

straight line described by P and whose magnitude is measured by the distance traversed by P.

If A, B, C be the principal moments of inertia at O, the vis viva of the body is clearly

$$2T = A\theta'^2 + B\phi'^2 + C\psi'^2.$$

Omitting the accents, the quadric $T = \alpha$ is evidently the momental ellipsoid at the fixed point.

Let the work of the forces as the co-ordinates change from zero to θ, ϕ, ψ be, as in Art. 444,

$$U = \frac{1}{2}a_{11}\theta^2 + a_{12}\theta\phi + \&c.$$

Then, following the analogy, as P moves along a radius vector OD' of the quadric $U = -\beta$, the work is $-\left(\frac{OP}{OD'}\right)^2\beta$. Hence this quadric possesses the property that the work done by the forces when the body is twisted through a given angle round any radius vector varies inversely as the square of that radius vector. If the equilibrium is stable, the work due to a rotation about every diameter must be negative, the quadric must therefore be an ellipsoid.

It now follows from the general theorem that the body will perform a principal oscillation if it is set in rotation about any one of the three conjugate diameters of the momental ellipsoid and the ellipsoid U, and will therefore continue to oscillate as if that diameter were fixed in space.

The quadric U has been called the *ellipsoid of the potential.* This name was given to it by Prof. Ball, who arrived at the theorem just proved by a different course of reasoning. See his *Theory of Screws*, Art. 126. The following application is also due to him.

456. When the only force acting on the body is gravity, the ellipsoid of the potential is a surface of revolution about a vertical axis. For the inverse square of any radius vector measures the work done in turning the body through a given small angle about that radius vector. But the work is also proportional to the vertical distance through which the centre of gravity has been elevated from its position in equilibrium vertically under the point of support. Hence all radii vectores which make the same angle with the vertical are equal. Further the vertical radius vector is infinite, for the work done in rotating the body about a vertical axis is zero. The ellipsoid of the potential is therefore a right circular cylinder with its axis vertical.

The common conjugate diameters of these two quadrics are obviously the vertical and the two common conjugate diameters of the two ellipses in which the diametral plane of the vertical with regard to the momental ellipsoid intersects the momental ellipsoid and the cylinder.

The principal oscillation about the vertical conjugate is performed in an infinite time and would therefore cause the body to depart far from the position of equilibrium. But this is contrary to supposition. The initial axis of rotation must therefore be in the plane of the other two conjugates, i.e. must be in the diametral plane of the vertical with regard to the momental ellipsoid, and it will remain in this plane throughout the whole of the subsequent motion.

Since these conjugate diameters project into the conjugate diameters of the

horizontal section of the cylinder, it is clear that two vertical planes each containing one of the principal or harmonic axes are at right angles to each other.

457. Ex. Show that the mean kinetic energy of a dynamical system oscillating about a position of equilibrium is equal to the mean potential energy, the mean being taken for any long period, and the position of equilibrium being the position of reference.

Refer the motion to principal co-ordinates and let

$$2T=\xi'^2+\eta'^2+\&c.,\quad 2(U-U_0)=-p_1^2\xi^2-p_2^2\eta^2-\&c.$$

Then we find $\xi=E\sin(p_1t+\alpha_1)$, $\eta=F\sin(p_2t+\alpha)$. Substituting these in T and U_0-U we have the instantaneous kinetic and potential energies. The means of these are obviously the same, and equal to $\frac{1}{4}(E^2p_1^2+F^2p_2^2+\&c.)$.

If the system remain in the position of equilibrium the Hamiltonian characteristic function $S=U_0t$. If the system be disturbed and after any time t again pass through the position of equilibrium, the value of S for these two neighbouring modes of passing from one position to another in the same time must be equal. Hence $\int_0^t (T+U)\,dt=U_0t$, i.e. the mean values of the kinetic and potential energies are equal.

458. Ex. Find the energy of a dynamical system oscillating about a position of equilibrium referred to any co-ordinates.

By referring the system to its principal co-ordinates, we can easily show that the energy is the sum of the energies of its principal oscillations. Let the system be referred to any co-ordinates θ, ϕ, &c. and let it perform the principal oscillation whose type is, by equation (5),

$$\frac{\theta}{L_1}=\frac{\phi}{M_1}=\&c.=\sin(p_1t+\alpha_1).$$

Substituting in the expression for T, we have $T=T_1p_1^2\cos^2(p_1t+\alpha_1)$. Repeating this for all the principal oscillations, we have

$$\text{kinetic energy}=T_1p_1^2\cos^2(p_1t+\alpha_1)+T_2p_2^2\cos^2(p_2t+\alpha_2)+\&c.$$

where T_1, T_2, &c. are the values of T when L_1, M_1, &c., L_2, M_2, &c. are substituted for θ', ϕ', &c. Similarly we find when the position of equilibrium is taken as the position of reference

$$\text{potential energy}=-U_1\sin^2(p_1t+\alpha_1)-U_2\sin^2(p_2t+\alpha_2)+\&c.$$

Adding these two, we have by Art. 445, Ex. 3,

$$\text{whole energy}=T_1p_1^2+T_2p_2^2+\dots$$

459. Ex. 1. A new constraint is introduced into a dynamical system, so that the general co-ordinates θ, ϕ, &c. are constrained to vary in the ratio l, m, &c. If we put $\theta=l\sin(p't+\alpha)$, $\phi=m\sin(p't+\alpha)$, &c., and if T', U' be the values of T and $U-U_0$ when l, m, &c. are substituted for θ', ϕ', &c., or θ, ϕ, &c., prove that

$$T'p'^2+U'=0.$$

A theorem similar to this is given by Lord Rayleigh in the *Proceedings of the Mathematical Society*, No. 63, 1873.

Ex. 2. Show how to find the possible displacements of a system which have a given time of oscillation.

Ex. 3. Show that all possible times of oscillation of a system caused by introducing any new constraints lie between the greatest and least of the times of its principal oscillations.

460. When a system starts from rest under the influence of any forces we may use Lagrange's equations to find the initial motion. Let the system be referred to any co-ordinates θ, ϕ, &c. which however do not necessarily vanish in the position of rest. As in Art. 444, let

$$2T = A_{11}\theta'^2 + 2A_{12}\theta'\phi' + A_{22}\phi'^2 + \ldots$$

where A_{11}, &c. are functions of θ, ϕ, &c. Since the system starts from rest, θ', ϕ', &c. will all be very small quantities in the beginning of the motion. If we reject all powers of θ', ϕ', &c. except the lowest which occur, we may regard A_{11}, &c. as constants whose values are found by substituting for θ, ϕ, &c. their initial values. Further, since the initial position of the system is not a position of equilibrium, the first differential coefficients of U with regard to θ, ϕ, &c. will not be zero. Let the initial values of these differential coefficients be respectively a_1, a_2, &c. The equations of motion are now

$$\left.\begin{aligned} A_{11}\theta'' + A_{12}\phi'' + \ldots &= a_1 \\ A_{12}\theta'' + A_{22}\phi'' + \ldots &= a_2 \\ \text{\&c.} &= \text{\&c.} \end{aligned}\right\}.$$

From these equations we may determine the initial values of θ'', ϕ'', &c. If x, y, z be the co-ordinates of any particle m of the system referred to any rectangular axes fixed in space, we have, by the geometry of the system, these co-ordinates expressed as known functions of θ, ϕ, ψ, &c., Art. 367. Thus if $x = f(\theta, \phi, \text{\&c.})$, we have initially

$$x'' = \frac{d^2 f}{d\theta^2}\theta'' + \frac{d^2 f}{d\phi^2}\phi'' + \ldots$$

with similar expressions for y and z. The quantities x'', y'', z'' are evidently proportional to the direction cosines of the initial direction of motion of m. In this way the initial direction of motion of every part of the system may be found.

Ex. A system has three co-ordinates θ, ϕ, ψ and starts from rest in a position in which these co-ordinates are all zero. Show that the representative point P (Art. 454) begins to move along the diametral line of the plane $a_1\theta + a_2\phi + a_3\psi = 0$ with regard to the ellipsoid $\frac{1}{2}A_{11}\theta^2 + A_{12}\theta\phi + \text{\&c.} = a$.

461. When the geometrical equations contain differential coefficients with regard to the time, or when we do not wish to express T and U in terms of independent co-ordinates, the Lagrangian equations must be modified in the manner explained in Art. 388. The equations (3) of Art. 444 must be replaced by the equations (4) of Art. 388. Since we reject all powers of the small quantities θ, ϕ, &c. except the lowest which occur, we may still use the expression for T given in (1) Art. 444, and treat the coefficients as constants. But, in making the position of the system depend on the quantities θ, ϕ, &c. (Art. 367), we may not have used all the available geometrical conditions, and therefore the first powers of θ, ϕ, &c. in the expansion of U may not be absent. Let

$$U = U_0 + a_1\theta + a_2\phi + \&c. + \tfrac{1}{2}a_{11}\theta^2 + a_{12}\theta\phi + \&c.$$

Also let the geometrical equations which are to be introduced by the method of indeterminate multipliers be

$$\left.\begin{aligned} E\theta' + F\phi' + \ldots &= 0 \\ H\theta' + K\phi' + \ldots &= 0 \\ \&c. &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where E, H, &c. are in general functions of θ, ϕ, &c., each of which may be expanded in the form

$$E = E_0 + E_1\theta + E_2\phi + \ldots$$

The equations of motion of Art. 388 will be

$$\left.\begin{aligned} A_{11}\theta'' + \&c. &= a_1 + a_{11}\theta + \&c. + \lambda E + \mu H + \&c. \\ A_{12}\theta'' + \&c. &= a_2 + a_{12}\theta + \&c. + \lambda F + \mu K + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \ldots\ldots\ldots (11).$$

Since the system has been disturbed from a position of equilibrium, these equations are satisfied by $\theta = 0$, $\phi = 0$, &c. We thus obtain the equilibrium values of λ, μ, &c. Let these be λ_0, μ_0, &c., then

$$\left.\begin{aligned} 0 &= a_1 + \lambda_0 E_0 + \mu_0 H_0 + \&c. \\ 0 &= \&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (12).$$

Let $\lambda = \lambda_0 + \lambda_1$, $\mu = \mu_0 + \mu_1$, &c. so that λ_1, μ_1, &c. are small quantities of the same order as θ, ϕ, &c. The equations of oscillation then become

$$\left.\begin{aligned} A_{11}\theta'' + \&c. &= a_{11}\theta + \&c. + \lambda_0(E_1\theta + E_2\phi + \&c.) + \lambda_1 E_0 + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \ldots (13).$$

Joining these to equations (10) we have a sufficient number of linear equations to find θ, ϕ, &c., λ_1, μ_1, &c. in terms of t. The solutions of these equations may evidently be conducted as in Art. 445.

The equations will be greatly simplified if the equilibrium values of λ, μ, &c. are all zero. This will generally be the case if θ, ϕ, &c. can be so chosen that the first powers in the expansion of U are absent. In this case E_1, E_2, &c. disappear from the equations, so that it is unnecessary to calculate the geometrical equations (10) beyond terms of the first order. The coefficients will then be constant, and the equations can be integrated. As explained in Art. 388, we may now reduce the number of variables θ, ϕ, &c. to the proper number of independent co-ordinates. We may therefore proceed as in Art. 444, without introducing λ, μ, &c. into the equations.

If, however, we prefer to retain the quantities λ_1, μ_1, &c., we see by equations (10) and (13) that we may obtain the periods exactly as in Art. 445, by equating the discriminant of $p^2T + U'$ to zero, where

$$U' = U + \lambda_1 (E_0\theta + F_0\phi + \ldots) + \mu_1 (H_0\theta + K_0\phi + \ldots) + \&c.$$

The determinant thus obtained has as many rows as there are quantities θ, ϕ, &c., λ_1, μ_1, &c.

The Energy test of Stability.

462. The principle of the Conservation of Energy may be conveniently used in some cases to determine whether a system of bodies *at rest* is in stable or unstable equilibrium.

Let the system be in equilibrium in any position and let V_0 be the potential energy of the forces in this position. Let the system be displaced into any initial position very near the position of equilibrium and be started with any very small initial kinetic energy T_1, and let V_1 be the potential energy of the forces in this position. At any subsequent time let T and V be the kinetic and potential energies. Then by the principle of energy

$$T + V = T_1 + V_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let V be an absolute minimum in the position of equilibrium, so that V is greater than V_0 for all neighbouring positions. The initial disturbed position being included amongst these, it follows that $V_1 - V_0$ is a small positive quantity. Now the kinetic energy T is necessarily a positive quantity, and since V is $> V_0$, the equation (1) shows that T is $< T_1 + V_1 - V_0$. Thus throughout the subsequent motion the vis viva is restricted between zero and a small positive quantity, and therefore the motion of the system can never be great.

Also, since T is necessarily positive, the system can never deviate so far from the position of equilibrium that V should become greater than $T_1 + V_1$. These two results may be stated thus.

If a system be in equilibrium in a position in which the potential energy of the forces is a minimum or the work a maximum for all displacements, then the system if slightly displaced will never acquire any large amount of vis viva, and will never deviate far from the position of equilibrium. The equilibrium is then said to be stable.

463. If the potential energy be an absolute maximum in the position of equilibrium, V is less than V_0 for all neighbouring positions. By the same reasoning we see that T is always greater than $T_1 + V_1 - V_0$, and the system cannot approach so near the position of equilibrium that V should become greater than $T_1 + V_1$. So far therefore as the equation of vis viva is concerned there is nothing to prevent the system from departing widely from the position of equilibrium. To determine this point we must examine the other equations of motion*.

If any principal oscillation could exist, let the system be placed at rest in an extreme position of that oscillation, then the system will describe that principal oscillation and will therefore pass through the position of equilibrium. But if T_1 be zero, V can never exceed V_1, and can therefore never become equal to V_0. Hence the system cannot pass through the position of equilibrium.

It is unnecessary to pursue this line of reasoning further, for the argument will be made clearer in the next proposition.

464. We may also deduce the test of stability from the equations which determine the small oscillations of a system about a position of equilibrium. Let the system be referred to its principal co-ordinates, and let these be θ, ϕ, &c. Then we have

$$2T = \theta'^2 + \phi'^2 + \ldots\ldots\ldots$$

$$2(U - U_0) = b_1\theta^2 + b_2\phi^2 + \ldots\ldots\ldots$$

where b_1, b_2, &c. are all constants, and U_0 is the value of U in the position of equilibrium. Taking as a type any one of Lagrange's equations

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta},$$

we have

$$\theta'' - b_1\theta = 0,$$

* This demonstration is twice given by Lagrange in his *Mecanique Analytique*. In the form in which it appears in the first part of that work, V is expanded in powers of the co-ordinates, which are supposed very small; but in Section VI. of the second part, this expansion is no longer used, and the proof appears almost exactly as it is given in this treatise up to the asterisk. The demonstration in the next proposition is simplified from that of Lagrange by the use of principal co-ordinates.

with similar equations for ϕ, ψ, &c. If b_1 is positive, this equation will give θ in terms of real exponentials, and the equilibrium will be unstable for all disturbances which affect θ, except such as make the coefficient of the term containing the positive exponent zero. If b_1 is negative, θ will be expressed by a trigonometrical term, and the equilibrium will be stable for all disturbances which affect θ only. In this demonstration the values of b_1, b_2, &c. are supposed not to be zero.

If in the position of equilibrium U is a maximum for all possible displacements of the system, we must have b_1, b_2, &c. all negative. Whatever disturbance is given to the system, it will oscillate about the position of equilibrium, and that position is then stable. If U is a maximum for some displacements and a minimum for others, some of the coefficients b_1, b_2, &c. will be negative and some positive. In this case if the system be disturbed in some directions, it will oscillate about the position of equilibrium; if disturbed in other directions, it may deviate more and more from the position of equilibrium. The equilibrium is therefore stable for all disturbances in certain directions, and unstable for disturbances in other directions. If U is a minimum in the position of equilibrium for all displacements, the coefficients b_1, b_2, &c. are all positive, the equilibrium will then be unstable for displacements in all directions. Briefly, we may sum up the results thus,

The system will oscillate about the position of equilibrium for all disturbances if the potential energy is a minimum for all displacements. It will oscillate for some disturbances and not for others if the potential energy is neither a maximum nor a minimum. It will not oscillate for any disturbance if the potential energy is a maximum for all displacements.

It appears from this theorem that the stability or instability of a position of equilibrium does not depend on the inertia of the system but only on the force function. The rule is, give the system a sufficient number of small arbitrary displacements, so that all possible displacements may be compounded of these. By examining the work done by the forces in these displacements we can determine whether the potential energy is a maximum or minimum or neither.

Ex. 1. A perfectly free particle is in equilibrium under the attraction of any number of fixed bodies. Show that if the law of attraction be the inverse square, the equilibrium is unstable. [*Earnshaw's Theorem.*]

Let O be the position of equilibrium, Ox, Oy, Oz any three rectangular axes, then if V be the potential of the bodies, $b_1 = \dfrac{d^2V}{dx^2}$, $b_2 = \dfrac{d^2V}{dy^2}$, $b_3 = \dfrac{d^2V}{dz^2}$. But since the sum of these is zero, b_1, b_2, b_3 cannot all have the same sign.

Ex. 2. Hence show that if any number of particles, mutually repelling each

other, be contained in a vessel, and be in equilibrium, the equilibrium will be unstable unless they all lie on the containing surface. [Sir W. Thomson, *Camb. Math. Journal*, 1845.]

465. We may in certain cases apply the energy criterion to determine when a *given motion* is stable. Let a dynamical system be in motion in any manner under a conservative system of forces, and let E be its energy. Then E is a known function of the co-ordinates θ, ϕ, &c. and their first differential coefficients θ', ϕ', &c.; this is constant and equal to h for the given motion. Suppose that either some or all of the other first integrals of the equations of motion are also known, let these be

$$F_1(\theta, \theta', \&c.) = C_1, \quad F_2(\theta, \theta', \&c.) = C_2, \quad \&c. = \&c.$$

For the purposes of this proposition, let us regard θ and θ', ϕ and ϕ', &c. as independent variables, except so far as they are connected by the equations just written down. Then if E be an absolute maximum, or an absolute minimum, for all variations of θ, θ', &c. (those corresponding to the given motion making E constant), the motion is stable for all disturbances which do not alter the constants C_1, C_2, &c.

This result follows from the same reasoning as in Art. 462, which we may briefly recapitulate thus. Let as many of the letters as is possible be found from the first integrals in terms of the rest, and substituted in the expression for E. Let ψ, ψ', &c. be these remaining letters, then we have

$$E = f(\psi, \psi', \&c., C_1, C_2, \&c.) = h.$$

Let the system be started in some manner slightly different from that given, then the constant h is altered into $h + \delta h$. First let E be a minimum along the given motion, then any change whatever of the letters ψ, ψ', &c. increases E, and it follows that the disturbed motion cannot deviate so far from the given motion that the change in E becomes greater than δh. Similarly, if E be an absolute maximum, the same result will follow.

The same argument will apply to any first integral of the equations of motion, besides the energy integral. If any one of the functions F_1, F_2, &c., which contains all the letters, be an absolute maximum or minimum, then the motion is stable for all displacements which do not alter the constants of the other integrals used.

When the system is disturbed from a position of equilibrium which is defined, as in Art. 444, by the vanishing of the co-ordinates θ, ϕ, &c., we have

$$E = \frac{1}{2} A_{11} \theta'^2 + A_{12} \theta' \phi' + \&c. - U,$$

where A_{11}, A_{12}, &c. are all constants, and U is independent of θ', ϕ', &c. Here the terms which constitute the kinetic energy, being necessarily positive and vanishing with θ', ϕ' &c., are evidently a minimum for all variations of θ', ϕ', &c. We see, without the use of any other integrals, that if $-U$ be a minimum for all variations of θ, ϕ, &c., E will be an absolute minimum, and that therefore the equilibrium is stable.

466. It often happens that the expression for the energy is not a function of some of the co-ordinates, though it is a function of the differential coefficients of all the co-ordinates with regard to the time. When this is the case, the system admits of what we shall call a steady motion. Let x, y, &c. be the co-ordinates which are absent from the expression for the energy E, and let ξ, η, &c. be the

remaining co-ordinates, then E is a function of ξ, η, &c., ξ', η', &c., x', y', &c. If we form the equations of motion by Lagrange's rule (Art. 369), these equations will contain $\xi, \eta, \xi', \eta', \xi'', \eta'', x', y', x'', y''$, &c., &c. Since these equations do not contain t explicitly, they may be satisfied by putting $x'=a$, $y'=b$, &c., $\xi=\alpha$, $\eta=\beta$, &c., where a, b, &c., α, β, &c. are constants to be determined by substituting in the equations. If θ stand for any one of the co-ordinates, it is evident that $\frac{dT}{d\theta}$ and $\frac{dT}{d\theta'}$, will both be constants after the substitution is made. The constants must therefore satisfy the typical equation $\frac{d(T+U)}{d\theta}=0$ (Art. 369). Since x, y, &c. are absent from the expressions for T and U, this is an identity if we write any of these co-ordinates for θ. Hence we have as many equations, viz.

$$\frac{d(T+U)}{d\xi}=0, \quad \frac{d(T+U)}{d\eta}=0 \quad\ldots\ldots\ldots\ldots (1),$$

as there are co-ordinates ξ, η, &c. present in the expressions for T and U. The quantities a, b, &c. are therefore undetermined except by the initial conditions, while α, β, &c. may be found in terms of a, b, &c. by these equations. These equations may be conveniently remembered by the following rule. *In the Lagrangian function, which is the difference between the kinetic and potential energies, write for the differential coefficients, their assumed constant values in the steady motion, viz.* $x'=a$, *&c.*, $\xi'=0$, *&c. Differentiating the result partially with regard to each of the remaining co-ordinates, we obtain the equations of steady motion.*

467. To determine if this motion is stable, we must by Art. 465 use the integrals $\frac{dT}{dx'}=u$, $\frac{dT}{dy'}=v$, &c., where u, v, &c. are constants. Let

$$T=\frac{1}{2}(xx)\,x'^2+(x\xi)\,x'\xi'+\&c. \quad\ldots\ldots\ldots\ldots (2),$$

where the coefficients of the accented letters, viz. the quantities in brackets, are all known functions of ξ, η, &c., but not of x, y, &c. The integrals may then be written in the form

$$\left.\begin{array}{c}(xx)\,x'+(xy)\,y'+\ldots=u-(x\xi)\,\xi'-(x\eta)\,\eta'-\&c.\\ (xy)\,x'+(yy)\,y'+\ldots=v-(y\xi)\,\xi'-(y\eta)\,\eta'-\&c.\\ \&c.=\&c.\end{array}\right\}\quad\ldots\ldots\ldots\ldots (3).$$

For the sake of brevity, let us call the right hand sides of these equations $u-X$, $v-Y$, &c. Since T is a quadratic function of the accented letters, we may write it in the form

$$T=\frac{1}{2}(\xi\xi)\,\xi'^2+(\xi\eta)\,\xi'\eta'+\&c.+\frac{1}{2}x'(u+X)+\frac{1}{2}y'(v+Y)+\&c.$$

If we substitute in the terms after the first &c. the values of x', y' given by (3) we obtain the determinant

$$-\frac{1}{2\Delta}\begin{vmatrix}0, & u+X, & v+Y, & \&c.\\ u-X, & (xx) & (xy), & \&c.\\ v-Y, & (xy), & (yy), & \&c.\\ \&c. & & & \end{vmatrix}$$

where Δ is the discriminant of T, when ξ', η', &c. have been put zero. If we change the signs of X, Y, &c., this determinant is unaltered, hence when expanded such terms as uX, vX, &c. cannot occur. If therefore, we put

$$F = -\frac{1}{2\Delta}\begin{vmatrix} 0 & u & v & \dots \\ u & (xx) & (xy) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad \text{.................................(4),}$$

and expand the first determinant, we have

$$T = F + \frac{1}{2}B_{11}\xi'^2 + B_{12}\xi'\eta' + \quad \text{.......................................(5),}$$

where the terms after F express some homogeneous quadratic function of ξ', η', &c.

When ξ', η', &c. are put zero, the process of finding F is exactly that described in Art. 378, as the Hamiltonian method of forming the reciprocal function. Following the same proof* as in that Article, we may show that if ξ be any letter contained in T, we have $\frac{dT}{d\xi} = -\frac{dF}{d\xi}$. Hence the *equations of steady motion* (1) *may also be written in the form*

$$\left.\begin{array}{ll} \dfrac{d(F-U)}{d\xi} = 0, & \dfrac{d(F-U)}{d\eta} = 0 \\ x' = \dfrac{d(F-U)}{du}, & y' = \dfrac{d(F-U)}{dv} \end{array}\right\} \quad \text{.............................(6),}$$

where F − U *is the energy expressed as a function of* u, v, &c. *instead of* x′, y′, &c., *the other accented letters, viz.* ξ', η', &c. *being put equal to zero either before or after differentiation.*

Further T is essentially positive for all values of x', y', &c. and therefore for such as make u, v, &c. all zero. Hence the quadratic expression $B_{11}\xi'^2 + \&c.$ is a minimum when ξ', η', &c. are zero. *If then the function* F − U *is a minimum for all variations of* ξ, η, &c., *the steady motion given by* (6) *is stable for all disturbances which do not alter the momenta* u, v, &c.

468. *If the energy be a function of one only of the co-ordinates, though it is a function of the differential coefficients of all of them, we may show conversely that the steady motion will not be stable unless* F − U *is a minimum.*

Let ξ be this single co-ordinate, then following the same notation as before, we have by Vis Viva

$$\frac{1}{2}B_{11}\xi'^2 + F - U = h.$$

Differentiating with regard to t, and treating B_{11} as constant because we shall neglect the square of ξ', we obtain

$$B_{11}\xi'' + \frac{d}{d\xi}(F - U) = 0.$$

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* Taking the notation of Art. 378, the proof is as follows. The total differential of T_2 when all the letters vary is

$$dT_2 = -\frac{dT_1}{d\theta}d\theta - \frac{dT_1}{d\xi}d\xi + \left(-\frac{dT_1}{d\theta'} + u\right)d\theta' + \theta' du + \&c.;$$

as before, the quantity in brackets vanishes, and hence when T_2 is expressed as a function of θ, ϕ, &c., u, v, &c. and ξ, we have $\frac{dT_2}{d\xi} = -\frac{dT_1}{d\xi}$.

To find the oscillation, let $\xi = a + p$, then by (6) we have

$$B_{11}\frac{d^2p}{dt^2} + \left[\frac{d^2(F-U)}{d\xi^2}\right]p = 0,$$

where a is to be written for ξ after differentiation in the quantity in square brackets. The motion is clearly stable or unstable according as the coefficient of p is positive or negative, *i.e.* according as $F - U$ is a minimum or maximum.

469. Ex. 1. Let us consider the simple case of a particle describing a circular orbit about a centre of attraction whose acceleration at a distance r is μr^n. If θ be the angle the radius vector r makes with the axis of x, we have here a steady motion in which $r' = 0$ and θ' is constant. Also

$$E = \frac{1}{2}(r'^2 + r^2\theta'^2) + \frac{\mu r^{n+1}}{n+1}.$$

We notice that θ is absent from this expression, hence by the rule we eliminate θ' also by the integral $r^2\theta' = h$, where h is the constant called u in Art. 467. We have then

$$E = \frac{1}{2}r'^2 + \frac{1}{2}\frac{h^2}{r^2} + \frac{\mu r^{n+1}}{n+1}.$$

Putting the remaining accented letters equal to zero according to the rule, we have in steady motion

$$\frac{dE}{dr} = -\frac{h^2}{r^3} + \mu r^n = 0,$$

and since

$$\frac{d^2E}{dr^2} = \frac{3h^2}{r^4} + \mu n r^{n-1} = \mu(n+3)r^{n-1},$$

this steady motion is stable or unstable according as $n+3$ is positive or negative for all disturbances which do not alter the angular momentum of the particle.

Ex. 2. Taking the example considered in Art. 374, show that a state of steady motion is given by θ constant and that it is stable if $C^2n^2 + 4MghA\cos\theta$ is positive. Hence if $\theta < \frac{\pi}{2}$ the motion is stable for all values of n.

Ex. 3. A solid of revolution moves in steady motion on a smooth horizontal plane, so that the inclination θ of its axis to the vertical is constant. Prove that the angular velocity μ of the axis about the vertical is given by

$$\mu^2 - \frac{Cn}{A\cos\theta}\mu + \frac{Mg}{A\sin\theta\cos\theta}\frac{dz}{d\theta} = 0,$$

where z is the altitude of the centre of gravity above the horizontal plane, n the angular velocity of the body about the axis, C, A and A the principal moments of inertia at the centre of gravity and M the mass. Find the least value of n which makes μ real and determine if the steady motion is stable.

Oscillations about Steady Motion.

470. The oscillations of a system about a state of steady motion may be found by methods analogous to those used in the oscillations about a position of equilibrium. Let the general equations of motion of the bodies be formed by any of the methods already described. If any reactions enter into these equations it

will be generally found advantageous to eliminate them as explained in Art. 428. Let the co-ordinates used in these equations to fix the positions of the bodies be called θ, ϕ, &c. Suppose the motion, about which the oscillation is required, to be determined by $\theta = f(t)$, $\phi = F(t)$, &c. Then exactly as in Art. 428, we substitute $\theta = f(t) + x$, $\phi = F(t) + y$, &c., in the equations of motion. The squares of x, y, &c. being neglected, we have certain linear equations to find x, y, &c. These equations can, however, seldom be solved unless we can make t disappear explicitly from them. When this can be done the linear equations can be solved by the usual known methods, and the required oscillations are then found.

In what follows we shall first illustrate the method just described by forming the equations in a few interesting cases from the beginning. We shall then generalize the process and obtain a determinantal equation analogous to that given by Lagrange for oscillations about a position of equilibrium. This equation will be adapted to all cases which lead to differential equations with constant coefficients.

471. Ex. 1. *To find the motion of the balls in Watt's Governor of the steam engine.*

The mode in which this works to moderate the fluctuations of the engine is well known. A somewhat similar apparatus has been used to regulate the motion of clocks, and in other cases where uniformity of motion is required. If there be any increase in the driving power of the engine, or any diminution of the load, so that the engine begins to move too fast, the balls, by their increased centrifugal force, open outwards, and by means of a lever either cut off the driving power or increase the load by a quantity proportional to the angle opened out. If on the other hand the engine goes too slow, the balls fall inward, and more driving power is called into action. In the case of the steam engine the lever is attached to the throttle-valve, and thus regulates the supply of steam. It is clear that a complete adaptation of the driving power to the load cannot take place instantaneously, but the machine will make a series of small oscillations about a mean state of steady motion. The problem to be considered may therefore be stated thus:—

Two equal rods OA, OA', each of length l, are connected with a vertical spindle by means of a hinge at O which permits free motion in the vertical plane AOA'. At A and A' are attached two balls, each of mass m. To represent the inertia of the other parts of the engine we shall suppose a horizontal fly-wheel attached to the spindle, whose moment of inertia about the spindle is I. When the machine is in uniform motion, the rods are inclined at some angle α to the vertical, and turn round it with uniform angular velocity n. If, owing to any disturbance of the motion, the rods have opened out to an angle θ with the vertical, a force is called into play whose moment about the spindle is $-\beta(\theta-\alpha)$. It is required to find the oscillations about the state of steady motion.

Let ϕ be the angle the plane AOA' makes with some vertical plane fixed in space. The equation of angular momentum about the spindle is

$$\frac{d}{dt}\left\{(I + 2mk^2 \sin^2\theta)\frac{d\phi}{dt}\right\} = -\beta(\theta-\alpha) \qquad\qquad (1),$$

where mk^2 is the moment of inertia of a rod and ball about a perpendicular to the rod through O, the balls being regarded as indefinitely small heavy particles. The semi Vis Viva of the system is

$$T=\tfrac{1}{2}I\left(\frac{d\phi}{dt}\right)^2+mk^2\left\{\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right\},$$

and the moment of the impressed forces on either rod and ball about a horizontal through O perpendicular to the plane AOA' is $\frac{1}{2}\frac{dU}{d\theta}=-mgh\sin\theta$, where h is the distance of the centre of gravity of a rod and ball from O. Hence by Lagrange's equation $\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}=\frac{dU}{d\theta}$, we have

$$\frac{d^2\theta}{dt}-\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2=-\frac{g}{a}\sin\theta \text{.........................(2)},$$

where a has been written for $\frac{k^2}{h}$. This equation might also have been obtained by taking the acceleration of either ball, treated as a particle, in a direction perpendicular to the rod in the plane in which θ is measured.

To find the steady motion we put $\theta=\alpha$, $\frac{d\phi}{dt}=n$, the second equation then gives $n^2\cos\alpha=\frac{g}{a}$. To find the oscillations, we put $\theta=\alpha+x$, $\frac{d\phi}{dt}=n+y$. The two equations then become

$$\left.\begin{aligned}(I+2mk^2\sin^2\alpha)\frac{dy}{dt}+2mk^2n\sin2\alpha\frac{dx}{dt}&=-\beta x\\ \frac{d^2x}{dt^2}-n\sin2\alpha y&=\left(n^2\cos2\alpha-\frac{g}{a}\cos\alpha\right)x\end{aligned}\right\}.$$

To solve these equations, we must write them in the form

$$\left.\begin{aligned}\left(\sin2\alpha D+\frac{\beta}{2mk^2n}\right)nx+\left(\frac{I}{2mk^2}+\sin^2\alpha\right)Dy&=0\\ (D^2+n^2\sin^2\alpha)x-n\sin2\alpha y&=0\end{aligned}\right\},$$

where the symbol D stands for the operation $\frac{d}{dt}$. Eliminating y by cross multiplication we have

$$\left[\left(\frac{I}{2mk^2}+\sin^2\alpha\right)D^3+n^2\sin^2\alpha\left(1+3\cos^2\alpha+\frac{I}{2mk^2}\right)D+\frac{\beta}{2mk^2}n\sin2\alpha\right]x=0.$$

The real root of this cubic equation is necessarily negative because the last term is positive. The other two roots are imaginary because the term D^2 has disappeared between two terms of like signs. Also the sum of the three roots being zero, the real parts of the two imaginary roots must be positive. Let these roots therefore be $-2p$ and $p\pm q\sqrt{-1}$. Then

$$x=He^{-2pt}+Ke^{pt}\sin(qt+L),$$

where H, K, L are three undetermined constants depending on the nature of the initial disturbance. Thus it appears that the oscillation is unstable. The balls will alternately approach and recede from the vertical spindle with increasing violence.

472. A common defect of governors is that they act too quickly, and thus produce considerable oscillation of speed in the engine. If the engine is working too violently, the governor cuts off the steam, but owing to the inertia of the parts of the machinery, the engine does not immediately take up the proper speed. The consequence is that the balls continue to separate after they have reduced the supply of steam to the proper amount, and thus too much steam is cut off. Similar remarks apply when the balls are approaching each other, and a considerable oscillation is thereby produced. This fault may be very much modified by applying some resistance to the motion of the governor.

In the same way when the motion of clock-work is regulated by centrifugal balls, it is found as a matter of observation that there is a strong tendency to irregularity. If the balls once receive in the slightest degree an elliptic motion, the resistance $\beta(\theta-a)$ by which the motion of the balls is regulated may tend to render the ellipse more and more elliptical. To correct this some other resistance must be called into play. This resistance should be of such a character that it does not affect the circular motion and is only produced by the ellipticity of the movement.

One method of effecting this has been suggested by Sir G. Airy. The elliptic motion of the balls may be made to cause a slider on the vertical spindle to rise and fall. If this be connected with a horizontal circular plate in a vertical cylinder of slightly greater radius, and filled with water, the slider may be made to move the plate up and down by its oscillations. Thus the slider may be subjected to a very great resistance, tending to diminish its oscillations, while its place of rest, as depending on statical, or slowly altering forces, is totally unaffected. *Memoirs of the Astronomical Society of London*, Vol. xx., 1851.

The general effect of the water will be to produce a resistance varying as the velocity, and may therefore be represented by a term $-\gamma\frac{d\theta}{dt}$ on the right hand of equation (2). The solution being continued as before, the cubic will now take the form

$$\left[\left(\frac{I}{2mk^2}+\sin^2 a\right)(D^3+\gamma D^2)+n\sin^2 a\left(1+3\cos^2 a+\frac{I}{2mk^2}\right)D+\frac{\beta}{2mk^2}n\sin 2a\right]x=0.$$

If the roots of this cubic are real, they are all negative, and the value of x takes the form

$$x=Ae^{-\lambda t}+Be^{-\mu t}+Ce^{-\nu t},$$

where $-\lambda$, $-\mu$, $-\nu$ are the roots, and A, B, C are three undetermined constants. If one root only is real, that root is negative, and if the other two be $p\pm q\sqrt{-1}$ the value of x takes the form

$$x=He^{-rt}+Ke^{pt}\sin(qt+L),$$

where H, K, L as before are undetermined constants.

In order that the motion may be stable it is necessary that p should be negative. The analytical condition* of this is

* If the roots of the cubic $ax^3+bx^2+cx+d=0$ be $x=\alpha\pm\beta\sqrt{(-1)}$ and γ, we have $-\frac{b}{a}=2\alpha+\gamma$, $\frac{c}{a}=2\gamma\alpha+\alpha^2+\beta^2$, $-\frac{d}{a}=(\alpha^2+\beta^2)\gamma$, whence we easily deduce $\frac{bc-ad}{a^2}=-2\alpha\{(\alpha+\gamma)^2+\beta^2\}$; hence $bc-ad$ and a have always opposite signs. See Art. 436.

$$\gamma\left(1+3\cos^2\alpha+\frac{I}{2mk^2}\right)>\frac{\beta}{2mk^2}2\cot\alpha.$$

If γ be sufficiently great this condition may be satisfied. The uniformity of motion of the rods round the vertical will then be disturbed by an oscillation whose magnitude is continually decreasing and whose period is $\frac{2\pi}{q}$. By properly choosing the magnitude of I when constructing the instrument, the period may sometimes be so arranged as to produce the least possible ill effect. If the period be made very long the instrument will work smoothly. If it can be made very short there will be less deviation from circular motion.

In this investigation no notice has been taken of the frictions at the hinge and at the mechanical appliances of the Governor, which may not be inconsiderable. These in many cases tend to reduce the oscillation and keep it within bounds.

473. In the case of Watt's Governor if any permanent change be made in the relation between the driving power and the load, the state of uniform motion which the engine will finally assume is different from that which it had before the change. Thus, when the engine is driving a given number of looms, let the rods OA, OA' of the Governor be inclined to each other at an angle 2α and be revolving about the vertical with an angular velocity n. If some large number of the looms is suddenly disconnected from the engine, the balls will separate from each other, and the rods will become inclined at some other angle $2\alpha'$. In this case, if n' be the angular velocity about the vertical, $n'^2\cos\alpha'=n^2\cos\alpha$. The rate of the engine is therefore altered, it works quicker with a less load than with a greater. This is a great defect of Watt's Governor. For this reason it has been suggested that the term *Governor* is inappropriate, the instrument being in fact only a *moderator* of the fluctuations of the engine.

This defect may be considerably decreased by the use of Huyghens' parabolic pendulum. In this instrument the centres of gravity A, A' of the balls are made to move along the arc of a parabola whose axis is the axis of revolution. Let AN be an ordinate of the parabola, AG the normal, then NG is constant and equal to L, where $2L$ is the latus rectum. Regarding the balls as particles, and neglecting the inertia of the rods which connect them with the throttle valve, we see by the triangle of forces that the balls will rest in any positions on the parabola, if $n^2L=g$, where n is the angular velocity of the balls about the vertical through O. It is also clear that when the angular velocity is not that given by this formula, the balls (unless placed at the vertex) must slide along the arc. Let us now consider how this modification of the governor affects the working of the engine. When the load is diminished the engine begins to quicken; the balls separate and the steam is cut off. It is clear that equilibrium will not be established until the quantity of steam admitted is just such as to cause the engine to move at exactly the same rate as before.

Ex. Show that when the inertia of the rod and balls are taken account of, the centre of gravity of either ball and rod must be constrained to describe a parabola whose latus rectum is independent of the radius of the ball, if the Governor is to cause the engine always to move at a given rate.

474. The reader who may be interested in the subject of Governors may refer to an article by *Sir G. Airy, Vol. XI. of the Memoirs of the Astronomical Society*, 1840, where four different constructions are considered. He may also consult an

article by *Mr Siemens in the Phil. Trans. for* 1866, and a brief sketch of several kinds of governors by *Prof. Maxwell* in the *Phil. Mag. for* 1868. An account of some experiments by *Mr Ellery*, on Huyghens' parabolic pendulum, may be found in the *Astronomical Notices for December*, 1875.

475. Ex. 2. It has been shown in Art. 282 that if three particles be placed at the corners of an equiangular triangle and properly projected, they will move under their mutual attractions so as always to remain at the angular points of an equilateral triangle. These we may call Laplace's three particles. It is our present object to determine if this motion is stable or unstable*.

Let the mass M of the particle to be reduced to rest be taken as unity, and let m, m' be the masses of the other two. Let r, r', R be the distances between the particles Mm, Mm', mm'; and let ϕ', ϕ, ψ be the angles opposite to these distances. If θ, θ' be the angles r, r' make with a straight line fixed in space, and if the law of attraction be the inverse κth power of the distance, the equations of motion are

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{1+m}{r^\kappa} + \frac{m'\cos\psi}{r'^\kappa} + \frac{m'\cos\phi}{R^\kappa} &= 0 \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) + \frac{m'\sin\psi}{r'^\kappa} - \frac{m'\sin\phi}{R^\kappa} &= 0 \end{aligned}\right\},$$

with two similar equations for the motion of m'.

Let us now put $r=a+x$, $r'=a+x+X$, and let the angle between these radii vectores be $\frac{\pi}{3}+Y$, also let $\theta=nt+y$, where x, y, X and Y, are all small quantities whose squares are to be neglected. It should be noticed that a variation of x, y alone, X and Y being zero, will represent a variation of steady motion in which the particles always keep at the corners of an equilateral triangle, while a variation of X, Y will represent a change from the equilateral form. The former of these we know by Art. 282 is a possible motion, hence the equations can be satisfied by some values of x, y joined to $X=0$, $Y=0$. By this choice of variables we may hope to discover some roots of the fundamental determinant previous to expansion, and thus save a great amount of numerical labour. If D stand for $\frac{d}{dt}$, the four equations will now become

$$\left\{bD^2-(\kappa+1)(1+m+m')\right\}x - 2abnDy - \frac{3}{4}m'(\kappa+1)X - \frac{\sqrt{3}}{4}m'(\kappa+1)aY=0,$$

$$2bnDx + abD^2y - \frac{\sqrt{3}}{4}m'(\kappa+1)X + \frac{3}{4}m'(\kappa+1)aY=0,$$

$$\left\{bD^2-(\kappa+1)(1+m+m')\right\}x - 2abnDy + \left\{bD^2-(\kappa+1)(1+\frac{m}{4}+m')\right\}X - \left\{2abnD+\frac{\sqrt{3}}{4}m(\kappa+1)a\right\}Y=0,$$

$$2bnDx + abD^2y + \left\{2bnD - \frac{\sqrt{3}}{4}(\kappa+1)m\right\}X + \left\{abD^2 - \frac{3}{4}m(\kappa+1)a\right\}Y=0.$$

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* In a brief note in Jullien's Problems, Vol. II. p. 29, it is mentioned that this question has been discussed by M. Gascheau in a Thèse de Mécanique, the particles being supposed to attract each other according to the law of nature. The result arrived at is that the motion is stable when the square of the sum of the masses is greater than 27 times the sum of the products of the masses taken two and two. No reference is given to where M. Gascheau's work can be found, and the author is therefore unable to give a description of the process employed.

476. To solve these we put $x=Ae^{\lambda t}$, $y=Be^{\lambda t}$, $X=Ge^{\lambda t}$, $Y=He^{\lambda t}$. Substituting and eliminating the ratios of A, B, G and H we obtain a determinantal equation whose constituents are the coefficients of x, y, X and Y with λ written for D. This equation will give six values of λ. We see at once that one factor is λ. This might have been expected, because we know that a variation of y with x, X and Y all zero, is a possible motion. Again, some variation of x and y with X and Y both zero is also a possible motion, hence some factor of the determinant can be found by examining the first two columns. By subtracting from the first $2n$ times the second column we find that this factor is $b\lambda^2-(\kappa-3)(1+m+m')=0$.

To find the other factors we divide the determinant by the factors already found. Then subtracting the first row from the third and the second from the fourth we have three zeros in the first column and two in the second. The expansion is then easy. We see that there is another factor λ, also

$$b^2\lambda^4+b\lambda^2(3-\kappa)(1+m+m')+\frac{3}{4}(1+\kappa)^2(m+m'+mm')=0.$$

The two zero roots give $x=A_1+A_2t$ with similar expressions y, X and Y. But by substitution in the equations of motion we see that $x=A_1$, $y=B_1-\dfrac{\kappa+1}{2}\dfrac{A_1}{a}nt$, $X=0$ and $Y=0$. These roots therefore indicate merely a permanent change in the size of the triangle. On examining the other values of λ^2, we find (1) The motion cannot be stable unless κ is less than 3. (2) The motion is stable whatever the masses may be, if the law of force be expressed by any positive power of the distance or any negative power less than unity. (3) The motion is stable to a first approximation if

$$\frac{(M+m+m')^2}{Mm+Mm'+mm'}>3\left(\frac{1+\kappa}{3-\kappa}\right)^2,$$

where M, m, m' are the masses. To express the co-ordinates in terms of the time, we must return to the differential equations of the second order. The results are rather long, and it may be sufficient to state that when, as in the solar system, two of the masses are much smaller than the third, the inequalities in their angular distances, as seen from the large body, have much greater coefficients than their linear distances from the same body.

477. To form the general equations of oscillation of a dynamical system about a state of steady motion.

Let the system be referred to any co-ordinates θ, ϕ, ψ, &c. Let the state of motion about which the system is oscillating be determined by $\theta=f(t)$, $\phi=F(t)$, &c., then as explained in Art. 470 we shall put $\theta=f(t)+x$, $\phi=F(t)+y$, &c. Let the Lagrangian function L (see Art. 381) be expanded in powers of x, y, &c., as follows:

$$\begin{aligned}L=L_0&+A_1x+A_2y+\&c.+B_1x'+B_2y'+\&c.\\&+\frac{1}{2}(A_{11}x^2+2A_{12}xy+\&c.)+\frac{1}{2}(B_{11}x'^2+2B_{12}x'y'+\&c.)\\&+C_{11}xx'+C_{12}xy'+C_{21}yx'+\&c.\end{aligned}$$

We shall now define a *steady motion* to be one in which all the coefficients in this expansion are independent of the time. The physical characteristic of such a motion is that when referred to proper co-ordinates the same oscillations follow from the same disturbance of the same co-ordinate at whatever instant it may be applied to the motion. If the coefficients are not constant for the co-ordinates chosen it may be possible to make them constant by a change of co-ordinates. There are obviously many systems of co-ordinates which may be chosen, and a set may generally be found by a simple examination of the steady motion. If there are any quantities which are constant during the steady motion, such as those called ξ, η, &c. in Art. 466, these may serve for some of the co-ordinates, others may be found by considering what quantities appear only as differential coefficients or velocities, for example those called x, y, &c. in the same Article. If none of these are obvious, we may sometimes obtain them by combining the existing co-ordinates. Practically these will be the most convenient methods of discovering the proper co-ordinates.

478. To obtain the equations of motion we must now substitute the value of L in the Lagrangian equations

$$\frac{d}{dt}\frac{dL}{dx'} - \frac{dL}{dx} = 0, \quad \&c. = 0,$$

and reject the squares of small quantities. The steady motion being given by x, y, &c. all zero, each of these must be satisfied when we omit the terms containing x, y, &c. We thus obtain the equations of steady motion, viz.

$$A_1 = 0, \quad A_2 = 0, \quad \&c. = 0,$$

which by Taylor's theorem are the same as the equations (1) of steady motion given in Art. 466.

Omitting these terms and retaining the first powers of all the small quantities we obtain the equations of small oscillations, of which the following is a specimen:

$$\left\{B_{11}\frac{d^2}{dt^2} - A_{11}\right\} x + \left\{B_{12}\frac{d^2}{dt^2} + (C_{21} - C_{12})\frac{d}{dt} - A_{12}\right\} y$$
$$+ \left\{B_{13}\frac{d^2}{dt^2} + (C_{31} - C_{13})\frac{d}{dt} - A_{13}\right\} z + \&c. = 0.$$

To solve these we write $x = Le^{\lambda t}$, $y = Me^{\lambda t}$, &c. Substituting and eliminating the ratios of L, M, &c. we obtain the following determinantal equation

$$\left|\begin{array}{c|c|c|c} B_{11}\lambda^2 - A_{11} & \begin{array}{c} B_{12}\lambda^2 - A_{12} \\ + (C_{21} - C_{12})\lambda \end{array} & \begin{array}{c} B_{13}\lambda^2 - A_{13} \\ + (C_{31} - C_{13})\lambda \end{array} & \&c. \\ \hline \begin{array}{c} B_{12}\lambda^2 - A_{12} \\ - (C_{21} - C_{12})\lambda \end{array} & B_{22}\lambda^2 - A_{22} & \begin{array}{c} B_{23}\lambda^2 - A_{23} \\ + (C_{32} - C_{23})\lambda \end{array} & \&c. \\ \hline \begin{array}{c} B_{13}\lambda^2 - A_{13} \\ - (C_{31} - C_{13})\lambda \end{array} & \begin{array}{c} B_{23}\lambda^2 - A_{23} \\ - (C_{32} - C_{23})\lambda \end{array} & B_{33}\lambda^2 - A_{33} & \&c. \\ \hline \&c. & \&c. & \&c. & \&c. \end{array}\right| = 0.$$

If in this equation we write $-\lambda$ for λ the rows of the new determinant are the same as the columns of the old, so that the determinant is unaltered. When expanded the equation contains only *even* powers of λ.

479. Regarding this as an equation to find λ^2, we notice that if the roots are all real and negative, each of the co-ordinates x, y, &c. can be expressed in a series of trigonometrical terms having different periods; the motion will therefore be stable. If any one of the roots is imaginary or if any one is real and positive, there will be both positive and negative real exponentials entering into the expressions for x, y, &c. and therefore the motion will be unstable. The condition of dynamical stability is therefore that the roots of this equation must all be of the form $\lambda = \pm \mu\sqrt{-1}$, where μ is some real quantity.

480. It follows also that when a system, under the action of forces which have a potential, oscillates about a stable state of steady motion, the oscillations of the co-ordinates are represented by trigonometrical terms of the form $A \sin (\lambda t + \alpha)$ which are not accompanied by any real exponential factors such as those which occurred in the problem of the Governor.

We see further that there will in general be as many finite values of λ^2 and therefore as many trigonometrical terms of different periods as there are co-ordinates. It often happens, as explained in Art. 477, that some of the co-ordinates are absent from the expression for L, appearing only as differential coefficients. Suppose for example θ to be absent; then A_{11}, A_{12}, &c. are all zero, and we may divide λ both out of the first line and the first column of the fundamental determinant. We therefore have two zero values of λ, while at the same time the number of finite values of λ^2 is diminished by unity. Hence the number of trigonometrical terms of different periods cannot exceed the number of

co-ordinates which *explicitly* enter into the Lagrangian function. For example in Art. 374, the function $T-U$ has only the co-ordinate θ explicitly expressed, the others ϕ' and ψ' appearing only as differential coefficients. It follows that if a top is disturbed from a state of steady motion, there will be but one period in the oscillation.

481. The relations between the coefficients L, M, &c. in the exponential values of x, y, &c. may be obtained without difficulty if we remember that the several lines of the fundamental determinant are really the equations of motion. Taking any one line; multiply the first constituent by L, the second by M, &c. and equate the sum to zero. We thus obtain as many equations as there are co-ordinates. On the whole we shall have, exactly as in Art. 445, twice as many arbitrary constants as there are co-ordinates, all the other constants being determined by the equations just found. The arbitrary constants are determined by the initial values of the co-ordinates and their differential coefficients.

But, unlike Art. 445, the quantity λ occurs in the first power in each of these equations, so that the ratios of L, M, &c. thus found may be imaginary. The expressions for the co-ordinates when rationalized may therefore take the form

$$\begin{aligned} x &= A_1 \sin(\lambda_1 t + \alpha_1) + A_2 \sin(\lambda_2 t + \alpha_2) + \dots \\ y &= B_1 \sin(\lambda_1 t + \beta_1) + B_2 \sin(\lambda_2 t + \beta_2) + \dots \\ z &= \&\text{c}. \end{aligned}$$

where α_1 is not necessarily equal to β_1, nor α_2 to β_2, &c., though they are connected together.

482. When the initial conditions are such that every co-ordinate is expressed by a trigonometrical term of one and the same period, the system is said to be performing a *principal* or *harmonic oscillation*. Thus each trigonometrical term corresponds to a principal oscillation, and any oscillation of the system is therefore said to be *compounded* of its principal oscillations. The physical characteristic of a principal oscillation is that the motion of every part of the system is repeated at a constant interval.

483. The stability of the motion depends on the nature of the roots of the fundamental determinant. If we expand the determinant we may use the methods given in the theory of equations to discover if the roots are all of the proper form. This however is often tedious and we may sometimes settle the point by a simple examination of the determinant as it stands.

In practice it frequently happens that the determinant is reduced to two rows. If the invariants be written

$$A = A_{11}A_{22} - A_{12}^2, \quad B = B_{11}B_{22} - B_{12}^2,$$
$$\Theta = A_{11}B_{22} + A_{22}B_{11} - 2A_{12}B_{12},$$

the conditions of stability are

(1) A is positive.

(2) $(C_{21} \sim C_{12})^2 - \Theta$ is positive and greater than $2\sqrt{AB}$.

These conditions may also be expressed thus. Let α and β be the roots of the quadratic formed by omitting the terms containing C_{12} and C_{21}. Then by Art. 448, α and β are real. If α and β are both negative the motion is stable. If both are positive, the motion is stable or unstable according as $\dfrac{C_{21} \sim C_{12}}{\sqrt{B}}$ is numerically greater or less than $\sqrt{\alpha} + \sqrt{\beta}$, the roots being taken positively. If α and β have opposite signs, the motion is unstable.

Whatever may be the number of co-ordinates, it may be shown that the motion cannot be stable unless the discriminant of $A_{11}x^2 + 2A_{12}xy + \&c.$ is positive or negative according as the number of rows is even or odd.

The following theorem is also useful. Beginning with the fundamental determinant we may form a series of determinants, each being obtained from the preceding by erasing the first line and the first column. As we may supplement the fundamental determinant with a row and a column of zeros added on at the bottom and right-hand side with unity at the right-hand bottom corner, we may suppose the series of determinants to terminate with unity. Let us substitute in the series any negative value of λ^2 and count the number of Variations of sign in the series. Then as λ^2 changes from $-\infty$ to 0, there cannot be fewer negative roots between any two given values of λ^2 than there are losses in the number of variations of sign corresponding to the two values of λ^2. If there be more negative roots than losses the excess must be an even number.

484. Ex. A homogeneous sphere of unit mass and radius a is suspended from a fixed point by a string of length b, and is set in rotation about the vertical diameter. When the sphere is slightly disturbed, let bx, by and b be the co-ordinates of the point on the surface to which the string is attached; $bx + a\xi$, $by + a\eta$, and $b + a$ the co-ordinates of the centre, the fixed point being the origin and the axis of z being vertical and downwards. Also let $\chi = \phi + \psi$ where ϕ and ψ have the same meaning as in Art. 235, so that before disturbance $\chi' = n$. Prove that the Lagrangian function is

$$L = \frac{a^2}{5}\left\{\left(\chi' - \frac{\xi\eta'}{2} + \frac{\xi'\eta}{2}\right)^2 + \xi'^2 + \eta'^2\right\} + \frac{1}{2}(a\xi' + bx')^2 + \frac{1}{2}(a\eta' + by')^2 - g\left\{b\frac{x^2+y^2}{2} + a\frac{\xi^2+\eta^2}{2}\right\}.$$

If the motion of the centre of gravity be represented by a series of terms of the form $M\cos(\mu t+N)$, prove that the values of μ are given by

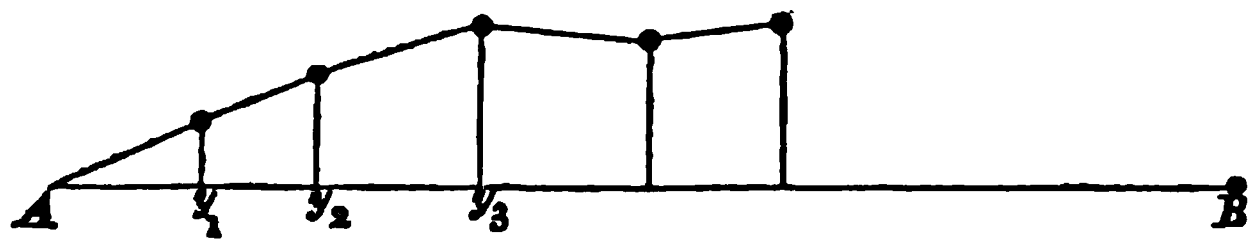
$$\left(\mu^2-\frac{g}{b}\right)\left(\mu^2-n\mu-\frac{5g}{2a}\right)=\frac{5g}{2b}\mu^2.$$

Show that whatever sign n may have this equation has two positive and two negative roots, which are separated by the roots of either of the factors on the left-hand side.

Application of the Calculus of Finite Differences.

485. We shall give some examples to illustrate the use of the Calculus of Finite Differences in cases in which there are an indefinite number of bodies similarly placed.

486. Ex. *A string of length* $(n+1)l$, *and insensible mass, stretched between two fixed points with a force* T, *is loaded at intervals* l *with* n *equal masses* m *not under the influence of gravity and is slightly disturbed; if* $\frac{T}{lm}=c^2$, *prove that the periodic times of the simple transversal vibrations which in general coexist are given by the formula* $\frac{\pi}{c}\operatorname{cosec}\frac{i\pi}{2(n+1)}$ *on putting in succession* $i=1, 2, 3 \ldots n$.



Let A, B be the fixed points; $y_1, y_2, \ldots y_n$ the ordinates at time t of the n particles. The motion of the particles parallel to AB is of the second order, and hence the tensions of all the strings must be equal, and in the small terms we may put this tension equal to T. Consider the motion of the particle whose ordinate is y_k. The equation of motion is

$$m\frac{d^2y_k}{dt^2}=\frac{y_{k+1}-y_k}{l}T-\frac{y_k-y_{k-1}}{l}T;$$

$$\therefore \frac{d^2y_k}{dt^2}=c^2(y_{k+1}-2y_k+y_{k-1}) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now the motion of each particle is vibratory, we may therefore expand y_k in a series of the form

$$y_k=\Sigma L\sin(pt+\alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where Σ implies summation for all values of p.

As there may be a term of the argument pt in every y, let L_1, L_2, ... be their respective coefficients. Then substituting, we have

$$L_{k+1} - 2L_k + L_{k-1} = -\frac{p^2}{c^2} L_k \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

To solve this linear equation of differences we follow the usual rule. Putting $L_k = Aa^k$, where A and a are two constants, we get after substitution and reduction $a - 2 + \frac{1}{a} = -\left(\frac{p}{c}\right)^2$, or

$$\sqrt{a} - \frac{1}{\sqrt{a}} = \frac{p}{c}\sqrt{-1}, \text{ and } \sqrt{a} + \frac{1}{\sqrt{a}} = \pm 2\sqrt{1 - \left(\frac{p}{2c}\right)^2};$$

$$\therefore \sqrt{a} = \pm\sqrt{1 - \left(\frac{p}{2c}\right)^2} + \frac{p}{2c}\sqrt{-1}.$$

Let these roots be called a_1 and a_2, then

$$L_k = Aa_1^k + Ba_2^k$$

is a solution, and since it contains two arbitrary constants, it is the general solution;

$$\therefore\ y_k = \Sigma\,[Aa_1^k + Ba_2^k] \sin(pt + \alpha) \quad\ldots\ldots\ldots\ldots\ldots(4).$$

The equations (1) and (3) will represent the motion of every particle from $k = 1$ to $k = n$, provided we suppose y_0 and y_{n+1} both zero, though there are no particles corresponding to values of k equal to 0 and $n + 1$. Since $y = 0$ when $k = 0$ for *all* values of t, every term of the series must vanish; $\therefore A + B = 0$. Also $y = 0$ when $k = n + 1$ for all values of t; $\therefore Aa_1^{n+1} + Ba_2^{n+1} = 0$. These equations give $a_1^{n+1} = a_2^{n+1}$. But if $\frac{p}{2c} > 1$, the ratio of a_1 to a_2 is real and different from unity. Hence we must have $\frac{p}{2c} < 1$. Let then $\frac{p}{2c} = \sin\theta$; and therefore $a = \cos 2\theta \pm \sin 2\theta\sqrt{-1}$.

Hence, by what we proved before,

$$(\cos 2\theta + \sin 2\theta\sqrt{-1})^{n+1} = (\cos 2\theta - \sin 2\theta\sqrt{-1})^{n+1};$$

$$\therefore \sin 2(n+1)\,\theta = 0, \text{ or } \frac{p}{2c} = \sin\frac{i\pi}{2(n+1)},$$

and the period of any term $= \dfrac{2\pi}{p}$.

If m and l be indefinitely small and n indefinitely large, the loaded string may be regarded as a uniform string of length $(n+1)\,l = L$ and mass $nm = M$ stretched between two fixed points

with a tension T. In this case the expression just found reduces to $p=\pi i\sqrt{\dfrac{T}{ML}}$.

487. If we substitute these values of θ in the expressions for a_1 and a_2, we easily find

$$y_k=\Sigma C_i \sin\frac{ki\pi}{n+1}\cdot\sin\left\{2ct\sin\frac{i\pi}{2(n+1)}+a_i\right\},$$

where C_i has been written for $2A\sqrt{-1}$, a_i for a, and the symbol Σ implies summation for all integer values of i from $i=1$ to $i=n$. This expression has n terms, and thus we have $2n$ arbitrary constants, viz. $C_1, C_2 \dots C_n$ and $a_1, a_2 \dots a_n$. These are to be determined by the known initial values of y_1, y_2, &c. and $\dfrac{dy_1}{dt}, \dfrac{dy_2}{dt}$, &c. To find these it will be more convenient to write the expression in the form

$$y_k=\Sigma E_i\sin\frac{ki\pi}{n+1}\sin\left\{2ct\sin\frac{i\pi}{2(n+1)}\right\}+\Sigma F_i\sin\frac{ki\pi}{n+1}\cos\left\{2ct\sin\frac{i\pi}{2(n+1)}\right\}.$$

Putting $t=0$, we have the two typical equations

$$[y_k]_0=\Sigma F_i\sin\frac{ki\pi}{n+1},$$

$$\frac{1}{2c}\left[\frac{dy_k}{dt}\right]_0=\Sigma E_i\sin\frac{ki\pi}{n+1}\sin\frac{i\pi}{2(n+1)}.$$

It is a theorem in Trigonometry that if i, i' be any integers between 0 and $n+1$, the sum of the series $\Sigma\sin\dfrac{ki\pi}{n+1}\sin\dfrac{ki'\pi}{n+1}$ taken from $k=1$ to $k=n$ is zero when i is different from i' and the sum is equal to $\dfrac{n+1}{2}$ when $i=i'$. This may be proved by expressing the general term of the series as the difference of two cosines, thus separating the given series into two series, each consisting of cosines of angles in arithmetical progression. Summing these from $k=0$ to $k=n$ when i and i' are both even or both odd, and from $k=1$ to $k=n$ when i is even and i' odd, we easily find the whole sum to be zero when i and i' are unequal. This change in the limits of the summation only adds a term which is zero to one end of the original series and therefore does not affect the sum. When i and i' are equal the value of the series may be found in a similar manner.

This theorem will at once enable us to find the general values of E_i and F_i. Let us multiply both sides of the first typical equation by the coefficient of F_i and sum all the series of which it is the type. We have

$$\Sigma\left\{[y_k]_0\sin\frac{ki\pi}{n+1}\right\}=\frac{n+1}{2}F_i,$$

where Σ implies summation for all values of k from $k=1$ to $k=n$. Treating the second equation in the same way, we have

$$\frac{1}{2c\sin\dfrac{i\pi}{2(n+1)}}\cdot\Sigma\left\{\left[\frac{dy_k}{dt}\right]_0\sin\frac{ki\pi}{n+1}\right\}=\frac{n+1}{2}E_i.$$

488. Lagrange in his *Mécanique Analytique* has applied his general equations of motion to the solution of the preceding problem. He has also determined the

oscillations of an inextensible string charged with any number of weights, and suspended by both ends or by one only. Though several solutions of these problems had been given before his time, he considers that they were all more or less incomplete.

489. Ex. 1. A light elastic string of length nl and coefficient of elasticity E is loaded with n particles each of mass m, ranged at intervals l along it beginning at one extremity. If it be suspended by the other extremity, prove that the periods of its vertical oscillations will be given by the formula $\pi\sqrt{\dfrac{lm}{E}}\operatorname{cosec}\dfrac{2i+1}{2n+1}\dfrac{\pi}{2}$, where $i=0, 1, 2 \dots n-1$ successively. Hence show that the periods of vertical oscillation of a heavy elastic string will be given by the formula $\dfrac{4}{2i+1}\sqrt{\dfrac{ML}{E}}$, where L is the length of the string, M its mass, and i is zero or any positive integer. [Math. Tripos, 1871.]

Ex. 2. An infinite number of equal particles, each of mass m, are placed in a row at distances each equal to l and mutually repel each other so that the force between any two is $m^2 f(D)$, where D is the distance between those two. A disturbance is given to the system such that each particle makes oscillations in the direction of the row whose extent is very small compared with l. Show that the disturbance of the k^{th} particle, counting from any one particle, is given by the series $\Sigma a \cos\dfrac{2\pi}{\lambda}(vt \pm kl)$, where Σ implies summation for all values of λ, and

$$v = l\sqrt{m}\left\{1^2 f'(h)\left(\frac{\sin\theta}{\theta}\right)^2 + 2^2 f'(2h)\left(\frac{\sin 2\theta}{2\theta}\right)^2 + \&c.\right\}^{\frac{1}{2}},$$

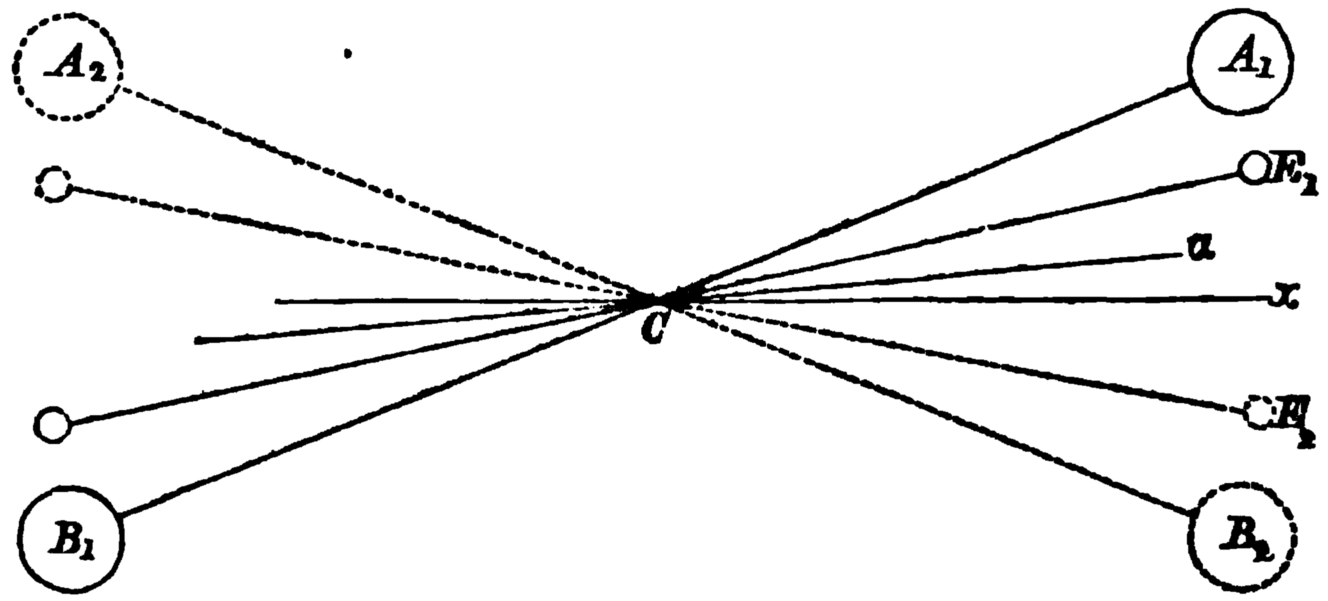
and $\theta = \dfrac{\pi l}{\lambda}$. Thence show that all very long waves travel with the same velocity. If $f(z) = \mu z^{-n}$, show that v is infinite unless n is greater than 3. [Phil. Mag.]

The Cavendish Experiment.

490. As an example of the mode in which the theory of small oscillations may be used as a means of discovery we have selected the Cavendish Experiment. The object of this experiment is to compare the mass of the earth with that of some given body. The plan of effecting this by means of a torsion-rod was first suggested by the Rev. John Michell. As he died before he had time to enter on the experiments, his plan was taken up by Mr Cavendish, who published the result of his labours in the *Phil. Trans.* for 1798. His experiments being few in number, it was thought proper to have a new determination. Accordingly in 1837, a grant of £500 was obtained from the Government to defray the expenses of the experiments. The theory and the analytical formulæ were supplied by Sir G. Airy, while the arrangement of the plan of operation and the task of making the experiments were undertaken by Mr Baily. Mr Baily made upwards of two thousand experiments with balls of different weights and sizes, and suspended in a variety of ways, a full account of which is

given in the *Memoirs of the Astronomical Society*, Vol. XIV. The experiments were, in general, conducted in the following manner.

491. Two small equal balls were attached to the extremities of a fine rod called the torsion-rod, and the rod itself was suspended by a string fixed to its middle point C. Two large spherical masses A, B were fastened on the ends of a plank which could turn freely about its middle point O. The point O was vertically under C and so placed that the four centres of gravity of the four balls were in one horizontal plane.



First, suppose the plank to be placed at right angles to the torsion-rod, then the rod will take up some position of equilibrium called the neutral position, in which the string has no torsion. Let this be represented in the figure by Cx. Now let the masses A and B be moved round O into some position B_1A_1, making a not very large angle with the neutral position of the torsion-rod. The attractions of the masses A and B on the balls will draw the torsion-rod out of its neutral position into a new position of equilibrium, in which the attraction is balanced by the torsion of the string. Let this be represented in the figure by CE_1. The angle of deviation E_1Cx and the time of oscillation of the rod about this position of equilibrium might be observed.

Secondly, replace the plank AB at right angles to the neutral position of the rod, and move it in the opposite direction until the masses A and B come into some position A_2B_2 near the rod but on the side opposite to B_1A_1. Then the torsion-rod will perform oscillations about another position of equilibrium CE_2 under the influence of the attraction of the masses and the torsion of the string. As before, the time of oscillation and the deviation E_2Ca might be observed.

In order to eliminate the errors of observation, this process was repeated over and over again, and the mean results taken.

The positions B_1A_1 and A_2B_2, into which the masses were alternately put, were as nearly as possible the same throughout all the experiments. The neutral position Ca of the rod very nearly bisected the angle between B_1A_1 and A_2B_2, but as this neutral position, possibly owing to changes in the torsion of the string, was found to undergo slight changes of position, it is not to be considered in any one experiment coincident with the bisector of the angle A_1CB_2.

Let Cx be any line fixed in space from which the angles may be measured. Let b be the angle xCa, which the neutral position of the rod makes with Cx; A and B the angles which the alternate positions, B_1A_1 and A_2B_2, of the straight line joining the centres of the masses, make with Cx; and let $a=\dfrac{A+B}{2}$. Also let x be the angle which the torsion-rod makes with Cx at the time t.

Supposing the masses to be in the position A_1B_1, the moment about CO of their attractions on the two balls and on the rod will be a function only of the angle between the rod and the line A_1B_1; let this moment be represented by $\phi(A-x)$. The whole apparatus was enclosed in a wooden casing to protect it from any currents of air. The attraction of this casing cannot be neglected. As it may be different in different positions of the rod, let the moment of its attraction about CO be $\psi(x)$. Also the torsion of the string will be very nearly proportional to the angle through which it has been twisted. Let its moment about CO be $E(x-b)$.

If then I be the moment of inertia of the balls and rod about the axis CO, the equation of motion will be

$$I\frac{d^2x}{dt^2}=\phi(A-x)+\psi(x)-E(x-b).$$

Now $a-x$ is a small quantity, let it be represented by ξ. Substituting for x and expanding by Taylor's theorem in powers of ξ, we get

$$-I\frac{d^2\xi}{dt^2}=\phi(A-a)+\psi(a)-E(a-b)+\{\phi'(A-a)-\psi'(a)+E\}\xi.$$

Let $$n^2=\frac{\phi'(A-a)-\psi'(a)+E}{I},$$

and $$e=a+\frac{\phi(A-a)+\psi(a)-E(a-b)}{In^2}.$$

Then $$x=e+L\sin(nt+L'),$$

where L and L' are two arbitrary constants. We see therefore that in the position of equilibrium the angle the torsion-rod

makes with the axis of x is e, and the time of oscillation about the position of equilibrium is $\frac{2\pi}{n}$.

Let us now suppose the masses to be moved into their alternate position A_2B_2; the moment of their attraction on the balls and rod will now be $-\phi(x-B)$. The equation of motion is therefore

$$I\frac{d^2x}{dt^2}=-\phi(x-B)+\psi(x)-E(x-b).$$

Let $a=x-\xi$, then substituting for B its value $2a-A$, we find by the same reasoning as before

$$x=e'+N\sin(nt+N'),$$

where n has the same value as before and

$$e'=a+\frac{-\phi(A-a)+\psi(a)-E(a-b)}{In^2}.$$

In these expressions, the attraction $\psi(a)$ of the casing, the coefficient of torsion E and the angle b are all unknown. But they all disappear together, if we take the difference between e and e'. We then find

$$\frac{\phi(A-a)}{I}=\frac{e-e'}{2}\cdot\left(\frac{2\pi}{T}\right)^2 \text{......................} (A),$$

where T is the time of a complete oscillation of the torsion-rod about either of the *disturbed* positions of equilibrium. Thus the attraction $\phi(A-a)$ can be found if the angle $e-e'$ between the two positions of equilibrium and also the time of oscillation about either can be observed.

492. The function $\phi(A-a)$ is the moment of the attraction of the masses and the plank on the balls and rod, when the rod has been placed in a position Cf, bisecting the angle A_1CB_2 between the alternate positions of the masses. Let M be the mass of either of the masses A and B, m that of one of the small balls, m' that of the rod. Let the attraction of M on m be represented by $\mu\frac{Mm}{D^2}$, where D is the distance between their centres. If (p, q) be the co-ordinates of the centre of A_1 referred to Cf as the axis of x, the moment about C of the attraction of both the masses on both the balls is

$$=2\mu Mm\left\{\frac{cq}{\{(p-c)^2+q^2\}^{\frac{3}{2}}}-\frac{cq}{\{(p+c)^2+q^2\}^{\frac{3}{2}}}\right\},$$

where c is the distance of the centre of either ball a, b from the centre C of motion. Let this be represented by μMmP. The moments of the attraction of the masses on the rod may by inte-

gration be found $=\mu Mm'Q$, where Q is a known function of the linear dimensions of the apparatus. The attraction of the plank might also be taken account of. Thus we find

$$\phi(A-a)=\mu M(mP+m'Q).$$

If r be the radius of either ball, we have

$$I=2m\left\{c^2+\frac{2}{5}r^2\right\}+m'\frac{(c-r)^2}{3},$$

which may be represented by $I=mP'+m'Q'$, where P' and Q' are known functions of the linear dimensions of the rod and balls. Hence we find by substituting in equation (A)

$$\mu M.\frac{mP+m'Q}{mP'+m'Q'}=\frac{e-e'}{2}.\left(\frac{2\pi}{T}\right)^2.$$

Let E be the mass of the earth, R its radius and g the force of gravity, then $g=\mu\frac{E}{R^2}$.* Substituting for μ, we find

$$\frac{M}{E}=\frac{e-e'}{2}.\left(\frac{2\pi}{T}\right)^2.\frac{1}{gR^2}.\frac{\frac{m}{m'}P'+Q'}{\frac{m}{m'}P+Q}.$$

The ratio $\frac{m}{m'}$ was taken equal to the ratio of the weights of the ball and rod *weighed in vacuo*, but it would clearly have been more accurate to have taken it equal to the ratio weighed in air. For since the masses attract the air as well as the balls, the pressure of the air on the side of a ball nearest the attracting mass is greater than that on the furthest side. The difference of these pressures is equal to the attraction of the mass on the air displaced by the ball.

493. By this theory the discovery of the mass of the earth has been reduced to the determination of two elements, (1) the time of oscillation of the torsion-rod, and (2) the angle $e-e'$ between its two positions of equilibrium when under the influence of the masses in their alternate positions. To observe these, a small mirror was attached to the rod at C with its plane nearly perpendicular to the rod. A scale was engraved on a vertical plate at a distance of 108 inches from the mirror, and the image of the scale formed by reflection on the mirror was viewed in a telescope placed just over the scale. The telescope was fur-

* In Baily's experiment, a more accurate value of g was used. If ϵ be the ellipticity of the earth, m the ratio of centrifugal force at the equator to equatoreal gravity, and λ the latitude of the place, we have $g=\mu\frac{E}{R^2}\left\{1-2\epsilon+\left(\frac{5}{2}m-\epsilon\right)\cos^2\lambda\right\}$.

nished with three vertical wires in its focus. As the torsion-rod turned on its axis, the image of the scale was seen in the telescope to move horizontally across the wires and at any instant the number of the scale coincident with the middle wire constituted the reading. The scale was divided by vertical lines one-thirteenth of an inch apart and numbered from 20 to 180 to avoid negative readings. The angle turned through by the rod when the image of the scale moved through a space corresponding to the interval of two divisions was therefore $\frac{1}{13} \cdot \frac{1}{108} \cdot \frac{1}{2} = 73''\cdot46$. But the division lines were cut diagonally and subdivided decimally by horizontal lines; so that not only could the tenth of a division be clearly distinguished, but, after some little practice, the fractional parts of these tenths. The arc of oscillation of the torsion-rod was so small that the square of its circular measure could be neglected; but as it extended over several divisions it is clear that it could be observed with accuracy. A minute description of the mode in which the observations were made would not find a fit place in a treatise on Dynamics, we must therefore refer the reader to Baily's Memoir.

In this investigation no notice has been taken of the effect of the resistance of the air on the arc of vibration. This was, to some extent at least, eliminated by a peculiar mode of taking the means of the observations. In this way also some allowance was made for the motion of the neutral position of the torsion-rod.

494. The density of water in which the weight of a cubic inch is 252·725 grains (7000 grains being equal to one pound avoirdupois) was taken as the unit of density. The final result of all the experiments was that the mean density of the earth is 5·6747.

495. Two other methods of finding the mean density have been employed. In 1772 Dr Maskelyne, then Astronomer Royal, suggested that the mass of the earth might be compared with that of a mountain by observing the deviation produced in a plumb-line by the attraction of the latter. The mountain chosen was Schehallien, and the density of the earth was found to be a little less than five times that of water. See *Phil. Trans.* 1778 and 1811. From some observations near Arthur's Seat, the mean density of the earth is given by Lieut.-Col. James, of the Ordnance Survey, as 5·316. See *Phil. Trans.* 1856.

The other method, used by Sir G. Airy, is to compare the force of gravity at the bottom of a mine with that at the surface, by observing the times of vibration of a pendulum. In this way the mean density of the earth was found to be 6·566. See *Phil. Trans.* 1856.

Oscillations of the Second Order.

496. The equations of small oscillations are formed on the following principle. Some small quantities are selected as the co-ordinates of the system, and all powers of these above the first are neglected. The assumption is tacitly made that the order of magnitude of the terms is not materially altered by the process of solving the equations; so that a small term, which should by the rule be neglected in forming the differential equations, cannot become of importance in the final integrals. This assumption, however, is not strictly correct. In the Lunar and Planetary theories, where something more is wanted than the mere periods of oscillations, there are many instances of small terms in the differential equations, which become of great magnitude in the result. We require some rule to distinguish the small terms which become of importance from those which remain insignificant. For the sake of simplicity we shall consider the case in which the system depends on two independent co-ordinates, though the remarks are for the most part quite general.

497. Referring to Art. 432, let $P\sin\lambda t$ be some small periodic term which occurs on the right-hand side of the first of the two differential equations of motion. To simplify the solution, let us write for the trigonometrical term its exponential value, and fix our attention on the part $\dfrac{1}{2\sqrt{-1}}Pe^{\lambda\sqrt{-1}t}$, or, as we shall write it, $Qe^{\mu t}$. Let $f(D)$ stand for the determinant which is the operator on x in the third equation of Art. 432. Also let $F(D)$ be the minor of the leading constituent; the value of x is then known to be

$$x=\frac{F(\mu)}{f(\mu)}Qe^{\mu t}+M_1e^{m_1t}+\dots\dots$$

The term $Qe^{\mu t}$ in the differential equation is the analytical representation of some small periodical force which acts on the system. The first term of the expression for x is the *direct* effect of the force, and is sometimes called the *forced vibration* in the co-ordinate x. The quantities m_1, m_2, &c. being generally imaginary, the remaining terms are also trigonometrical and are sometimes called the *free* or *natural vibrations* in the co-ordinate. In the analytical theory of linear differential equations, the forced vibration is called the *particular integral* and the free vibration the *complementary function*.

498. If we examine the coefficient of the forced vibration in x we shall see that it is large only if $f(\mu)$ is very small or zero. Since the roots of the equation $f(\mu)=0$ are m_1, m_2, &c. the rule may be simply stated thus: *any small periodical term whose coefficient in the differential equation is less than the standard of quantities to be neglected may rise into importance if its period is nearly equal to one of the free vibrations of the system.*

Suppose the dynamical system to have two of its free periods equal and let it be acted on by a small force whose period is nearly equal to this free period. The divisor $f(\mu)$ of the forced vibration will be a small quantity of the second order and the magnitude of the term may be much greater than if the free periods were unequal. When such a case occurs in the Lunar theory, the term is said to *rise two orders*.

499. This principle admits of an elementary explanation in some cases. Let a system oscillating with one degree of freedom be acted on by a small periodical force at some point A. The force will act sometimes to accelerate the motion of A and sometimes to retard it, and thus the magnitude of the vibration will not become very great. But if the period of the force be equal to that of the point A, the force may continually act to increase the motion of A in whatever direction A is moving. Thus the extent of the vibration will be continually increasing. For example, every one knows how a heavy swing can be set in violent oscillation by a series of small pushes and pulls applied at the proper times.

If the period of the force be only nearly equal to that of the point A, a time will come when the force acts continually to decrease the motion of A. Thus the oscillation will not increase indefinitely, but will alternately slowly increase and as slowly decrease.

500. A remarkable use of this principle was made by Capt. Kater in his experiments to determine the length of the seconds' pendulum. It was important to determine if the support of his pendulum was perfectly firm. He had recourse to a delicate and simple instrument invented by Mr Hardy a clockmaker, the sensibility of which is such that had the slightest motion taken place in the support it must have been instantly detected. The instrument consists of a steel wire, the lower part of which is inserted in the piece of brass which forms its support, and is flattened so as to form a delicate spring. On the wire a small weight slides by means of which it may be made to vibrate in the *same time* as the pendulum to which it is to be applied as a test. When thus adjusted it is placed on the material to which the pendulum is attached, and should this not be perfectly firm, the motion will be communicated to the wire, which in a little time will accompany the pendulum on its vibrations. This ingenious contrivance appeared fully adequate to the purpose for which it was employed, and afforded a satisfactory proof of the stability of the point of suspension. See *Phil. Trans.* 1818.

501. It generally happens that the small terms rejected in the equations of motion are functions of the co-ordinates and their differential coefficients. To take account of these terms we proceed by successive approximation. Suppose the co-ordinates x, y to determine the oscillation about some state of steady motion, and to be zero for that motion. As a first approximation we obtain (Art. 432)

$$x = M_1 e^{m_1 t} + M_2 e^{m_2 t} + \ldots\ldots$$

with a corresponding expression for y, where m_1, m_2, &c. give the free periods, and M_1, M_2, &c. are all small quantities of the first order. If we now substitute these values of x and y in any small term of a high order which occurs in the differential equation, it becomes a series of exponentials of the form

$$Pe^{(pm_1 + qm_2 + \ldots)t},$$

where p, q, &c. are positive integers whose sum is equal to the order of the term. By the principle explained in Art. 498, the corresponding forced vibration cannot be important unless $pm_1 + qm_2 + \ldots$ is very nearly equal to one of the quantities m_1, m_2, &c. In the same way, in any approximation, if the periods of the terms are not such that an equality of this nature can be very nearly true, the next approximation to the motion will not produce any important terms. Even if such a relation does approximately hold, yet, if the order of the term to be examined is great, the term will probably remain insignificant.

502. As an example let us consider the case of a planet describing a circle about the sun, considered as fixed in the centre. If slightly disturbed the changes in the radius vector and longitude will be very small and will correspond to what we have called x and y. From the theory of elliptic motion, we know that these will be approximately

$$x = a - ae\cos(nt + \alpha), \quad y = bt + c + 2e\sin(nt + \alpha),$$

where a, b, c and e are all small quantities, and $\frac{2\pi}{n}$ is the period of the planet. Comparing these with the expressions for x and y given in Art. 432, we see that the free periods for x are given by $m=0$, $m=\pm n\sqrt{-1}$, and for y, by $m=0$, $m=0$, $m=\pm n\sqrt{-1}$, one period being absent from x. We infer that any small periodical force may produce a considerable disturbance both in the radius vector and in the longitude of the planet, if its period is nearly equal to that of the planet or is very long. Since there are two equal free periods in the longitude corresponding to $m=0$, those small forces whose periods are very long may be expected to rise two orders in the longitude. If any such forces act on the planet it will be necessary to examine into their effects. Small forces, whose periods are different from these, and whose magnitude is beneath the standard of quantities to be retained, may be disregarded.

503. If the period of the small disturbing force $Qe^{\mu t}$ be equal to one of the free periods, the solution changes its character. The forced vibration now takes the form $\frac{F(\mu)}{f'(\mu)}Qte^{\mu t}$. This may indicate that the motion of the system will, after a time, become very different from that which we took as a first approximation. We may have therefore to amend our first approximation by including in it the effect of this force. We may then enquire how far this modified first approximation indicates that the undisturbed motion is stable or unstable. When this force is included in the equations, the equations will probably be no longer linear, and it may be impossible to solve them or to find a solution sufficiently accurate to serve as a first approximation throughout the whole motion.

504. In many cases however the effects of some of these forces may be included in the first approximation by slightly altering the free periods. Referring to Art. 432, let us suppose that on substituting our first approximation in the small terms, we have on the right-hand side of the two first equations

$$\left.\begin{array}{l} Q_1e^{m_1t} + Q_2e^{m_2t} + \dots \\ R_1e^{m_1t} + R_2e^{m_2t} + \dots \end{array}\right\} \dots\dots\dots\dots (1).$$

These are supposed to have arisen from some relations of the form

$$pm_1 + qm_2 + \dots = m_1 \dots\dots\dots\dots (2).$$

Let us take as our amended first approximation

$$\left.\begin{array}{l} x = N_1e^{n_1t} + N_2e^{n_2t} + \dots \\ y = N_1'e^{n_1t} + N_2'e^{n_2t} + \dots \end{array}\right\} \dots\dots\dots\dots (3),$$

where N_1, &c. N_1', &c. are, as before, small quantities of the first order, and $n_1 = m_1 + \delta m_1$, $n_2 = m_2 + \delta m_2$, &c. where δm_1, δm_2, &c. are quantities of the order Q_1, &c. R_1, &c. If we substitute the amended values of x, y in the small terms, they will become

$$\left.\begin{array}{l} Q_1'e^{n_1t} + Q_2'e^{n_2t} + \dots \\ R_1'e^{n_1t} + R_2'e^{n_2t} + \dots \end{array}\right\} \dots\dots\dots\dots (4),$$

instead of (1), provided the relations represented by (2) apply also to the indices n_1, n_2, &c. Here Q_1', &c. R_1', &c. differ from Q_1, &c. R_1, &c. by quantities of the order Q_1^2. Substituting the values (3) in the differential equations of Art. 432, and rejecting the squares of Q_1, &c. R_1, &c., we obtain

$$\left.\begin{aligned}(An^2+Bn+C)\,N+(Fn^2+Gn+H)\,N'=Q\\(A'n^2+B'n+C')\,N+(F'n^2+G'n+H')\,N'=R\end{aligned}\right\}\text{..................}(5),$$

where the suffixes have been dropped for the sake of generality. These two equations determine n and N', leaving N to be determined by the initial conditions. The test of the success of the amended first approximation is that the values of n thus found satisfy the relation (2).

505. The condition may also be stated thus. Consider the determinant given in Art. 432, which when expanded is equal to $f(D)$. After substitution of the first approximation in the small terms of the higher orders in the equations, perform on these equations the operations indicated by the minors of the constituents in the first column, and add the results together. We have an equation of the form

$$f(D)\,x=\Delta_1 e^{m_1t}+\Delta_2 e^{m_2t}+\dots$$

where the coefficients Δ_1, Δ_2, &c. are all functions of M_1, M_2, &c., m_1, m_2, &c. Following the same reasoning as in the last Article, and amending our first approximation, we find

$$\delta m_1=\frac{\Delta_1}{M_1 f'(m_1)},\quad \delta m_2=\frac{\Delta_2}{M_2 f'(m_2)},\ \text{\&c.}$$

If these satisfy the relations typified by

$$p\delta m_1+q\delta m_2+\dots=\delta m_1,$$

the effect of the disturbing cause is to modify the free periods of the system without affecting the stability of the undisturbed motion.

506. Having in this way amended the first approximation, we may proceed to the second by substitution in the small term, and so on. If the several stages can be so arranged that no term makes its appearance which can become greater than our previous approximation, we may consider that we have obtained a correct representation of the motion.

507. Ex. 1. *A pendulum swings in a very rare medium, resisting partly as the velocity and partly as the square of the velocity, to find the motion.*

Let θ be the angle the straight line joining the point O of support to the centre of gravity G of the pendulum makes with the vertical. Then the equation of motion is

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=-2\kappa\frac{d\theta}{dt}-\mu\left(\frac{d\theta}{dt}\right)^2\text{.............................}(1),$$

where l is the length of the simple equivalent pendulum, 2κ and μ the coefficients of the resistance divided by the moment of inertia of the pendulum about the axis of suspension. Let $g=ln^2$. Since θ is small we may write the equation in the form

$$\frac{d^2\theta}{dt^2}+n^2\theta=-2\kappa\frac{d\theta}{dt}-\mu\left(\frac{d\theta}{dt}\right)^2+n^2\frac{\theta^3}{6}-\dots$$

Since κ and θ are very small, we might at first suppose that it would be sufficient as a first approximation to reject all the terms on the right-hand side.

This gives $\theta=a\sin nt$, the origin of measurement of t being so chosen that t and θ vanish together. If we substitute this in the small terms we get

$$\frac{d^2\theta}{dt^2}+n^2\theta=-2\kappa n\,.\,a\cos nt+\frac{1}{8}n^2a^3\sin nt+\&c.,$$

which gives

$$\theta=a\sin nt-\kappa a\,.\,t\sin nt+\frac{1}{16}na^3t\cos nt+\&c.$$

These additional terms contain t as a factor, and show that our first approximation was not sufficiently near the truth to represent the motion except for a short time. To obtain a sufficiently near first approximation we must include in it the small term $2\kappa\frac{d\theta}{dt}$, we have therefore

$$\frac{d^2\theta}{dt^2}+2\kappa\frac{d\theta}{dt}+n^2\theta=0.$$

This gives $\theta=ae^{-\kappa t}\,.\,\sin mt$, where for the sake of brevity we have put $n^2-\kappa^2=m^2$.

In our second approximation we shall reject all terms of the order a^3 or $a^2\kappa$ unless they are such that after integration they rise in importance in the manner explained in Art. 498. We thus get

$$\frac{d^2\theta}{dt^2}+2\kappa\frac{d\theta}{dt}+n^2\theta=-\frac{\mu a^2m^2}{2}e^{-2\kappa t}(1+\cos 2mt)+\frac{n^2}{6}a^3\frac{e^{-3\kappa t}}{4}(3\sin mt-\sin 3mt)$$
$$-\mu a^2\kappa e^{-2\kappa t}\left(-\frac{\kappa}{2}+\frac{\kappa}{2}\cos 2mt+m\sin 2mt\right),$$

where all the terms on the right-hand side after the first are of the third order, and are to be rejected unless they rise in importance. To solve this, let us first consider the general case

$$\frac{d^2\theta}{dt^2}+2\kappa\frac{d\theta}{dt}+n^2\theta=e^{-p\kappa t}\,.\,(A\sin rmt+B\cos rmt).$$

Put $\theta=e^{-p\kappa t}(L\sin rmt+M\cos rmt)$. Substituting we get

$$\left.\begin{aligned}L\{(p-1)^2\kappa^2+m^2(1-r^2)\}+2(p-1)\kappa rmM=A\\ M\{(p-1)^2\kappa^2+m^2(1-r^2)\}-2(p-1)\kappa rmL=B\end{aligned}\right\}.$$

Now κ is very small. If then r be not equal to unity, we have $L=\frac{A}{m^2(1-r^2)}$, $M=\frac{B}{m^2(1-r^2)}$ nearly; but if $r=1$, we have $L=\frac{-B}{2(p-1)\kappa m}$, $M=\frac{A}{2(p-1)\kappa m}$ nearly. The case of $p=1$ does not occur in our problem. It appears that those terms only in the differential equation which have $r=1$ give rise to terms in the value of x which have the small quantity κ in the denominator. Hence in the differential equation the only term of the third order which should be retained is the first. We thus find, putting successively $r=0$, $r=2$, $r=1$,

$$\theta=ae^{-\kappa t}\sin mt-\frac{\mu a^2}{2}e^{-2\kappa t}+\frac{\mu a^2}{6}e^{-2\kappa t}\cos 2mt+\frac{n^2a^3}{32\kappa m}e^{-3\kappa t}\cos mt.$$

This equation determines the motion only during any one swing of the pendulum; when the pendulum turns to go back μ changes sign. Let us suppose the pendulum to be moving from left to right, and let us find the lengths of the arcs of descent and ascent. To do this, we must put $\frac{d\theta}{dt}=0$. Let the equation be written

in the form $\theta=f(t)$, then if we neglect all the small terms, $\frac{d\theta}{dt}$ vanishes when $mt=\pm\frac{\pi}{2}$. Put then $mt=-\frac{\pi}{2}+x$ where x is a small quantity, we have

$$f'(t)=f'\left(-\frac{\pi}{2m}\right)+f''\left(-\frac{\pi}{2m}\right)\frac{x}{m}=0.$$

Now

$$f'(t)=ae^{-\kappa t}(m\cos mt-\kappa\sin mt)-\frac{\mu a^2}{2}e^{-2\kappa t}\left(-2\kappa+\frac{2\kappa}{3}\cos 2mt+\frac{2m}{3}\sin 2mt\right)$$
$$+\frac{n^2a^3}{32\kappa m}e^{-3\kappa t}(-m\sin mt-3\kappa\cos mt).$$

A sufficiently near approximation to the value of $f''(t)$ may be found by differentiating the first term of the value of $f'(t)$. We thus find $x=-\frac{\kappa}{m}-\frac{4}{3}\frac{\mu a\kappa}{m}-\frac{n^2a^2}{32\kappa m}$; the second of these terms being smaller than the other two might be neglected. We also find as the arc of descent

$$\theta=f\left(-\frac{\pi}{2m}\right)+f'\left(-\frac{\pi}{2m}\right)\frac{x}{m}=-\left[ae^{\frac{\kappa\pi}{2m}}+\frac{2}{3}\mu a^2e^{\frac{\kappa\pi}{m}}-x\left\{\kappa ae^{\frac{\kappa\pi}{2m}}+\frac{n^2a^3}{32\kappa m}e^{\frac{3\kappa\pi}{2m}}\right\}\right].$$

Similarly to find the arc of ascent we put $mt=\frac{\pi}{2}+y$. This gives $y=-\frac{\kappa}{m}-\frac{n^2a^2}{32\kappa m}$, and the arc of ascent is

$$\theta=ae^{-\frac{\kappa\pi}{2m}}-\frac{2}{3}\mu a^2e^{-\frac{\kappa\pi}{m}}-y\left\{\kappa ae^{-\frac{\kappa\pi}{2m}}+\frac{n^2a^3}{32\kappa m}e^{-\frac{3\kappa\pi}{2m}}\right\}.$$

In these expressions for the arcs of descent and ascent the terms containing x and y are very small, and assuming κ not to be extremely small, these terms will be neglected*.

Now a is different for every swing of the pendulum, we must therefore eliminate a. Let u_n and u_{n+1} be two successive arcs of descent and ascent, and let $\lambda=e^{-\frac{\kappa\pi}{2m}}$, so that λ is a little less than unity. Then we have

$$u_n=a\frac{1}{\lambda}+\frac{2}{3}\mu a^2\frac{1}{\lambda^2},\quad u_{n+1}=a\lambda-\frac{2}{3}\mu a^2\lambda^2;$$

eliminating a we have very nearly

$$\frac{1}{u_{n+1}}+\frac{1}{c}=\frac{1}{\lambda^2}\left(\frac{1}{u_n}+\frac{1}{c}\right),$$

where $$c=\frac{3}{2\mu}\frac{1-\lambda^2}{1+\lambda^2}=\frac{3\kappa\pi}{4\mu m}\text{ nearly.}$$

* If these terms are not neglected the equation connecting the successive arcs of descent and ascent becomes

$$\frac{1}{u_n}-\frac{\lambda^2}{u_{n+1}}=-\frac{2}{3}\mu(1+\lambda^2)+\frac{n^2x}{32\kappa m}\frac{1-\lambda^4}{\lambda}.$$

Now $1-\lambda^4=\frac{2\kappa\pi}{m}$ nearly, so that this additional term is very small compared with that retained.

The successive arcs are, therefore, such that $\frac{1}{u_n}+\frac{1}{c}$ is the general term of a geometrical series whose ratio is $e^{\frac{\kappa\pi}{m}}$. The ratio of any arc u_n to the following arc u_{n+1} is

$$\frac{u_n}{u_{n+1}}=e^{\frac{\kappa\pi}{m}}+\frac{u_n}{c}\left(e^{\frac{\kappa\pi}{m}}-1\right),$$

which continually decreases with the arc. In any series of oscillations the ratio is at first greater and afterwards less than its mean value. This result seems to agree with experiment.

To find the time of oscillation. Let t_1, t_2 be the times at which the pendulum is at the extreme left and right of its arc of oscillation. Then

$$mt_1=-\frac{\pi}{2}-\frac{\kappa}{m}-\frac{n^2a^2}{32m\kappa},\quad mt_2=\frac{\pi}{2}-\frac{\kappa}{m}-\frac{n^2a^2}{32m\kappa}.$$

The time of oscillation from one extreme position to the other is t_2-t_1 which is equal to $\frac{\pi}{m}$. This result is independent of the arc, so that the time of oscillation remains constant throughout the motion. The time is however not exactly the same as in vacuo, but is a little longer; the difference depending on the *square* of the small quantity κ.

Ex. 2. If in Art. 418 a first approximation to the motion is $\theta=A\sin(qt+B)$, show that a second will be

$$\theta=A\sin(at+B)+\tfrac{1}{2}(b+c)A^2+\tfrac{1}{6}(3b+c)A^2\cos 2(at+B)$$

where $\quad b=\frac{rs\sin\alpha}{k^2+r^2},\quad c=\frac{1}{2}\frac{s^2}{s\cos\alpha-r}\left\{-\frac{\cos\alpha}{s}\frac{ds}{d\sigma}+\frac{\sin 2\alpha}{r}-\frac{\sin\alpha}{\rho'}\right\},$

and σ is the length of the arc of either cylinder.

A general method of solving problems of this kind, both for two and three dimensions, is given in the *Proceedings of the London Mathematical Society*, Vol. v. page 101, 1874.

Ex. 3. A rigid body is suspended by two equal and parallel threads attached to it at two points symmetrically situated with respect to a principal axis through the centre of gravity which is vertical, and being turned round that axis through a small angle is left to perform small *finite* oscillations. Investigate the reduction to infinitely small oscillations. [Smith's Prize.]

EXAMPLES*.

1. A uniform rod of length $2c$ rests in stable equilibrium with its lower end at the vertex of a cycloid whose plane is vertical and vertex downwards, and passes through a small smooth fixed ring situated in the axis at a distance b from the vertex. Show that if the equilibrium be slightly disturbed, the rod will perform

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* These examples are taken from the Examination Papers which have been set in the University and in the Colleges.

small oscillations with its lower end on the arc of the cycloid in the time $4\pi\sqrt{\dfrac{a\{c^2+3(b-c)^2\}}{3g(b^2-4ac)}}$, where $2a$ is the length of the axis of the cycloid.

2. A small smooth ring slides on a circular wire of radius a which is constrained to revolve about a vertical axis in its own plane, at a distance c from the centre of the wire, with a uniform angular velocity $\sqrt{\dfrac{g\sqrt{2}}{c\sqrt{2}+a}}$; show that the ring will be in a position of stable relative equilibrium when the radius of the circular wire passing through it is inclined at an angle 45° to the horizon; and that if the ring be slightly displaced, it will perform a small oscillation in the time

$$2\pi\left\{\frac{a\sqrt{2}}{g}\cdot\frac{c\sqrt{2}+a}{c\sqrt{8}+a}\right\}^{\frac{1}{2}}.$$

3. A uniform bar of length $2a$ suspended by two equal parallel strings each of length b from two points in the same horizontal line is turned through a small angle about the vertical line through the middle point, show that the time of a small oscillation is $2\pi\sqrt{\dfrac{bk^2}{gk^2}}$.

4. Two equal heavy rods connected by a hinge which allows them to move in a vertical plane rotate about a vertical axis through the hinge, and a string whose length is twice that of either rod is fastened to their extremities and bears a weight at its middle point. If M, M' be the masses of a rod and the particle, and $2a$ the length of a rod, prove that the angular velocity about the vertical axis when the rods and string form a square is $\sqrt{\dfrac{3g}{2a\sqrt{2}}\cdot\dfrac{M+2M'}{M}}$, and if the weight be slightly depressed in a vertical direction the time of a small oscillation is $2\pi\sqrt{\dfrac{4a\sqrt{2}}{15g}\cdot\dfrac{M+3M'}{M+2M'}}$.

5. A ring of weight W which slides on a rod inclined to the vertical at an angle α is attached by means of an elastic string to a point in the plane of the rod so situated that its least distance from the rod is equal to the natural length of the string. Prove that if θ be the inclination of the string to the rod when in equilibrium, $\cot\theta-\cos\theta=\dfrac{W}{w}\cos\alpha$, where w is the modulus of elasticity of the string. And if the ring be slightly displaced the time of a small oscillation will be $2\pi\sqrt{\dfrac{Wl}{wg}\dfrac{1}{1-\sin^3\theta}}$, where l is the natural length of the string.

6. A circular tube of radius a contains an elastic string fastened at its highest point equal in length to $\dfrac{1}{8}$ of its circumference, and having attached to its other extremity a heavy particle which hanging vertically would double its length. The system revolves about the vertical diameter with an angular velocity $\sqrt{\dfrac{g}{a}}$. Find the position of relative equilibrium and prove that if the particle be slightly disturbed the time of a small oscillation is $\dfrac{2\pi\sqrt{\pi}}{\sqrt{\pi+4}}\sqrt{\dfrac{a}{g}}$.

7. A heavy uniform rod AB has its lower extremity A fixed to a vertical axis and an elastic string connects B to another point C in the axis such that $AC = \frac{AB}{\sqrt{2}} = a$; the whole is made to revolve round AC with such angular velocity that the string is double its natural length, and horizontal when the system is in relative equilibrium and then left to itself. If the rod be slightly disturbed in a vertical plane, prove that the time of a small oscillation is $2\pi\sqrt{\frac{4a}{21g}}$, the weight of the rod being sufficient to stretch the string to twice its length.

8. Three equal elastic strings AB, BC, CA surround a circular arc, the end A being fixed. At B and C two equal particles of mass m are fastened. If l be the natural length of each string supposed always stretched and λ the modulus of elasticity, show that if the equilibrium be disturbed the particles will be at equal distances from A after intervals $\pi\sqrt{\frac{ml}{\lambda}}$.

9. A particle of mass M is placed near the centre of a smooth circular horizontal table of radius a, strings are attached to the particle and pass over n smooth pullies which are placed at equal intervals round the circumference of the circle; to the other end of each of these strings a particle of mass M is attached; show that the time of a small oscillation of the system is $2\pi\left(\frac{2+n}{n}\frac{a}{g}\right)^{\frac{1}{2}}$.

10. In a circular tube of uniform bore containing air, slide two discs exactly fitting the tube. The two discs are placed initially so that the line joining their centres passes through the centre of the tube, and the air in the tube is initially of its natural density. One disc is projected so that the initial velocity of its centre is a small quantity w. If the inertia of the air be neglected, prove that the point on the axis of the tube equidistant from the centre of the discs moves uniformly and that the time of an oscillation of each disc is $2\pi\sqrt{\frac{Ma\pi}{4P}}$, where M is the mass of each disc, a the radius of the axis of tube, P the pressure of air on the disc in its natural state.

11. A uniform beam of mass M and length $2a$ can turn round a fixed horizontal axis at one end; to the other end of the beam a string of length l is attached and at the other end of the string a particle of mass m. If, during a small oscillation of the system, the inclination of the string to the vertical is always twice that of the beam, then $M(3l-a)=6m(l+a)$.

12. A conical surface of semivertical angle a is fixed with its axis inclined at an angle θ to the vertical, and a smooth cone of semivertical angle β is placed within it so that the vertices coincide. Show that the time of a small oscillation $=2\pi\sqrt{\frac{a\sin(\alpha-\beta)}{g\sin\theta}}$, where a is the distance of the centre of gravity of the cone from the vertex.

13. A number of bodies, the particles of which attract each other with forces varying as the distance, are capable of motion on certain curves and surfaces. Prove that if A, B, C be the moments of inertia of the system about three axes mutually at right angles through its centre of gravity, the positions of stable equilibrium will be found by making $A+B+C$ a minimum.

CHAPTER IX.

MOTION OF A BODY UNDER THE ACTION OF NO FORCES.

Solution of Euler's Equations.

508. *To determine the motion of a body about a fixed point, in the case in which there are no impressed forces.*

The equations of motion are by Art. 230,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 &= 0 \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 &= 0 \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 &= 0 \end{aligned}\right\};$$

multiplying these respectively by ω_1, ω_2, ω_3; adding and integrating, we get

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T \qquad\qquad (1),$$

where T is an arbitrary constant.

Again, multiplying the equations respectively by $A\omega_1$, $B\omega_2$, $C\omega_3$, we get, similarly,

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2 \qquad\qquad (2),$$

where G is an arbitrary constant.

To find a third integral, let

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2 \qquad\qquad (3);$$

$$\therefore\ \omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt} + \omega_3\frac{d\omega_3}{dt} = \omega\frac{d\omega}{dt};$$

then multiplying the original equations respectively by $\frac{\omega_1}{A}$, $\frac{\omega_2}{B}$, $\frac{\omega_3}{C}$, and adding, we get

$$\omega\frac{d\omega}{dt} = \left(\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right)\omega_1\omega_2\omega_3 \qquad (4)$$

$$= -\frac{(B-C)(C-A)(A-B)}{ABC}\,\omega_1\omega_2\omega_3.$$

But solving the equations (1), (2), (3), we get

$$\left.\begin{aligned} \omega_1^2 &= \frac{BC}{(A-C)(A-B)}\cdot(-\lambda_1+\omega^2) \\ \omega_2^2 &= \frac{CA}{(B-A)(B-C)}\cdot(-\lambda_2+\omega^2) \\ \omega_3^2 &= \frac{AB}{(C-B)(C-A)}\cdot(-\lambda_3+\omega^2) \end{aligned}\right\} \text{............(5)},$$

where $\lambda_1 = \dfrac{T(B+C)-G^2}{BC}$, with similar expressions for λ_2 and λ_3. Substituting in equation (4), we have

$$\omega\frac{d\omega}{dt} = \sqrt{(\lambda_1-\omega^2)(\lambda_2-\omega^2)(\lambda_3-\omega^2)} \text{ .... (6)}.$$

The integration of equation (6)* can be reduced without difficulty to depend on an elliptic integral. The integration can be effected in finite terms in two cases; when $A=B$, and when $G^2=TB$, where B is neither the greatest nor the least of the three quantities A, B, C. Both these cases will be discussed further on.

Ex. If right lines are measured along the three principal axes of the body from the fixed point, and inversely proportional to the radii of gyration round those axes, the sum of the squares of the velocities of their extremities is constant throughout the motion.

509. It will generally be supposed that A, B, C are in order of magnitude, so that A is greater than B, and B than C. The axis of B will be called the axis of mean moment. If we eliminate ω_1 from the equations (1) and (2), we have

$$AT-G^2=B(A-B)\,\omega_2^2+C(A-C)\,\omega_3^2,$$

which is essentially positive. In the same way we can show that $CT-G^2$ is negative. Thus the quantity $\dfrac{G^2}{T}$ may have any value lying between the greatest and least moments of inertia.

The three quantities $\lambda_1, \lambda_2, \lambda_3$ in Art. 508 are all positive quantities; for since $B+C-A$ is positive, and $\dfrac{G^2}{T}<A$, it follows that λ_1 is positive. The numerators of λ_2 and λ_3 are each greater than that of λ_1, and are therefore positive, the denominators are also positive; hence λ_2 and λ_3 are both positive. Also $\lambda_1-\lambda_2=\dfrac{TC-G^2}{ABC}(A-B)$, with similar expressions for $\lambda_2-\lambda_3$ and $\lambda_3-\lambda_1$. It easily follows that λ_3 is the greatest of the three, and λ_1 or λ_2 is the least according as $\dfrac{G^2}{T}$ is $>$ or $<$ B.

It follows from equations (5) that throughout the motion ω^2 must lie between λ_3 and the greater of the quantities λ_1 and λ_2.

* Euler's solution of these equations is given in the *ninth volume of the Quarterly Journal*, p. 361, by Prof. Cayley. Kirchhoff's and Jacobi's integrations by elliptic functions are given in an improved form by Prof. Greenhill in the fourteenth volume, pages 182 and 265. 1876.

510. The solution in terms of elliptic integrals has been effected in the following manner by Kirchhoff. If we put

$$\Delta(\phi) = \sqrt{1 - k^2 \sin^2 \phi}, \quad F(\phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

then k is called the modulus of F, and must be less than unity if F is to be real for all values of ϕ. The upper limit ϕ is called the amplitude of the elliptic integral F and is usually written am F. In the same way $\sin \phi$, $\cos \phi$, and $\Delta(\phi)$ are written sin am F, cos am F, and Δ am F.

We have by differentiation

$$\left.\begin{aligned} \frac{d\cos\phi}{dF} &= -\sin\phi \frac{d\phi}{dF} = -\sin\phi\Delta(\phi) \\ \frac{d\sin\phi}{dF} &= \cos\phi\frac{d\phi}{dF} = \cos\phi\Delta(\phi) \\ \frac{d\Delta(\phi)}{dF} &= -\frac{k^2\sin\phi\cos\phi}{\Delta(\phi)}\frac{d\phi}{dF} = -k^2\sin\phi\cos\phi \end{aligned}\right\} \quad \text{.................(1).}$$

These equations may be made identical with Euler's equations if we put $F = \lambda(t - \tau)$ and

$$\left.\begin{aligned} \omega_1 &= a\Delta \text{ am } \lambda(t-\tau) \\ \omega_2 &= b \sin \text{am } \lambda(t-\tau) \\ \omega_3 &= c \cos \text{am } \lambda(t-\tau) \end{aligned}\right\} \quad \text{.................................(2),}$$

$$\frac{A-B}{C} = -\frac{c\lambda}{ab}, \quad \frac{A-C}{B} = -\frac{b\lambda}{ca}, \quad \frac{B-C}{A} = -k^2\frac{a\lambda}{bc} \quad \text{.................(3).}$$

We have introduced here six new constants, viz. a, b, c, λ, k and τ. With these we may satisfy the three last equations and also any initial values of ω_1, ω_2, ω_3. The solution if real will also be complete.

When $t = \tau$ we have from (2) $\omega_1 = a$, $\omega_2 = 0$, and $\omega_3 = c$. Hence by Art. 508

$$Aa^2 + Cc^2 = T, \qquad A^2a^2 + C^2c^2 = G^2;$$

$$\therefore\ a^2 = \frac{G^2 - CT}{A(A-C)}, \qquad c^2 = \frac{AT - G^2}{C(A-C)}.$$

Dividing the second of equations (3) by the first, we have

$$\frac{b^2}{c^2} = \frac{A-C}{A-B}\frac{C}{B}; \quad \therefore\ b^2 = \frac{AT - G^2}{B(A-B)}.$$

Multiplying the first and second of equations (3), we obtain

$$\lambda^2 = \frac{(A-B)(G^2 - CT)}{ABC}.$$

The ratios of the right-hand sides of (3) are as $c^2 : b^2 : k^2a^2$, and these have just been found. Hence if the signs of a, b, c, λ be chosen to satisfy any one of the three equalities, the signs of all will be satisfied.

Dividing the last of equations (3) by either of the other two, we find

$$k^2 = \frac{B-C}{A-B}\frac{AT - G^2}{G^2 - CT}; \quad \therefore\ 1 - k^2 = \frac{A-C}{(A-B)}\frac{G^2 - BT}{(G^2 - CT)}.$$

If $G^2 > BT$ and A, B, C are in descending order of magnitude, the values of a^2, b^2, c^2 and λ^2 are all positive. Also k^2 is positive and less than unity. The solution is therefore real and complete.

If $G^2 < BT$ we must suppose A, B, C to be in ascending order of magnitude to obtain a real solution. If we may anticipate a phrase used by Poinsot, and which will be explained a little further on, we may say that the expression for ω_1 in this solution is to be taken for the angular velocity about that principal axis which is enclosed by the polhode.

If $G^2 = BT$ we have $k^2 = 1$ and

$$F = \int_0^\phi \frac{d\phi}{\cos\phi} = \frac{1}{2}\log\frac{1+\sin\phi}{1-\sin\phi};$$

$$\therefore \sin \operatorname{am} F = \frac{e^F - e^{-F}}{e^F + e^{-F}}.$$

Substituting in equations (2) the elliptic functions become exponential.

If $B = C$ we have $k^2 = 0$ and in this case $F = \phi$, so that $\operatorname{am} F = F$. If we again substitute in equations (2) the elliptic functions become trigonometrical.

The geometrical meaning of this solution will be given a little further on.

Poinsot's and MacCullagh's constructions for the motion.

511. The fundamental equations of motion of a body about a fixed point are

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2 \quad\text{.....................}\quad (1),$$

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T \quad\text{.....................}\quad (2).$$

These have been already obtained by integrating Euler's equations, but they also follow very easily from the principles of Angular Momentum, and Vis Viva.

Let the body be set in motion by an impulsive couple whose moment is G. Then we know by Art. 279, that throughout the whole of the subsequent motion, the moment of the momentum about every straight line which is fixed in space, and passes through the fixed point O, is constant, and is equal to the moment of the couple G about that line. Now by Art. 241, the moments of the momentum about the principal axes at any instant are $A\omega_1$, $B\omega_2$, $C\omega_3$. Let α, β, γ be the direction angles of the normal to the plane of the couple G referred to these principal axes as co-ordinate axes. Then we have

$$\left.\begin{aligned} A\omega_1 &= G\cos\alpha \\ B\omega_2 &= G\cos\beta \\ C\omega_3 &= G\cos\gamma \end{aligned}\right\} \quad\text{.......................}\quad (3),$$

adding the squares of these we get equation (1).

Throughout the subsequent motion the whole momentum of the body is equivalent to the couple G. It is therefore clear that if at any instant the body were acted on by an impulsive couple equal and opposite to the couple G, the body would be reduced to rest.

512. It follows from Art. 290, that the plane of this couple is the Invariable plane and the normal to it the Invariable line. This line is absolutely fixed in space, and the equations (3) give the direction cosines of this line* referred to axes moving in the body.

It appears from these equations, that if the body be set in rotation about an axis whose direction cosines are (l, m, n) when referred to the principal axes at the fixed point, then the direction cosines of the invariable line are proportional to Al, Bm, Cn. If the axes of reference are not the principal axes of the body at the fixed point, the direction cosines of the invariable line will, by Art. 240, be proportional to $Al - Fm - En$, $Bm - Dn - Fl$, and $Cn - El - Dm$, where the letters have the meaning given to them in Art. 15.

513. Since the body moves under the action of no impressed forces, we know that the Vis Viva will be constant throughout the motion. Hence by Art. 348, we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T,$$

where T† is a constant to be determined from the initial values of ω_1, ω_2, ω_3.

The equations (1), (2), (3) will suffice to determine the path in space described by every particle of the body, but not the position at any given time.

* That the straight line whose equations referred to the moving principal axes are $\frac{x}{A\omega_1} = \frac{y}{B\omega_2} = \frac{z}{C\omega_3}$ is absolutely fixed in space may be also proved thus, if we assume the truth of equation (1) in the text. Let x, y, z be the co-ordinates of any point P in the straight line at a given distance r from the origin, then each of the equalities in the equation to the straight line is equal to $\frac{r}{G}$ and is therefore constant. The actual velocity of P in space resolved parallel to the instantaneous position of the axis of x is

$$= \frac{dx}{dt} - y\omega_3 + z\omega_2 = \frac{r}{G}\left\{A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3\right\}.$$

But this is zero, by Euler's equation. Similarly the velocities parallel to the other axes are zero.

† It should be observed that in this Chapter T represents the whole vis viva of the body. In treating of Lagrange's equations in Chapter VII. it was convenient to let T represent *half* the vis viva of the system.

514. *To explain Poinsot's representation of the motion by means of the momental ellipsoid.*

Let the momental ellipsoid at the fixed point be constructed, and let its equation be

$$Ax^2 + By^2 + Cz^2 = M\epsilon^4.$$

Let r be the radius vector of this ellipsoid coinciding with the instantaneous axis, and p the perpendicular from the centre on the tangent plane at the extremity of r. Also let ω be the angular velocity about the instantaneous axis.

The equations to the instantaneous axis are $\dfrac{x}{\omega_1} = \dfrac{y}{\omega_2} = \dfrac{z}{\omega_3}$, and if (x, y, z) be the co-ordinates of the extremity of the length r, each of these fractions is equal to $\dfrac{r}{\omega}$.

Substituting in the equation to the ellipsoid, we have

$$(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)\frac{r^2}{\omega^2} = M\epsilon^4;$$

$$\therefore\ \omega = \sqrt{\frac{T}{M\epsilon^2}}\,\frac{r}{\epsilon}.$$

Again the expression for the perpendicular on the tangent plane at (x, y, z) is known to be $\dfrac{1}{p^2} = \dfrac{A^2x^2 + B^2y^2 + C^2z^2}{M^2\epsilon^8}$, substituting as before we get

$$\frac{1}{p^2} = \frac{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}{M^2\epsilon^8}\cdot\frac{r^2}{\omega^2} = \frac{G^2}{M^2\epsilon^8}\cdot\frac{M\epsilon^4}{T};$$

$$\therefore\ p = \frac{\sqrt{MT}}{G}\cdot\epsilon^2.$$

The equation to the tangent plane at the point (x, y, z) is

$$Ax\xi + By\eta + Cz\zeta = M\epsilon^4,$$

substituting for (x, y, z) we see that the equations to the perpendicular from the origin are

$$\frac{\xi}{A\omega_1} = \frac{\eta}{B\omega_2} = \frac{\zeta}{C\omega_3};$$

but these are the equations to the invariable line. Hence this perpendicular is fixed in space.

From these equations we infer

(1) *The angular velocity about the radius vector round which the body is turning varies as that radius vector.*

(2) *The resolved part of the angular velocity about the perpendicular on the tangent plane at the extremity of the instantaneous axis is constant.* This theorem is due to Lagrange.

For the cosine of the angle between the perpendicular and the radius vector $= \frac{p}{r}$. Hence the resolved angular velocity is $= \omega \frac{p}{r} = \frac{T}{G}$, which is constant.

(3) *The perpendicular on the tangent plane at the extremity of the instantaneous axis is fixed in direction, viz. normal to the invariable plane, and constant in length.*

The motion of the momental ellipsoid is therefore such that, its centre being fixed, it always touches a fixed plane, and the point of contact, being in the instantaneous axis, has no velocity. *Hence the motion may be represented by supposing the momental ellipsoid to roll on the fixed plane with its centre fixed.*

515. Ex. 1. If the body while in motion be acted on by any impulsive couple whose plane is perpendicular to the invariable line, show that the momental ellipsoid will continue to roll on the same plane as before, but the rate of motion will be altered.

Ex. 2. If a plane be drawn through the fixed point parallel to the invariable plane, prove that the area of the section of the momental ellipsoid cut off by this plane is constant throughout the motion.

Ex. 3. The sum of the squares of the distances of the extremities of the principal diameters of the momental ellipsoid from the invariable line is constant throughout the motion. This result is due to Poinsot.

Ex. 4. A body moves about a fixed point O under the action of no forces. Show that if the surface $Ax^2 + By^2 + Cz^2 = M(x^2 + y^2 + z^2)^2$ be traced in the body, the principal axes at O being the axes of co-ordinates, this surface throughout the motion will roll on a fixed sphere.

516. To assist our conception of the motion of the body, let us suppose it so placed, that the plane of the couple G, which would set it in motion, is horizontal. Let a tangent plane to the momental ellipsoid be drawn parallel to the plane of the couple G, and let this plane be fixed in space. Let the ellipsoid roll on this fixed plane, its centre remaining fixed, with an angular velocity which varies as the radius vector to the point of contact, and let it carry the given body with it. We shall then have constructed the motion which the body would have assumed if it had been left to itself after the initial action of the impulsive couple G*.

* Prof. Sylvester has pointed out a *dynamical* relation between the free rotating body and the ellipsoidal top, as he calls Poinsot's central ellipsoid. If a *material*

The point of contact of the ellipsoid with the plane on which it rolls traces out two curves, one on the surface of the ellipsoid, and one on the plane. The first of these is fixed in the body and is called the *polhode*, the second is fixed in space and is called the *herpolhode*. The equations to any polhode referred to the principal axes of the body may be found from the consideration that the length of the perpendicular on the tangent plane to the ellipsoid at any point of the polhode is constant. Hence its equations are

$$\left.\begin{aligned} A^2x^2 + B^2y^2 + C^2z^2 &= \frac{MG^2\epsilon^4}{T} \\ Ax^2 + By^2 + Cz^2 &= M\epsilon^4 \end{aligned}\right\}.$$

Eliminating y, we have

$$A(A-B)\,x^2 + C(C-B)\,z^2 = \left(\frac{G^2}{T} - B\right) M\epsilon^4.$$

Hence if B be the axis of greatest or least moment of inertia, the signs of the coefficients of x^2 and z^2 will be the same, and the projection of the polhode will be an ellipse. But if B be the axis of mean moment of inertia, the projection is an hyperbola.

A polhode is therefore a closed curve drawn round the axis of greatest or least moment, and the concavity is turned towards the axis of greatest or least moment according as $\frac{G^2}{T}$ is greater or less than the mean moment of inertia. The boundary line which separates the two sets of polhodes is that polhode whose projection on the plane perpendicular to the axis of mean moment is an hyperbola whose concavity is turned neither to the axis of greatest, nor to the axis of least moment. In this case $G^2 = BT$, and the projection consists of two straight lines whose equation is

$$A(A-B)\,x^2 - C(B-C)\,z^2 = 0.$$

This polhode consists of two ellipses passing through the axis of mean moment, and corresponds to the case in which the perpendicular on the tangent plane is equal to the mean axis of the ellipsoid. This polhode is called the *separating polhode*.

Since the projection of the polhode on one of the principal planes is always an ellipse, the polhode must be a re-entering curve.

ellipsoidal top be constructed of uniform density, similar to Poinsot's central ellipsoid, and if with its centre fixed it be set rolling on a perfectly rough horizontal plane, it will represent the motion of the free rotating body not in space only, but also in time: the body and the top may be conceived as continually moving round the same axis, and at the same rate, at each moment of time. The reader is referred to the memoir in the *Philosophical Transactions* for 1866.

517. To find the motion of the extremity of the instantaneous axis along the polhode which it describes we have merely to substitute from the equations

$$\frac{\omega_1}{x} = \frac{\omega_2}{y} = \frac{\omega_3}{z} = \frac{\omega}{r} = \sqrt{\frac{T}{M}}\frac{1}{\epsilon^2},$$

in any of the equations of Art. 508. For example we thus obtain

$$\frac{dx}{dt} = \sqrt{\frac{T}{M}}\frac{B-C}{A}\frac{yz}{\epsilon^2}, \text{ \&c., \&c.,}$$

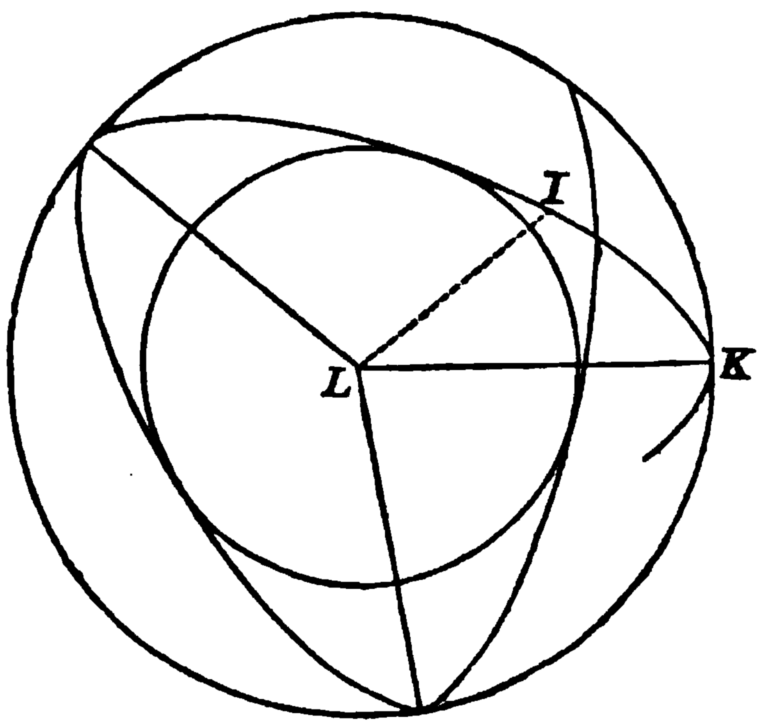
$$x^2 = \frac{BC}{(A-C)(A-B)}(-\lambda_1' + r^2), \text{ \&c., \&c.}$$

Ex. 1. A point P moves along a polhode traced on an ellipsoid, show that the length of the normal between P and any one of the principal planes at the centre is constant. Show also that the normal traces out on a principal plane a conic similar to the focal conic in that plane. Also the measure of curvature of an ellipsoid along any polhode is constant.

Ex. 2. Show that the line OJ used in Art. 234 to find the pressure on the fixed point is at right angles to the invariable line, and parallel to the tangent plane to the momental ellipsoid at the point where the invariable line cuts it. Show also that $\Omega'^4 = -\omega^4 + \omega^2\frac{2Tp_2 - G^2p_1}{p_3} - \frac{p_2^2T^2 - (p_1p_2 + p_3)G^2T + p_2G^4}{p_3^2}$, where p_1, p_2, p_3 are the sum of the products A, B, C taken respectively one, two and three together.

518. Since the herpolhode is traced out by the points of contact of an ellipsoid rolling about its centre on a fixed plane, it is clear that the herpolhode must always lie between two circles which it alternately touches. The common centre of these circles will be the foot of the perpendicular from the fixed centre O on the fixed plane. To find the radii let OL be this perpendicular, and I be the point of contact. Let $LI = \rho$. Then we have

$$\rho^2 = r^2 - p^2 = \frac{M\epsilon^4}{T}\left(\omega^2 - \frac{T^2}{G^2}\right).$$



The radii will therefore be found by substituting for ω^2 its greatest and least values. But by Art. 509, these limits are λ_2, and the greater of the two quantities λ_1, λ_3.

The herpolhode is not in general a re-entering curve; but if the angular distance of the two points in which it successively touches the same circle be commensurable with 2π, it will be re-entering, i.e. the same path will be traced out repeatedly on the fixed plane by the point of contact.

519. *To explain MacCullagh's representation of the motion by means of the ellipsoid of gyration.*

This ellipsoid is the reciprocal of the momental ellipsoid, and the motion of the one ellipsoid may be deduced from that of the other by reciprocating the properties proved in the preceding Articles. We find,

(1) The equation to the ellipsoid referred to its principal axes is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M}.$$

(2) This ellipsoid moves so that its superficies always passes through a point fixed in space. This point lies in the invariable line at a distance $\dfrac{G}{\sqrt{MT}}$ from the fixed point. By Art. 509 we know that this distance is less than the greatest, and greater than the least semi-diameter of the ellipsoid.

(3) The perpendicular on the tangent plane at the fixed point is the instantaneous axis of rotation, and the angular velocity of the body varies inversely as the length of this perpendicular. If p be the length of this perpendicular, then $\omega = \dfrac{1}{p}\sqrt{\dfrac{T}{M}}$.

(4) The angular velocity about the invariable line is constant and $= \dfrac{T}{G}$.

The corresponding curve to a polhode is the path described on the moving surface of the ellipsoid by the point fixed in space. This curve is clearly a sphero-conic. The equations to the sphero-conic described under any given initial conditions are easily seen to be

$$x^2 + y^2 + z^2 = \frac{G^2}{MT}, \quad \frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M}.$$

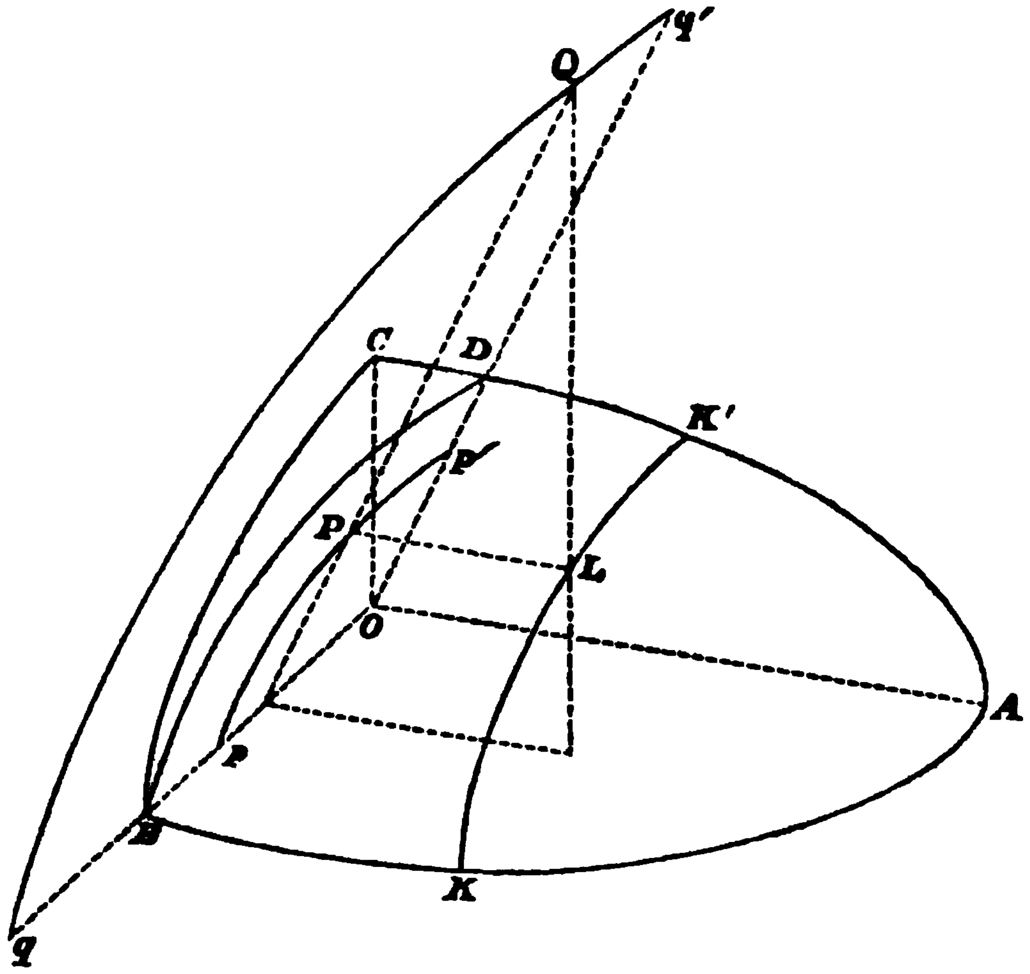
These sphero-conics may be shown to be closed curves round the axes of greatest and least moment. But in one case, viz.

when $\frac{G^2}{T}=B$, where B is neither the greatest nor least moment of inertia, the sphero-conic becomes the two central circular sections of the ellipsoid of gyration.

The motion of the body may thus be constructed by means of either of these ellipsoids. The momental ellipsoid resembles the general shape of the body more nearly than the ellipsoid of gyration. It is protuberant where the body is protuberant, and compressed where the body is compressed. The exact reverse of this is the case in the ellipsoid of gyration.

520. MacCullagh has used the ellipsoid of gyration to obtain a geometrical interpretation of the solution of Euler's equations in terms of elliptic integrals.

The ellipsoid of gyration moves so as always to touch a point L fixed in space. Let us now project the point L on a plane passing through the axis of mean moment and making an angle α with the axis of greatest moment. This projection may be effected by drawing a straight line parallel to either the axis of greatest moment or least moment. We thus obtain two projections which we will call P and Q. These points will be in a plane PQL which is always perpendicular to the axis of mean moment. As the body moves about O the point L describes on the surface of the ellipsoid of gyration a sphero-conic KK', and the points P, Q describe two curves pp', qq' on the plane of projection OBD. If the sphero-conic as in the figure enclose the extremity A of the axis of greatest moment, the curve inside the ellipsoid is formed by the projection parallel to the axis of greatest moment, but if the sphero-conic enclose the axis of least moment, the inner curve



is formed by the projection parallel to that axis. The point P which describes the inner curve will obviously travel round its projection, while the point Q which describes the outer curve will oscillate between two limits obtained by drawing tangents to the inner projection at the points where it cuts the axis of mean moment.

Since the direction cosines of OL are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$ it is easy to see that, if x, y, z are the co-ordinates of L,

$$\frac{x}{A\omega_1}=\frac{y}{B\omega_2}=\frac{z}{C\omega_3}=\frac{r}{G}=\frac{1}{\sqrt{MT}} \quad\text{.......................... (1).}$$

Let $OP=\rho$, $OQ=\rho'$, and let the angles these radii vectores make with the plane containing the axes of greatest and least moment be ϕ and ϕ' measured in the direction BD so that $DOP=-\phi$, $DOQ=-\phi'$: we then have

$$\left.\begin{aligned}-\rho\sin\phi=y=B\omega_2(MT)^{-\frac{1}{2}}\\ \rho\cos\phi\sin\alpha=z=C\omega_3(MT)^{-\frac{1}{2}}\end{aligned}\right\} \quad\text{..................................(2),}$$

$$\left.\begin{aligned}\rho'\cos\phi'\cos\alpha=x=A\omega_1(MT)^{-\frac{1}{2}}\\ -\rho'\sin\phi' \qquad =y=B\omega_2(MT)^{-\frac{1}{2}}\end{aligned}\right\} \quad\text{..................................(3).}$$

It is proved in treatises on solid geometry that, if the plane on which the projection is made is one of the circular sections of the ellipsoid, the projections will be circles. This result may be verified by finding ρ or ρ' from these equations. Remembering that ρ and ρ' are constants, let us substitute in Euler's equation

$$B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\omega_1=0$$

from (2) and the first of equations (3). We have

$$\rho\frac{d\phi}{dt}=\frac{A-C}{AC}\sqrt{MT}\,\rho\rho'\sin\alpha\cos\alpha\cos\phi'.$$

Since $\rho'\cos\phi'$ is the ordinate of Q, we see that *the velocity of* P *varies as the ordinate of* Q, *and in the same way the velocity of* Q *varies as the ordinate of* P.

To find the constants ρ, ρ' we notice that ρ is the value of y obtained from the equations to the sphero-conic when $z=0$. We thus have

$$\rho^2=\frac{(AT-G^2)B}{MT(A-B)},\quad \rho'^2=\frac{(G^2-CT)B}{MT(B-C)},$$

the latter being obtained from the former by interchanging the letters A and C. Hence

$$\begin{pmatrix}\text{velocity}\\ \text{of } P\end{pmatrix}=\frac{\sqrt{B-C}}{\sqrt{ABC}}\sqrt{AT-G^2}\begin{pmatrix}\text{ordinate}\\ \text{of } Q\end{pmatrix},$$

$$\begin{pmatrix}\text{velocity}\\ \text{of } Q\end{pmatrix}=\frac{\sqrt{A-B}}{\sqrt{ABC}}\sqrt{G^2-CT}\begin{pmatrix}\text{ordinate}\\ \text{of } P\end{pmatrix}.$$

521. Since $\rho'\sin\phi'=\rho\sin\phi$, we have by substitution

$$\frac{d\phi}{dt}=\lambda\sqrt{1-\frac{\rho^2}{\rho'^2}\sin^2\phi},$$

where λ^2 has the same value as in Art. 510. Let us suppose ϕ expressed in terms of t by the elliptic integral

$$\lambda(t-\tau)=\int_0^\phi\frac{d\phi}{\sqrt{1-\frac{\rho^2}{\rho'^2}\sin^2\phi}},$$

so that $\phi=\operatorname{am}\lambda(t-\tau)$. Substituting this value of ϕ in equations (2) or (3), we obtain the values of ω_1, ω_2, ω_3 expressed in terms of the time.

Ex. Investigate the corresponding theorem for the momental ellipsoid.

522. If a body be set in rotation about any principal axis at a fixed point, it will continue to rotate about that axis as a permanent axis. But the three principal axes at the fixed point do not possess equal degrees of stability. If any small disturbing cause act on the body, the axis of rotation will be moved into a neighbouring polhode. If this polhode be a small nearly circular curve enclosing the original axis of rotation, the instantaneous axis will never deviate far in the body from the principal axis which was its original position. The herpolhode also will be a curve of small dimensions, so that the principal axis will never deviate far from a straight line fixed in space. In this case the rotation is said to be *stable*. But if the neighbouring polhode be not nearly circular, the instantaneous axis will deviate far from its original position in the body. In this case a very small disturbance may produce a very great change in the subsequent motion, and the rotation is said to be *unstable*.

If the initial axis of rotation be the axis OB of mean moment, the neighbouring polhodes all have their convexities turned towards B. Unless, therefore, the cause of disturbance be such that the axis of rotation is displaced along the separating polhode, the rotation must be unstable. If the displacement be along the separating polhode, the axis may have a tendency to return to its original position. This case will be considered a little further on, and for this particular displacement the rotation may be said to be stable.

If the initial axis of rotation be the axis of greatest or least moment, the neighbouring polhodes are ellipses of greater or less eccentricity. If they be nearly circular, the rotation will certainly be stable; if very elliptical, the axis will recede far from its initial position, and the rotation may be called unstable. If OC be the axis of initial rotation, the ratio of the squares of the axes of the neighbouring polhode is ultimately $\dfrac{A(A-C)}{B(B-C)}$. It is therefore necessary for the stability of the rotation that this ratio should not differ much from unity.

It is well known that the steadiness or stability of a moving body is much increased by a rapid rotation about a principal axis. The reason of this is evident from what precedes. If the body be set rotating about an axis very near the principal axis of greatest or least moment, both the polhode and herpolhode will generally be very small curves, and the direction of that principal axis of the body will be very nearly fixed in space. If now a small impulse f act on the body, the effect will be to alter slightly the position of the instantaneous axis. It will be moved from one polhode to another very near the former, and thus the angular position of the axis in space will not be much affected. Let Ω be the angular velocity of the body, ω that generated by the im-

pulse, then, by the parallelogram of angular velocities, the change in the position of the instantaneous axis cannot be greater than $\sin^{-1}\frac{\omega}{\Omega}$. If therefore Ω be great, ω must also be great, to produce any considerable change in the axis of rotation. But if the body has no initial rotation Ω, the impulse may generate an angular velocity ω about an axis not nearly coincident with a principal axis. Both the polhode and the herpolhode may then be large curves, and the instantaneous axis of rotation will move about both in the body and in space. The motion will then appear very unsteady. In this manner, for example, we may explain why in the game of cup and ball, spinning the ball about a vertical axis makes it more easy to catch on the spike. Any motion caused by a wrong pull of the string or by gravity will not produce so great a change of motion as it would have done if the ball had been initially at rest. The fixed direction of the earth's axis in space is also due to its rotation about its axis of figure. In rifles, a rapid rotation is communicated to the bullet about an axis in the direction in which the bullet is moving. It follows, from what precedes, that the axis of rotation will be nearly unchanged throughout the motion. One consequence is that the resistance of the air acts in a known manner on the bullet, the amount of which may therefore be calculated and allowed for.

On the Cones described by the Invariable and Instantaneous Axes.

523. It is clear from what precedes that there are two important straight lines whose motions we should consider. These are the invariable line and the instantaneous axis. The first of these is fixed in space, but as the body moves the invariable line describes a cone in the body, which by Art. 519 intersects the ellipsoid of gyration in a sphero-conic. This cone is usually called the *Invariable Cone*. The instantaneous axis describes both a cone in the body and a cone in space. By Art. 514, the cone described in the body intersects the momental ellipsoid in a polhode, and the cone described in space intersects the fixed plane on which the momental ellipsoid rolls in a herpolhode. These two cones may be called respectively the *instantaneous cone* and the *cone of the herpolhode*.

524. There are various ways in which we may study the properties of these cones. We may have recourse to solid geometry. We may examine their curves of intersection with the momental ellipsoid or the ellipsoid of gyration, as Poinsot and MacCullagh have done. We may also examine by the help of spherical trigonometry their curves of intersection with a sphere

whose centre is at the fixed point, and which is either fixed in the body or fixed in space at our pleasure. This will be found convenient when we wish to use a diagram.

525. Let the principal axes at the fixed point be taken as the axes of co-ordinates. The axes of reference are therefore fixed in the body but moving in space. By Art. 512, the direction-cosines of the invariable line are $\frac{A\omega_1}{G}$, $\frac{B\omega_2}{G}$, $\frac{C\omega_3}{G}$; and the direction-cosines of the instantaneous axis are $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$. From the equations (1) and (2) of Art. 511, we easily find

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = (A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2)\frac{T}{G^2}.$$

If we take the co-ordinates x, y, z to be proportional to the direction-cosines of either of these straight lines and eliminate ω_1, ω_2, ω_3 by the help of this equation, we obtain the equation to the corresponding cone described by that straight line. In this way we find that the cones described in the body by the invariable line and the instantaneous axis are respectively

$$\frac{AT - G^2}{A}x^2 + \frac{BT - G^2}{B}y^2 + \frac{CT - G^2}{C}z^2 = 0,$$

$$A(AT - G^2)x^2 + B(BT - G^2)y^2 + C(CT - G^2)z^2 = 0.$$

These cones become two planes when the initial conditions are such that $G^2 = BT$.

Ex. 1. Show that the circular sections of the invariable cone are parallel to those of the ellipsoid of gyration and perpendicular to the asymptotes of the focal conic of the momental ellipsoid.

526. There is a third straight line whose motion it is sometimes convenient to consider, though it is not nearly so important as either the invariable line or the instantaneous axis. If x, y, z be the co-ordinates of the extremity of a radius vector of an ellipsoid referred to its principal diameters as axes and if a, b, c be the semi-axes, the straight line whose direction-cosines are $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ is called the *eccentric line* of that radius vector. Taking this definition, it is easy to see that the direction-cosines of the eccentric line of the instantaneous axis with regard to the momental ellipsoid are $\omega_1\sqrt{\frac{A}{T}}$, $\omega_2\sqrt{\frac{B}{T}}$, $\omega_3\sqrt{\frac{C}{T}}$. These are also the direction-cosines of the eccentric line of the invariable line with regard to the ellipsoid of gyration. This straight line may therefore be called simply the *eccentric line* and the cone described by it in the body may be called the *eccentric cone*.

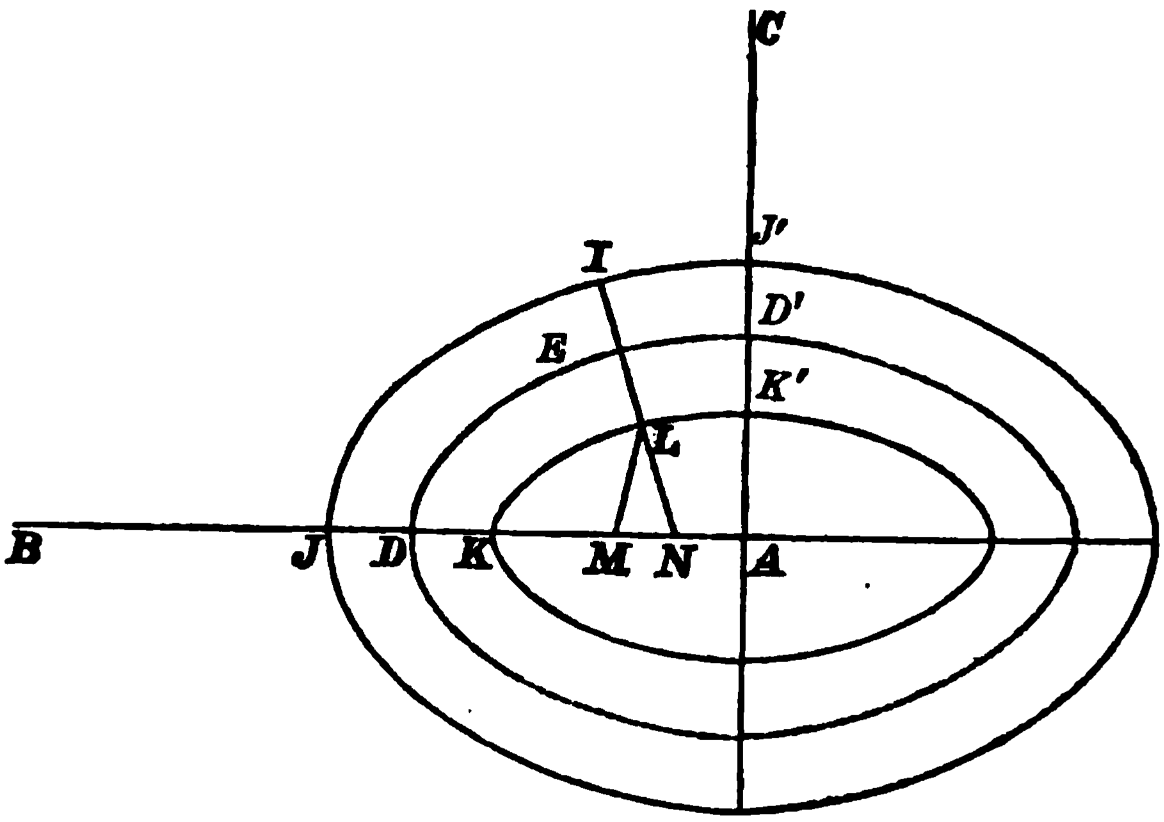
Ex. 1. The equation to the eccentric cone referred to the principal axes at the fixed point is

$$(AT - G^2)x^2 + (BT - G^2)y^2 + (CT - G^2)z^2 = 0.$$

This cone has the same circular sections as the momental ellipsoid and cuts that ellipsoid in a sphero-conic.

Ex. 2. The polar plane of the instantaneous axis with regard to the eccentric cone touches the invariable cone along the corresponding position of the invariable line. Thus the invariable and instantaneous cones are reciprocals of each other with regard to the eccentric cone.

527. Let a sphere of radius unity be described with its centre at the fixed point O about which the body is free to turn. Let this sphere be fixed in the body, and therefore move with it in space. Let the invariable line, the instantaneous axis, and the eccentric line cut this sphere in the points L, I, and E respectively. Also let the principal axes cut the sphere in A, B, C. It is clear that the intersections of the invariable, instantaneous, and eccentric cones with this sphere will be three sphero-conics which are represented in the figure by the lines KK', JJ', DD', respectively. The eye is supposed to be situated on the axis OA, viewing the sphere from a considerable distance. All great circles on the sphere are represented by straight lines. Since the cones are co-axial with the momental ellipsoid, these sphero-conics are symmetrical about the principal planes of the body. The intersections of these principal planes with the sphere will be three arcs of great circles, and the portions of these arcs cut off by any sphero-conic are called axes of that sphero-conic. If we put $z = 0$ in the equations to any one of the three cones, the value of $\frac{y}{x}$ is the tangent of that semi-axis of the sphero-conic which lies in the



plane of xy. Similarly, putting $y = 0$, we find the axis in the plane of xz. If (a, b), (a', b'), (α, β) be the semi-axes of the invariable, instantaneous, and eccentric sphero-conics respectively, we thus find

$$\frac{\tan a}{B}=\frac{\tan a'}{A}=\frac{\tan \alpha}{\sqrt{AB}}=\frac{\sqrt{AT-G^2}}{\sqrt{G^2-BT}}\frac{1}{\sqrt{AB}},$$

$$\frac{\tan b}{C}=\frac{\tan b'}{A}=\frac{\tan \beta}{\sqrt{AC}}=\frac{\sqrt{AT-G^2}}{\sqrt{G^2-CT}}\frac{1}{\sqrt{AC}}.$$

The first of these two sets gives the axes in the plane AOB, the second those in the plane AOC. The former will be imaginary if $G^2 < BT$. In this case the sphero-conics do not cut the plane AOB. The sphero-conics will therefore have their concavities turned towards the extremities of the axes OA or OC, i.e. towards the extremities of the axes of greatest or least moment according as G^2 is $>$ or $<$ BT.

Ex. 1. If we put $1-e^2=\frac{\sin^2 b}{\sin^2 a}$ we may define e to be the eccentricity of the sphero-conic whose semi-axes are a and b. If e and e' be the eccentricities of the invariable and eccentric sphero-conics respectively, prove that $e^2=\frac{A}{B}\frac{B-C}{A-C}$ and $e'^2=\frac{B-C}{A-C}$ so that both these eccentricities are independent of the initial conditions.

Ex. 2. If the radius of the sphere had been taken equal to $\left(\frac{G^2}{MT}\right)^{\frac{1}{2}}$ instead of unity, show that it would have intersected the ellipsoid of gyration along the invariable ellipse, and if the radius had been $\left(\frac{MTe^4}{G^2}\right)^{\frac{1}{2}}$, it would have intersected the momental ellipsoid along the eccentric ellipse.

Ex. 3. A body is set rotating with an initial angular velocity n about an axis which very nearly coincides with a principal axis OC at a fixed point O. The motion of the instantaneous axis in the body may be found by the following formulæ. Let a sphere be described whose centre is O, and let I be the extremity of the radius vector which is the instantaneous axis at the time t. If (x, y) be the co-ordinates of the projection of I on the plane AOB referred to the principal axes OA, OB, then

$$x=\sqrt{B(B-C)}\,L\sin(pnt+M),$$

$$y=\sqrt{A(A-C)}\,L\cos(pnt+M),$$

where $p^2=\frac{(B-C)(A-C)}{AB}$, and L, M are two arbitrary constants depending on the initial values of x, y.

Ex. 4. If in the last question L be the point in which the sphere cuts the invariable line, if (ρ, θ) be the spherical polar co-ordinates of C with regard to L as origin, and a the radius of the sphere, then

$$\rho^2=n^2\frac{AB}{2G^2}L^2\{2AB-C(A+B)+(A-B)C\cos 2(pnt+M)\},$$

$$\theta=\frac{T}{G}t+\frac{CT-G^2}{CG}\int\frac{a^2dt}{\rho^2}.$$

528. *To find the motion of the invariable line and the instantaneous axis in the body.*

Since the invariable line OL is fixed in space and the body is turning about OI as instantaneous axis, it is evident that the direction of motion of OL in the body is perpendicular to the plane IOL. Hence on a sphere whose centre is at O *the arc IL is normal to the sphero-conic described by the invariable line.* This simple relation will serve to connect the motions of the invariable line and the instantaneous axis along their respective sphero-conics.

529. Let v be the velocity of the invariable line along its sphero-conic, then since the body is turning about OI with angular velocity ω, and OL is unity, we have $v = \omega \sin LOI$. But by Art. 514 $\frac{T}{G} = \omega \cos LOI$. Eliminating ω we have $v = \frac{T}{G} \tan LOI$.

530. Produce the arc IL to cut the axis AK in N, so that LN is a normal to the sphero-conic described by the invariable line. Taking the principal axes at the fixed point O as axes of reference, the direction-cosines of OL and OI are respectively proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$, and ω_1, ω_2, ω_3. The equation to the plane LOI is

$$(B - C)\,\omega_2\omega_3 x + (C - A)\,\omega_3\omega_1 y + (A - B)\,\omega_1\omega_2 z = 0.$$

This plane intersects the plane of xy in the straight line ON, hence putting $z = 0$, we find the direction-cosines of ON to be proportional to $(A - C)\,\omega_1$, $(B - C)\,\omega_2$, and 0. Hence

$$\cos LON = \frac{A\,(A - C)\,\omega_1^2 + B\,(B - C)\,\omega_2^2}{G\sqrt{(A - C)^2\,\omega_1^2 + (B - C)^2\,\omega_2^2}}.$$

The numerator of this expression is easily seen to be $G^2 - CT$. Expanding the quantity under the root we have

$$A^2\omega_1^2 + B^2\omega_2^2 - 2C\,(A\omega_1^2 + B\omega_2^2) + C^2\,(\omega_1^2 + \omega_2^2),$$

which is clearly the same as

$$G^2 - C^2\omega_3^2 - 2C\,(T - C\omega_3^2) + C^2\,(\omega^2 - \omega_3^2).$$

Substituting we find

$$\cos LON = \frac{G^2 - CT}{G\sqrt{G^2 - 2CT + C^2\omega^2}};$$

$$\therefore \tan LON = \frac{C\sqrt{G^2\omega^2 - T^2}}{G^2 - CT}.$$

But $\frac{T}{G} = \omega \cos LOI$, $\therefore \tan LOI = \frac{\sqrt{G^2\omega^2 - T^2}}{T}$. *Hence the ratio* $\frac{\tan LOI}{\tan LON} = \frac{G^2 - CT}{CT}$, *and is therefore constant throughout the motion.*

Treating the other principal planes in the same way, we see that this proposition supplies us with a geometrical meaning for the three expressions $\frac{G^2}{AT} - 1$, $\frac{G^2}{BT} - 1$, and $\frac{G^2}{CT} - 1$.

Combining this result with that given in the last Article, we see that the

$$\left.\begin{array}{r}\text{velocity of } L\\ \text{along its conic}\end{array}\right\} = \frac{G^2 - CT}{CG} \tan n,$$

where n is the angle LON. If we adopt the conventions of spherical trigonometry, n is also the length of the arc normal to the sphero-conic intercepted between the curve and the principal plane AB of the body.

Ex. 1. If the focal lines of the invariable cone cut the sphere in S and S', these points are called the foci of the sphero-conic. Prove that the velocity of L resolved perpendicular to the arc SL is constant throughout the motion and equal to $\frac{1}{G}\left\{\frac{(G^2 - BT)(AT - G^2)}{AB}\right\}^{\frac{1}{2}}$. If LM be an arc of a great circle perpendicular to the axis containing the foci, and ρ be the arc SL, prove also that

$$\frac{d\rho}{dt} = -\frac{G}{C}\left\{\frac{(A - C)(B - C)}{AB}\right\}^{\frac{1}{2}} \sin LM.$$

Ex. 2. Prove that the velocity of L resolved perpendicular to the central radius vector AL is $\frac{AT - G^2}{AG} \cot AL$.

Ex. 3. If r, r', r'' be the lengths of the arcs joining the extremity A of a principal axis to the extremities L, I, E of the invariable line, instantaneous axis, and eccentric line respectively; θ, θ', θ'' the angles these arcs make with any principal plane AOB, prove that

$$\frac{\cos r}{AT} = \frac{\cos r'}{G^2 \cos \zeta} = \frac{\cos r''}{G\sqrt{AT}},$$

$$\frac{\tan \theta}{C} = \frac{\tan \theta'}{B} = \frac{\tan \theta''}{\sqrt{BC}},$$

where $\zeta = \text{arc } LI$. This theorem will enable us to discover in what manner the motions of the three points L, I, E are related to each other.

Ex. 4. Show that the velocity of the instantaneous axis along its sphero-conic is $\frac{G}{T}\frac{G^2 - CT}{AB} \tan n' \cos \zeta$, where n' is the length of the normal to the instantaneous sphero-conic intercepted between the curve and the arc AB, and $\zeta = \text{arc } LI$.

Comparing this result with the corresponding formula for the motion of L given in Art. 530, we see that for every theorem relating to the motion of L in its sphero-conic there is a corresponding theorem for the motion of I. For example, if S' be a focus of the instantaneous sphero-conic, we see that the velocity of I resolved perpendicular to the focal radius vector $S'I$ bears a constant ratio to $\cos LI$. This constant ratio is $\dfrac{G}{CT}\left\{\dfrac{(AT-G^2)(G^2-BT)}{AB}\right\}^{\frac{1}{2}}$.

Ex. 5. Show that the velocity of the eccentric line along its sphero-conic is $\dfrac{G^2-CT}{\sqrt{ABCT}}\tan n''$, where n'' is the length of the arc normal to the sphero-conic intercepted between the curve and the principal arc AB.

Ex. 6. Prove that (velocity of $E)^2$ − (velocity of $L)^2$ = constant. Show also that this constant $=\dfrac{(AT-G^2)(BT-G^2)(CT-G^2)}{ABCG^2T}$.

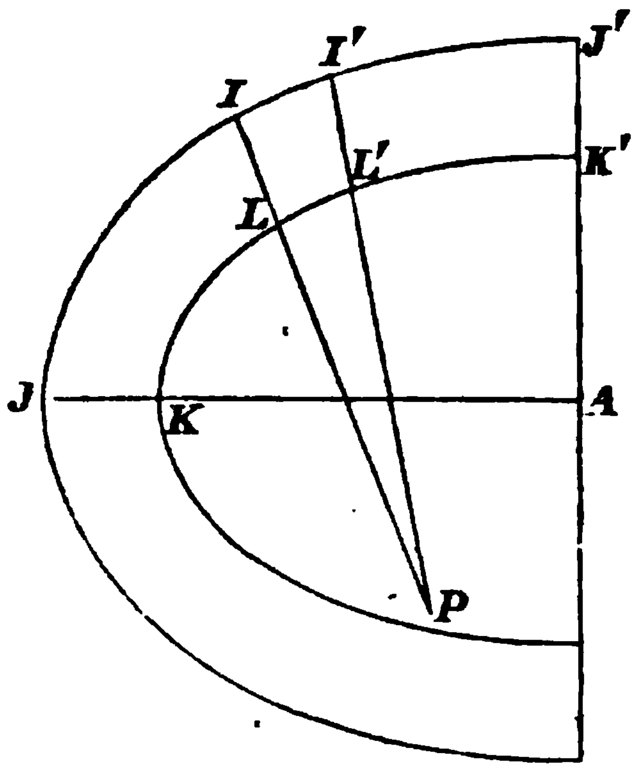
Ex. 7. The motion of L along its sphero-conic is the same as that of a particle acted on by two forces whose directions are the tangents at L to the arcs LS, LS' joining L to the foci of the sphero-conic and whose magnitudes are respectively proportional to $\sin LS \cos LS'$ and $\sin LS' \cos LS$.

531. The instantaneous axis describes a cone in space, which has been called the cone of the herpolhode. The equation of this cone cannot generally be found, but when it can be determined we have another geometrical representation of the motion. For suppose the two cones described by the instantaneous axis in space and in the body to be constructed. Since each of these cones will contain two consecutive positions of their common generator, they will touch each other along the instantaneous axis. Then the points of contact having no velocity the motion will be represented by making the cone fixed in the body roll on the cone fixed in space.

532. *To find the motion of the instantaneous axis in space.*

Since the invariable line OL is fixed in space, it will be convenient to refer the motion to OL as one axis of co-ordinates. Let the angle the instantaneous axis OI makes with OL be called ζ, and let ϕ be the angle the plane IOL makes with any plane passing through OL and fixed in space.

During the motion the cone described by OI in the body rolls on the cone described by OI in space. It is therefore clear that the angular velocity of the instantaneous axis in space is the same as its angular velocity in the body. Describe a sphere whose centre is at O and radius unity, and let this sphere be fixed in the body. Let L, I be the intersections of the invariable line and instantaneous axis with the sphere at the time t, L', I' their intersections at the time $t+dt$. Then IL, $I'L'$ are consecutive normals to the sphero-conic KK' traced out by the invariable line and therefore intersect each other in some point P



which may be regarded as a centre of curvature of the sphero-conic. Let $\rho = PL$. Then clearly

$$\left.\begin{array}{l}\text{velocity of } I \text{ resolved}\\ \text{perpendicularly to } IL\end{array}\right\} = \left(\begin{array}{c}\text{velocity}\\ \text{of } L\end{array}\right) \cdot \frac{\sin(\rho+\zeta)}{\sin\rho}$$

$$= \frac{T}{G}\tan\zeta(\cos\zeta + \cot\rho\sin\zeta);$$

$$\therefore \frac{d\phi}{dt} = \frac{T}{G}\left(1 + \frac{\tan\zeta}{\tan\rho}\right).$$

But it may be proved that in any sphero-conic $\tan\rho = \dfrac{\tan^3 n}{\tan^2 l}$, where n is the length of the normal intercepted between the curve and that axis which contains the foci, and $2l$ is the length of the ordinate through either focus, and is usually called the latus rectum. Substituting for $\tan\rho$, and remembering that

$\dfrac{\tan\zeta}{\tan n} = \dfrac{G^2 - CT}{CT}$, by Art. 530, and $\tan l = \dfrac{\tan^2 b}{\tan a}$, we get

$$\frac{d\phi}{dt} = \frac{T}{G} + \frac{T}{G}\left(\frac{G^2 - CT}{CT}\right)^3 \cdot \left(\frac{\tan^2 b}{\tan a}\right)^2 \cot^2\zeta.$$

If we substitute for $\tan a$ and $\tan b$ their values, we get

$$\frac{d\phi}{dt} = \frac{T}{G} + \frac{(AT - G^2)(BT - G^2)(CT - G^2)}{ABCGT^2}\cot^2\zeta.$$

This result was first discovered by Poinsot.

533. Since the resolved angular velocity about the invariable line is constant, we easily find $\omega = \dfrac{T}{G}\sec\zeta$. Substituting this value of ω in equation (6) of Art. 508, we find a relation between ζ and $\dfrac{d\zeta}{dt}$, which however is too complicated to be of much use.

The values of $\frac{d\phi}{dt}$ and $\frac{d\zeta}{dt}$ in terms of ζ have now both been found; from these the motion of the instantaneous axis in space can be deduced.

Ex. 1. Show that the angular velocity v' of the instantaneous axis in space or in the body is given by

$$\omega^2 v'^2 = \frac{T^2}{ABC}\left(A+B+C-2\frac{G^2}{T}\right) - \frac{\lambda_1\lambda_2\lambda_3}{\omega^2},$$

where ω is the resultant angular velocity of the body and λ_1, λ_2, λ_3 have the meanings given to them in Art. 508. This result is due to Poinsot.

Ex. 2. The length of the spiral between two of its successive apsides, described in absolute space, on the surface of a fixed concentric sphere, by the instantaneous axis of rotation, is equal to a quadrant of the spherical ellipse described by the same axis on an equal sphere moving with the body. This is Booth's Theorem.

Ex. 3. If the eccentric line intersect in the point E the unit sphere which is fixed in the body and has its centre at the fixed point, prove that

$$\begin{pmatrix}\text{velocity}\\ \text{of } E\end{pmatrix}^2 = \frac{T}{G}\frac{d\phi}{dt}\tan^2\zeta.$$

534. Let O be the fixed point, OI the instantaneous axis. Let the angular velocity ω about OI be resolved into two, viz. a uniform angular velocity $\frac{T}{G}$ about the invariable line OL, and an angular velocity $\omega \sin IOL$ about a line OH lying in a plane fixed in space perpendicular to the invariable line, and passing through the fixed point O. Let this fixed plane be called the invariable plane at O. As the body moves, OH will describe a cone in the body which will always touch this fixed plane. The velocity of any point of the body lying for a moment in OH is unaffected by the rotation about OH, and the point has therefore only the motion due to the uniform angular velocity about OL. We have thus a new representation of the motion of the body. Let the cone described by OH in the body be constructed, and let it roll on the invariable plane at O with the proper angular velocity, while at the same time this plane turns round the invariable line with a uniform angular velocity $\frac{T}{G}$. The cone described by OH in the body has been called by Poinsot the *Rolling and Sliding Cone.*

535. *To find a construction for the sliding cone.* Its generator OH is at right angles to OL, and lies in the plane IOL. Now OL is fixed in space; let OL' be the line in the body which, after an interval of time dt, will come into the position OL. Since the body is turning about OI, the plane LOL' is perpendicular to the plane LOI, and hence OH is perpendicular to both OL and OL'. That is, OH is perpendicular to the tangent plane to the cone

described by OL in the body. The cone described by OH in the body is therefore the *reciprocal cone* of that described by OL. The equation to the cone described by OL has been shown to be

$$\frac{AT-G^2}{A}x^2+\frac{BT-G^2}{B}y^2+\frac{CT-G^2}{C}z^2=0.$$

Hence the equation to the cone described by OH is

$$\frac{A}{AT-G^2}x^2+\frac{B}{BT-G^2}y^2+\frac{C}{CT-G^2}z^2=0.$$

The focal lines of the cone described by OH are perpendicular to the circular sections of the reciprocal cone, that is the cone described by OL. And these circular sections are the same as the circular sections of the ellipsoid of gyration. Hence the focal lines lie in the plane containing the axes of greatest and least moment, and are independent of the initial conditions.

This cone becomes a straight line in the case in which the cone described by OL becomes a plane, viz. when the initial conditions are such that $G^2=BT$.

536. *To find the motion of* OH *in space and in the body.*

Since OL, OH and OI are always in the same plane the motion of OH in space round the fixed straight line OL is the same as that of OI, and is given by the expression for $\frac{d\phi}{dt}$ in Art. 532.

To find the motion of OH in the body it will be convenient to refer to the figure of Art. 532. Produce the arcs PL, PL' to H and H' so that LH and $L'H'$ are each quadrants. Then H and H' are the points in which the axis OH intersects the unit sphere at the times t and $t+dt$.

We have therefore

$$\begin{pmatrix}\text{velocity}\\ \text{of } H\end{pmatrix}=\begin{pmatrix}\text{velocity}\\ \text{of } L\end{pmatrix}\cdot\frac{\sin\left(\rho+\frac{\pi}{2}\right)}{\sin\rho}=\frac{T}{G}\tan\zeta\cot\rho.$$

Substituting for $\tan\rho$ as before we may express the result in terms of ζ or n at our pleasure.

Since the cone described by OH in the body rolls on a plane which also turns round a normal to itself at O, it is clear that the angular velocity of OH in the body is less than the angular velocity of OH in space by the angular velocity of the plane, i. e.

$$\begin{pmatrix}\text{velocity}\\ \text{of } H\end{pmatrix}=\frac{d\phi}{dt}-\frac{T}{G}\,;$$

Motion of the Principal Axes.

537. *To find the angular motions in space of the principal axes.*

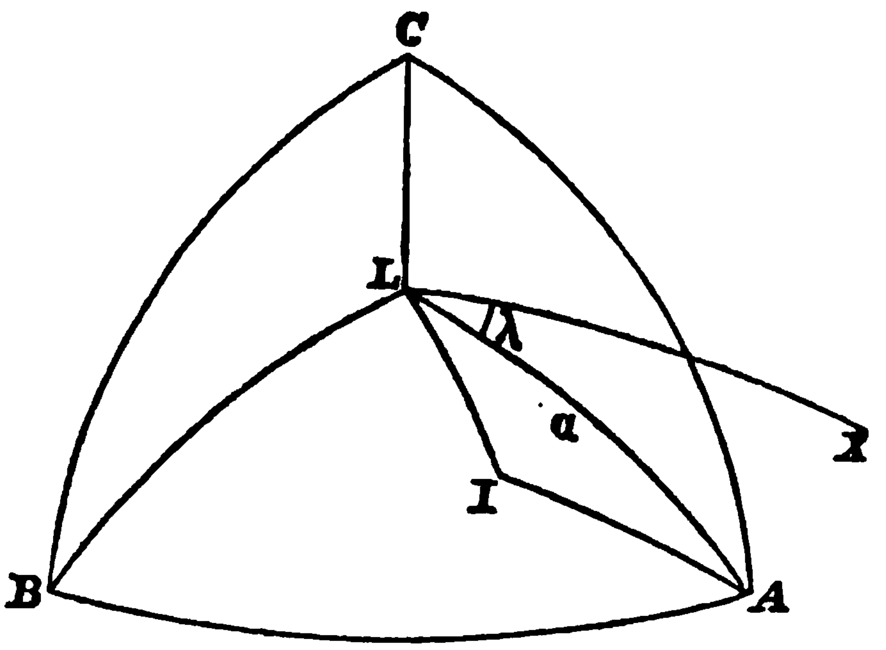
Since the invariable line OL is fixed in space it will be convenient to refer the motion to this straight line as axis of z. Let OA, OB, OC be the principal axes at the fixed point O, and let, as before, α, β, γ be their inclinations to the axis OL or OZ. Let λ, μ, ν be the angles the planes LOA, LOB, LOC make with some fixed plane LOX passing through OL. Our object is to find $\frac{d\alpha}{dt}$ and $\frac{d\lambda}{dt}$ with similar expressions for the other axes. This problem is really the same as that discussed in Art. 235, but it will be found advantageous to make a slight variation on the demonstration.

Describe a sphere whose centre is at the fixed point, and whose radius is unity. Let the invariable line, the instantaneous axis and the principal axes cut this sphere in the points L, I, A, B, C respectively. The velocity of A resolved perpendicular to LA will then be $\sin\alpha \frac{d\lambda}{dt}$. But since the body is turning round OI as instantaneous axis, the point A is moving perpendicularly to the arc IA, and its velocity is $\omega \sin IA$. Resolving this perpendicular to the arc LA, we have

$$\sin\alpha \frac{d\lambda}{dt} = \omega \sin AI \cos LAI$$

$$= \omega \frac{\cos LI - \cos LA \cos IA}{\sin LA},$$

by a fundamental formula in spherical trigonometry. But $\omega \cos LI$ is the resolved part of the angular velocity about OL, which is equal to $\frac{T}{G}$, and $\omega \cos IA$ is the resolved part of the angular



velocity about OA, which is ω_1. We have therefore

$$\sin^2\alpha \frac{d\lambda}{dt} = \frac{T}{G} - \omega_1 \cos\alpha,$$

a result which follows immediately from Art. 249. Since $G\cos\alpha = A\omega_1$, we have

$$\sin^2\alpha \frac{d\lambda}{dt} = \frac{T}{G} - \frac{G\cos^2\alpha}{A} \quad \text{..................(1)}.$$

This result may also be written in the form

$$\frac{d\lambda}{dt} = \frac{T}{G} + \frac{AT - G^2}{AG}\cot^2\alpha \quad \text{..................(2)}.$$

538. To find $\frac{d\alpha}{dt}$ we may proceed in the following manner. We have $\cos\alpha = \frac{A\omega_1}{G}$, $\cos\beta = \frac{B\omega_2}{G}$, $\cos\gamma = \frac{C\omega_3}{G}$.

Substituting in Euler's equation

$$A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = 0,$$

we have $$\sin\alpha \frac{d\alpha}{dt} = \left(\frac{1}{B} - \frac{1}{C}\right) G \cos\beta \cos\gamma \quad \text{...........(3)}.$$

But by Art. 508 $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are connected by the equations

$$\left.\begin{aligned} \frac{\cos^2\alpha}{A} + \frac{\cos^2\beta}{B} + \frac{\cos^2\gamma}{C} = \frac{T}{G^2} \\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \end{aligned}\right\} \quad \text{...............(4)}.$$

If we solve these equations so as to express $\cos\beta$, $\cos\gamma$ in terms of $\cos\alpha$, we easily find

$$\sin^2\alpha\left(\frac{d\alpha}{dt}\right)^2 = -\frac{G^2}{BC}\left(\frac{G^2 - CT}{G^2} - \frac{A - C}{A}\cos^2\alpha\right)\left(\frac{G^2 - BT}{G^2} - \frac{A - B}{A}\cos^2\alpha\right) \text{...(5)}.$$

539. Since the left-hand side of equation (5) is necessarily real, we see that the values of $\cos^2\alpha$ are restricted to lie between certain limits. If the axis whose motion we are considering is the axis of greatest or least moment let B be the axis of mean moment. In this case $\cos^2\alpha$ must lie *between* the limits $\frac{G^2 - CT}{G^2}\frac{A}{A - C}$ and $\frac{G^2 - BT}{G^2}\frac{A}{A - B}$ if both be positive. By Art. 509 the former of these two is positive and less than unity; this is easily shown by dividing the numerator and the denominator by ACG^2. If the latter is positive the spiral described by the principal axes on the surface of a sphere whose centre is at the fixed point lies between two concentric circles which it alternately touches. If the latter limit is negative $\cos\alpha$ has no inferior limit. In this case the spiral always lies between two small circles on the sphere, one of which is exactly opposite the other.

If the axis considered is the axis of mean moment, $\cos^2\alpha$ must lie *outside* the same two limits as before. Both these are positive, but one is greater and the other less than unity. The spiral therefore lies between two small circles opposite each other.

In order that $\frac{d\lambda}{dt}$ may vanish we must have $G^2\cos^2\alpha = AT$, but this by substitution makes $\frac{d\alpha}{dt}$ imaginary. Thus $\frac{d\lambda}{dt}$ always keeps one sign. It is easy to see that if the initial conditions are such that $\frac{G^2}{T}$ is less than the moment of inertia about the axis which describes the spiral we are considering, the angular velocity will be greatest when the axis is nearest the invariable line and least when the axis is furthest. The reverse is the case if $\frac{G^2}{T}$ is greater than the moment of inertia.

540. Ex. 1. Let OM be any straight line fixed in the body and passing through O and let it cut the ellipsoid of gyration at O in the point M. Let OM' be the perpendicular from O on the tangent plane at M. If $OM=r$, $OM'=p$, and if i, i' be the angles OM, OM' make with the invariable line OL, prove that

$$\sin^2 i\,\frac{dj}{dt} = \frac{T}{G} - \frac{G}{pr}\cos i\cos i',$$

where j is the angle the plane LOM makes with some plane fixed in space passing through OL. This follows from Art. 249 or from Art. 537.

Ex. 2. If KLK' be the spiral traced out by the invariable line in the manner described in Art. 527, show that

$$\lambda = \frac{T}{G}t + L\hat{A}K - \begin{pmatrix}\text{vectorial area}\\ LAK\end{pmatrix},$$

where λ is the angle described by the plane containing the invariable line and the principal axis OA.

Ex. 3. If ψ be the angle described in space by the plane containing the invariable line and any straight line OM, fixed in the body, passing through O and cutting the sphere in M, prove that

$$\psi = \frac{T}{G}t + L\hat{M}N - \begin{pmatrix}\text{vectorial area}\\ LMN\end{pmatrix},$$

where MN is any spherical arc fixed in the body and cutting in N the sphero-conic described by the invariable line.

Ex. 4. If we draw three straight lines OA, OB, OC along the principal axes at the fixed point O of equal lengths, the sum of the areas conserved by these lines on the invariable plane is proportional to the time. [Poinsot.]

Ex. 5. If the lengths OA, OB, OC be proportional to the radii of gyration about the axes respectively, the sum of the areas conserved by these lines on the invariable plane will also be proportional to the time. [Poinsot.]

Motion of the body when two principal axes are equal.

541. Let the body be rotating with an angular velocity ω about an instantaneous axis OI. Let OL be the perpendicular on the invariable line. The momental ellipsoid is in this case a spheroid, the axis of which is the axis of unequal moment in the body. Let the equal moments of inertia be A and B. From the symmetry of the figure it is evident that as the spheroid rolls on the invariable planes, the angles LOC, LOI are *constant*, and the three axes OI, OL, OC are always in one plane. Let the angles $LOC = \gamma$, $IOC = i$.

Following the same notation as in Art. 508, we have

$$\omega_3 = \omega \cos i,\ \ \omega_1^2 + \omega_2^2 = \omega^2 \sin^2 i,$$
$$G^2 = (A^2 \sin^2 i + C^2 \cos^2 i)\,\omega^2,$$
$$T = (A \sin^2 i + C \cos^2 i)\,\omega^2.$$

We therefore have

$$\cos\gamma = \frac{C\omega_3}{G} = \frac{C\cos i}{\sqrt{A^2\sin^2 i + C^2\cos^2 i}}.$$

This result may also be obtained as follows. In any conic if i and γ be the angles a central radius vector and the perpendicular on the tangent at its extremity make with the minor axis, and if a, b be the semi-axes, then $\tan\gamma = \frac{b^2}{a^2}\tan i$. Applying this to the momental spheroid, we have

$$\tan\gamma = \frac{A}{C}\tan i.$$

The angle i being known from the initial conditions, the angle γ can be found from either of these expressions. The peculiarities of the motion will then be as follows.

The invariable line describes a right cone in the body whose axis is the axis of unequal moment, and whose semi-angle is γ.

The instantaneous axis describes a right cone in the body whose axis is the axis of unequal moment, and whose semi-angle is i.

The instantaneous axis describes a right cone in space, whose axis is the invariable line, and whose semi-angle is $i \sim \gamma$.

The axis of unequal moment describes a right cone in space whose axis is the invariable line, and whose semi-angle is γ.

The angular velocity of the body about the instantaneous axis varies as the radius vector of the spheroid, and is therefore constant.

542. The rate of motion of the invariable line and the instantaneous axis in the body may be found most readily by referring to the original equations of motion in Art. 508. We have in this case

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(A-C)\,\omega_2\omega\cos i=0 \\ A\frac{d\omega_2}{dt}+(A-C)\,\omega_1\omega\cos i=0 \end{aligned}\right\}.$$

Solving these by differentiating the first and eliminating ω_2, we find

$$\omega_1=F\cos\left(\frac{A-C}{A}\omega t\cos i+f\right),$$

$$\omega_2=-F\sin\left(\frac{A-C}{A}\omega t\cos i+f\right),$$

where F and f are arbitrary constants. Let the projection of either the instantaneous axis or the invariable line on the plane perpendicular to the axis of unequal moment make an angle χ with any fixed straight line which may be taken as axis OA. Then $\tan\chi=\frac{\omega_2}{\omega_1}$. Hence we find

$$\frac{d\chi}{dt}=-\frac{A-C}{A}\omega\cos i.$$

543. To find the common rate of motion in space of the instantaneous axis and the axis of unequal moment.

Let C be the extremity of the axis of figure of the momental ellipsoid, and let Ω be the rate at which the plane LOC is turning round OL. Let CM, CN be perpendiculars on CL and CI. Then since the body is turning round CI, the velocity of C is $CN.\omega$. But this is also $CM.\Omega$. Since $CM=OC\sin\gamma$, $CN=OC\sin i$, we have at once

$$\Omega\sin\gamma=\omega\sin i,$$

whence Ω can be found.

544. Ex. 1. If a right circular cone whose altitude a is double the radius of its base turn about its centre of gravity as a fixed point, and be originally set in motion about an axis inclined at an angle α to the axis of figure, the vertex of the cone will describe a circle whose radius is $\frac{3}{4}a\sin\alpha$. [Coll. Exam.]

Ex. 2. A circular plate revolves about its centre of gravity as a fixed point. If an angular velocity ω were originally impressed on it about an axis making an angle α with its plane, a normal to the plane of the disc will make a revolution in space in time $\frac{2\pi}{\omega\sqrt{1+3\sin^2\alpha}}$. [Coll. Exam.]

Ex. 3. A body which can turn freely about a fixed point at which two of the principal moments are equal and less than the third, is set in rotation about any axis. Owing to the resistance of the air and other causes, it is continually acted on by a retarding couple whose axis is the instantaneous axis of rotation and whose magnitude is proportional to the angular velocity. Show that the axis of rotation will continually tend to become coincident with the axis of unequal moment. In the case of the earth therefore, a near coincidence of the axis of rotation and axis of figure is not a proof that such coincidence has always held. Astronomical Notices, March 8, 1867.

Motion when $G^2 = BT$.

545. The peculiarities of this case have been already alluded to in Art. 508. When the initial conditions are such that this relation holds between the Vis Viva and the Momentum of the body the whole discussion of the motion becomes more simple*.

The fundamental equations of motion are

$$\left.\begin{array}{r} A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T \\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2 = BT \end{array}\right\} \qquad (1).$$

Solving these, we have

$$\left.\begin{array}{l} \omega_1^2 = \dfrac{B-C}{A-C} \cdot \dfrac{G^2 - B^2\omega_2^2}{AB} \\ \omega_3^2 = \dfrac{A-B}{A-C} \cdot \dfrac{G^2 - B^2\omega_2^2}{BC} \end{array}\right\} \qquad (2).$$

But $$\frac{d\omega_2}{dt} = \frac{C-A}{B}\,\omega_1\omega_3;$$

$$\therefore \frac{d\omega_2}{dt} = \mp \sqrt{\frac{(A-B)(B-C)}{AC}} \cdot \frac{G^2 - B^2\omega_2^2}{B^2}.$$

When the initial values of ω_1 and ω_3 have like signs, $(C-A)\,\omega_1\omega_3$ is negative and therefore $\dfrac{d\omega_2}{dt}$ must be negative, hence in this expression the upper or lower sign is to be used according as the initial values of ω_1, ω_3 have like or unlike signs.

$$\therefore \frac{B^2}{G^2 - B^2\omega_2^2}\,\frac{d\omega_2}{dt} = \mp \sqrt{\frac{(A-B)(B-C)}{AC}}.$$

* This case appears to have been considered by nearly every writer on this subject. As examples of different methods of treatment the reader may consult *Legendre, Traité des Fonctions Elliptiques*, 1825, Vol. I. page 382, and *Poinsot Théorie Nouvelle de la Rotation des corps*, 1852, page 104.

If we put $\mp n$ for the right-hand side and integrate we have

$$\frac{G + B\omega_2}{G - B\omega_2} = E \,.\, e^{\mp\frac{2G}{B}nt},$$

where E is some undetermined constant.

$$\therefore \frac{B\omega_2}{G} = \frac{E \,.\, e^{\mp\frac{2G}{B}nt} - 1}{E \,.\, e^{\mp\frac{2G}{B}nt} + 1}.$$

As t increases indefinitely, ω_2 approaches $\mp \dfrac{G}{B}$ as its limit and therefore by (2) ω_1 and ω_3 approach zero.

The conclusion is that the instantaneous axis ultimately approaches to coincidence with the mean axis of principal moment, but never actually coincides with it. It approaches the positive or negative end of the mean axis according as the initial value of $(C-A)\,\omega_1, \omega_3$ is positive or negative.

546. *To find what the cones traced out in the body by the invariable line and instantaneous axis become when* $G^2 = BT$.

Eliminating ω_2 from the fundamental equations of the last Article we have

$$A\,(A - B)\,\omega_1^2 = C\,(B - C)\,\omega_3^2.$$

Taking the principal axes at the fixed point as axes of reference, the equations of the invariable line are $\dfrac{x}{A\omega_1} = \dfrac{y}{B\omega_2} = \dfrac{z}{C\omega_3}$. Eliminating ω_1 and ω_3 the locus of the invariable line is one of the two planes

$$\sqrt{\frac{A-B}{A}}\,x = \pm\sqrt{\frac{B-C}{C}}\,z.$$

The equations of the instantaneous axes are $\dfrac{x}{\omega_1} = \dfrac{y}{\omega_2} = \dfrac{z}{\omega_3}$. Eliminating ω_1 and ω_3 the locus of the instantaneous axis is one of the two planes

$$\sqrt{A\,(A-B)}\,x = \pm\sqrt{C\,(B-C)}\,z.$$

In these equations since $\dfrac{z}{x}$ follows the sign of $\dfrac{\omega_3}{\omega_1}$ the upper or lower sign is to be taken according as the initial values of ω_1, ω_3 have like or unlike signs. These planes pass through the mean axis, and are independent of the initial conditions except so far that $G^2 = BT$.

The rolling and sliding cone is the reciprocal of that described by the invariable plane, and is therefore the straight line perpendicular to that plane which is traced out by the invariable line.

Ex. 1. Show that the planes described by the invariable line coincide with the central circular sections of the ellipsoid of gyration and are perpendicular to the asymptotes of that focal conic of the momental ellipsoid which lies in the plane of the greatest and least moments.

Ex. 2. The planes described by the instantaneous axis are perpendicular to the umbilical diameters of the ellipsoid of gyration and are the diametral planes of the asymptotes of the focal conic in the momental ellipsoid.

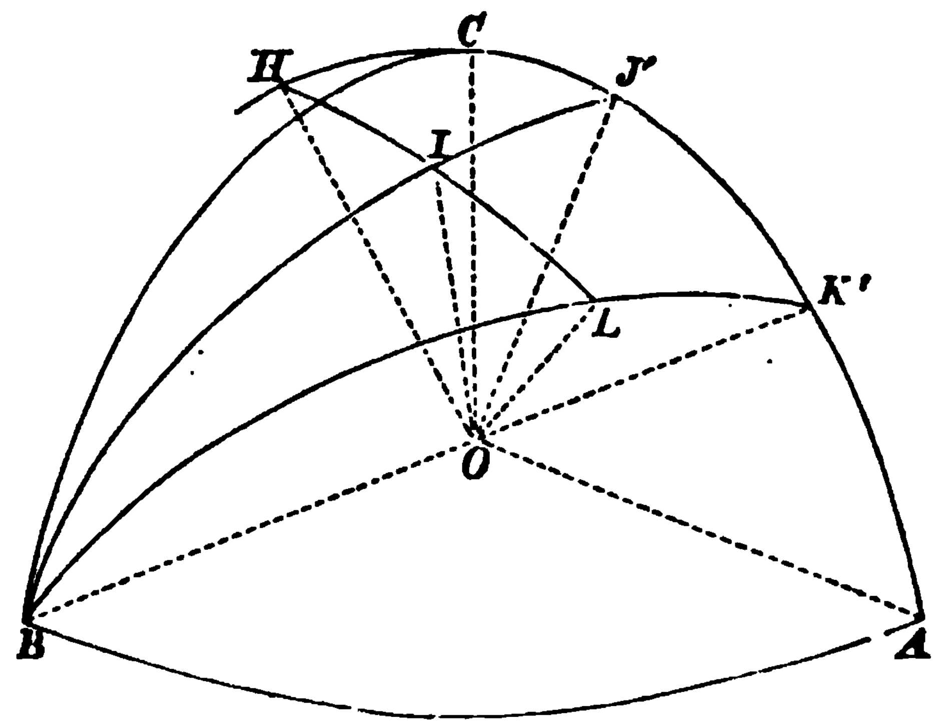
547. The relations to each other of the several planes fixed in the body may be exhibited by the following figure. Let A, B, C be the points in which the principal axes of the body cut a sphere whose centre is O, and radius unity. Let BLK', BIJ' be the planes traced out by the invariable line and the instantaneous axis respectively. Then by the last Article

$$\tan CK' = \sqrt{\frac{A}{C} \cdot \frac{B-C}{A-B}}, \quad \tan CJ' = \sqrt{\frac{C}{A} \cdot \frac{B-C}{A-B}}.$$

Hence we find

$$\tan K'J' = \tan LBI = \sqrt{\frac{(B-C)(A-B)}{AC}}.$$

This is the quantity which has been called n in Art. 545.



Exactly as in Art. 528 the direction of motion of L is perpendicular to IL and hence the angle ILB is a right angle. Thus the spherical triangle ILB has one angle right, and another constant and independent of all initial conditions.

Exactly as in Art. 528, the velocity of L along LB is equal to

$\omega \sin IL$ which, by Art. 514, is equal to $\frac{T}{G} \tan IL$. But from the spherical triangle ILB

$$n \sin BL = \tan IL.$$

If then we put as before $\beta = BL$, we have

$$\frac{d\beta}{dt} = \pm \frac{T}{G} n \sin \beta.$$

If the initial values of ω_1, ω_3 have the same sign, the body is turning round I from K' to B. Hence, since L is fixed in space, BL is increasing and therefore the upper sign must be used in this figure. See also Art. 545.

We may also find an expression for β in terms of the time. Since $\cos \beta = \frac{B\omega_3}{G}$ we have, by Art. 545,

$$\frac{1 + \cos \beta}{1 - \cos \beta} = E e^{\mp \frac{2G}{B} nt},$$

$$\therefore \cot \frac{\beta}{2} = \sqrt{E} e^{\mp \frac{G}{B} nt}.$$

Ex. Show that the eccentric line describes a great circle passing through B and cutting AC in some point D' where $\tan^2 CD' = \tan CJ' \tan CK'$. If E be the intersection of the eccentric line with the sphere, show that the arcs BE and BL are always equal.

548. *To find the motion of the body in space.*

We have already seen that the motion is such that a plane fixed in the body, viz. the plane BK', contains a straight line fixed in space, viz. the invariable line OL. Since the body is brought from any position into the next by an angular velocity $\omega \cos IOL = \frac{T}{G}$ about OL, and an angular velocity $\omega \sin IOL$ about a perpendicular to OL, viz. OH, it follows that the plane fixed in the body turns round the line fixed in space with a uniform angular velocity $\frac{T}{G}$ or $\frac{G}{B}$. At the same time the plane moves so that the line fixed in space appears to describe the plane with a variable velocity $\omega \sin IOL$. If β be the angle BL, this has been proved in the last Article to be $\frac{T}{G} n \sin \beta$.

549. The cone described by OH in the body is the reciprocal cone of that described by OL, and from it we may deduce reciprocal theorems. The motion is therefore such that a straight line fixed in the body, viz. OH, describes a plane fixed in space, viz. the plane perpendicular to OL. The straight line moves

along this plane with a uniform angular velocity equal to $\frac{T}{G}$ or $\frac{G}{B}$, while the angular velocity of the body about this straight line is $\pm\frac{G}{B} n \sin\beta$.

550. The motion of the principal axes may be deduced from the general results given in Art. 537. But we may also proceed thus. Since the body is turning about OI, the point B on the sphere is moving perpendicularly to the arc IB. Hence the tangent to the path of B makes with LB an angle which is the complement of the constant angle IBL. The path traced out by the axis of mean moment on a sphere whose centre is at O is a rhumb line which cuts all the great circles through L at an angle whose cotangent is $\pm n$.

551. *To find the motion of the instantaneous axis in space.*

This problem is the same as that considered in Art. 532. We may however deduce the result at once from Art. 548. The angle ILB is always a right angle, it therefore follows that the angular velocity of I round L is the same as that of the arc BL round L. But the angular velocity of the latter is constant and equal to $\frac{T}{G}$. If then ϕ be the angle the plane LOI containing the instantaneous axis and the invariable line makes with some fixed plane passing through the invariable line, we have $\frac{d\phi}{dt} = \frac{T}{G}$.

552. To find the equation of the cone described by the instantaneous axis in space, we require a relation between ζ and ϕ, where ζ is the arc IL on the sphere. From the right-angled triangle ILB we have $n \sin\beta = \tan\zeta$, and by Art. 547,

$$\cot\frac{\beta}{2} = \sqrt{E}e^{\mp\frac{G}{B}nt}.$$

Eliminating β, we shall have an expression for ζ in terms of t. We find

$$\frac{2n}{\tan\zeta} = \cot\frac{\beta}{2} + \tan\frac{\beta}{2} = \sqrt{E}e^{\mp\frac{G}{B}nt} + \frac{1}{\sqrt{E}}e^{\pm\frac{G}{B}nt}.$$

By the last Article $\phi = \frac{T}{G}t + F$, where F is some constant. Let us substitute for t in terms of ϕ, and let us choose the plane from which ϕ is measured so that $\sqrt{E}e^{\mp nF} = 1$.

The equation to the cone traced out in space by the instantaneous axis is

$$2n\cot\zeta = e^{n\phi} + e^{-n\phi}.$$

When $\phi = 0$, we have $\tan \zeta = n$. Therefore the plane fixed in space from which ϕ is measured is the plane containing the axes of greatest and least moment at the instant when that plane contains the invariable line.

On tracing this cone, we see that it cuts a sphere whose centre is at the fixed point in a spiral curve. The branches determined by positive and negative values of ϕ are perfectly equal. As ϕ increases positively the radial arc ζ continually decreases, the spiral therefore makes an infinite number of turns round the point L, the last turn being infinitely small.

Ex. In the herpolhode $\frac{2mb}{r} = e^{m\theta} + e^{-m\theta}$, if the locus of the extremity of the polar subtangent of this curve be found and another curve be similarly generated from this locus, the curve thus obtained will be similar to the herpolhode. [Math. Tripos, 1863.]

On Correlated and Contrarelated Bodies.

553. *To compare the motions of different bodies acted on by initial couples whose planes are parallel.*

Let α, β, γ be the angles the principal axes OA, OB, OC of a body at the fixed point O make with the invariable line OL. Then by Art. 511, Euler's equations may be put into the form

$$\frac{d\cos\alpha}{dt} + G\left(\frac{1}{B} - \frac{1}{C}\right)\cos\beta\cos\gamma = 0 \quad\ldots\ldots\ldots\ldots(1),$$

with two similar equations. Let λ, μ, ν be the angles the planes LOA, LOB, LOC make with any plane fixed in space, and passing through OL. Then

$$\sin^2\alpha\,\frac{d\lambda}{dt} = \frac{T}{G} - \frac{G\cos^2\alpha}{A} \quad\ldots\ldots\ldots\ldots(2),$$

with similar equations for μ and ν.

If accented letters denote similar quantities for some other body, the corresponding equations will be

$$\frac{d\cos\alpha'}{dt} + G'\left(\frac{1}{B'} - \frac{1}{C'}\right)\cos\beta'\cos\gamma' - 0 \quad\ldots\ldots\ldots\ldots(3),$$

$$\sin^2\alpha'\,\frac{d\lambda'}{dt} = \frac{T'}{G'} - \frac{G'\cos^2\alpha'}{A'} \quad\ldots\ldots\ldots\ldots(4).$$

If then the bodies are such that

$$G\left(\frac{1}{B} - \frac{1}{C}\right) = G'\left(\frac{1}{B'} - \frac{1}{C'}\right), \quad \&c. = \&c. \quad\ldots\ldots\ldots\ldots(5),$$

the equations (1) to find α, β, γ are the same as the equations (3) to find α', β', γ'. Therefore if these two bodies be initially placed with their principal axes parallel and be set in motion by impulsive couples whose magnitudes are G and G', and whose planes are parallel, then after the lapse of any time t the principal axes of the two bodies will still be equally* inclined to the common axis of the couples.

The equations (5) may be put into the form

$$\frac{G}{A}-\frac{G'}{A'}=\frac{G}{B}-\frac{G'}{B'}=\frac{G}{C}-\frac{G'}{C'}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Since by Art. 513 the Vis Viva is given by

$$\frac{T}{G^2}=\frac{\cos^2\alpha}{A}+\frac{\cos^2\beta}{B}+\frac{\cos^2\gamma}{C}\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

we see that each of the expressions in (6) is equal to $\dfrac{T}{G}-\dfrac{T'}{G'}$.

It immediately follows by subtracting equations (2) and (4) and dividing by $\sin^2\alpha$ that

$$\frac{d\lambda}{dt}-\frac{d\lambda'}{dt}=\frac{T}{G}-\frac{T'}{G'},$$

with similar equations for μ and ν. Thus the two bodies being started as before with their principal axes parallel each to each, the parallelism of the principal axes may be restored by turning

* In order that the angles which the principal axes make with the axis of the couple may be the same in each body, it is necessary that the cones described by the axis OL in the body should be the same. Hence by Art. 525, the two ellipsoids of gyration must have the same circular sections, or which is the same thing, the two momental ellipsoids must have the same asymptotes to their hyperbolic focal conics. Also in order that the cones may be the same we must have

$$\frac{\frac{1}{A}-\frac{T}{G^2}}{\frac{1}{A'}-\frac{T'}{G'^2}}=\frac{\frac{1}{B}-\frac{T}{G^2}}{\frac{1}{B'}-\frac{T'}{G'^2}}=\frac{\frac{1}{C}-\frac{T}{G^2}}{\frac{1}{C'}-\frac{T'}{G'^2}}.$$

If we put each of these equal to some quantity r, we easily find

$$\frac{\frac{1}{A}-\frac{1}{B}}{\frac{1}{A'}-\frac{1}{B'}}=\frac{\frac{1}{B}-\frac{1}{C}}{\frac{1}{B'}-\frac{1}{C'}}=\frac{\frac{1}{C}-\frac{1}{A}}{\frac{1}{C'}-\frac{1}{A'}}=r.$$

If in the two bodies the angles between the principal axes and the axis of the couple are to be equal each to each at the *same time*, the equations (1) and (3) of Art. 553 show that we must have in addition $\dfrac{G'}{G}=r$. This leads to the generalization of Prof. Sylvester's theory given in the text.

the body whose principal axes are A', B', C' about the common axis of the impulsive couples through an angle $\left(\frac{T}{G}-\frac{T'}{G'}\right)t$ in the direction in which positive impulsive couples act*.

554. When the couples G and G' are equal the condition (6) becomes

$$\frac{1}{A}-\frac{1}{A'}=\frac{1}{B}-\frac{1}{B'}=\frac{1}{C}-\frac{1}{C'}=\frac{T-T'}{G^2},$$

the bodies are then said to be *correlated*. If momental ellipsoids of the two bodies be taken so that the moment of inertia in each bears the same ratio to the square of the reciprocal of the radius vector these ellipsoids are clearly confocal.

When the couples G and G' are equal and opposite, the equation (6) becomes

$$\frac{1}{A}+\frac{1}{A'}=\frac{1}{B}+\frac{1}{B'}=\frac{1}{C}+\frac{1}{C'}=\frac{T+T'}{G^2},$$

and the bodies are said to be *contrarelated*.

555. *To compare the angular velocities of the two bodies at any instant.*

Let ω be the angular velocity of one body at any instant, then following the usual notation we have

$$\omega^2=\omega_1^2+\omega_2^2+\omega_3^2=G^2\left(\frac{\cos^2\alpha}{A^2}+\frac{\cos^2\beta}{B^2}+\frac{\cos^2\gamma}{C^2}\right).$$

If the same letters accented denote similar quantities for the other body

$$\omega'^2=G'^2\left(\frac{\cos^2\alpha}{A'^2}+\frac{\cos^2\beta}{B'^2}+\frac{\cos^2\gamma}{C'^2}\right).$$

But remembering the condition (6) these give

$$\omega^2-\omega'^2=\left(\frac{T}{G}-\frac{T'}{G'}\right)\left[\cos^2\alpha\left(\frac{G}{A}+\frac{G'}{A'}\right)+\cos^2\beta\left(\frac{G}{B}+\frac{G'}{B'}\right)+\cos^2\gamma\left(\frac{G}{C}+\frac{G'}{C'}\right)\right].$$

* Since the cones described by the invariable line in the two bodies are identical, their reciprocal cones, i. e. Poinsot's rolling and sliding cones, are also identical in the two bodies. Thus in the two bodies, the rolling motions of these cones are equal, but the sliding motions may be different. The sliding motions represent angular velocities about the invariable line respectively equal to $\frac{T}{G}$ and $\frac{T'}{G'}$. Hence we have

$$\frac{d\lambda}{dt}-\frac{d\lambda'}{dt}=\frac{d\mu}{dt}-\frac{d\mu'}{dt}=\frac{d\nu}{dt}-\frac{d\nu'}{dt}=\frac{T}{G}-\frac{T'}{G'}.$$

This remark on the former note is due to Prof. Cayley.

By referring to (7) the quantity in square brackets is easily seen to be $\frac{T}{G}+\frac{T'}{G'}$,

$$\therefore\ \omega^2-\omega'^2=\frac{T^2}{G^2}-\frac{T'^2}{G'^2}.*$$

Ex. If two bodies be so related that their ellipsoids of gyration are confocal, and be initially so placed that the angles (α, β, γ) $(\alpha', \beta', \gamma')$ their principal axes make with the invariable line of each are connected by the equations

$$\frac{\cos\alpha}{\sqrt{A}}=\frac{\cos\alpha'}{\sqrt{A'}},\quad \frac{\cos\beta}{\sqrt{B}}=\frac{\cos\beta'}{\sqrt{B'}},\quad \frac{\cos\gamma}{\sqrt{C}}=\frac{\cos\gamma'}{\sqrt{C'}},$$

and if these bodies be set in motion by two impulsive couples G, G' respectively proportional to $\sqrt{ABC}$ and $\sqrt{A'B'C'}$, then the above relations will always hold between the angles (α, β, γ) $(\alpha', \beta', \gamma')$. If p and p' be the reciprocals of $\frac{d\lambda}{dt}$ and $\frac{d\lambda'}{dt}$, then $Gp-G'p'$ will be constant throughout the motion, where λ, λ', &c., are the angles the planes LOA, $L'O'A'$ make at the time t with their positions at the time $t=0$.

556. When a body turns about a fixed point its motion in space is represented by making its momental ellipsoid roll on a fixed plane. This gives no representation of the *time* occupied by the body in passing from any position to any other. The preceding Articles will enable us to supply this defect.

To give distinctness to our ideas let us suppose the momental ellipsoid to be rolling on a horizontal plane underneath the fixed point O, and that the instantaneous axis OI is describing a polhode about the axis of A. Let us now remove that half of the ellipsoid which is bounded by the plane of BC, and which does not touch the fixed plane. Let us replace this half by the half of another smaller ellipsoid which is confocal with the first. Let a plane be drawn parallel to the invariable plane to touch this ellipsoid in I' and suppose this plane also to be fixed in space. These two semi-ellipsoids may be considered as the momental ellipsoids of two correlated bodies. If they were not attached to each other

* This result may also be obtained in the following manner. By Art. 534 the angular velocity ω of one body is equivalent to an angular velocity $\frac{T}{G}$ about the invariable line and an angular velocity Ω about a straight line OH which is a generator of the rolling and sliding cone. Hence $\omega^2=\frac{T^2}{G^2}+\Omega^2$. A similar equation with accented letters will hold for the other body. Since in the two bodies the angles between the principal axes and the invariable line are equal each to each throughout the motion, the rolling motions of the two cones must be equal, hence $\Omega=\Omega'$. It follows immediately that $\omega^2-\omega'^2=\frac{T^2}{G^2}-\frac{T'^2}{G'^2}$.

and were free to move without interference, each would roll, the one on the fixed plane which touches at I, and the other on that which touches at I'. By what has been shown the upper ellipsoid (being the smallest) may be brought into parallelism with the lower by a rotation $Gt\left(\frac{1}{A}-\frac{1}{A'}\right)$ about the invariable line. If then the upper plane on which the upper ellipsoid rolls be made to turn round the invariable line as a fixed axis with an angular velocity $G\left(\frac{1}{A}-\frac{1}{A'}\right)$, the two ellipsoids will always be in a state of parallelism, and may be supposed to be rigidly attached to each other.

Suppose then the upper tangent plane to be perfectly rough and capable of turning in a horizontal plane about a vertical axis which passes through the fixed point. As the nucleus is made to roll with the under part of its surface on the fixed plane below, the friction between the upper surface and the plane will cause the latter* to rotate about its axis. Then the time elapsed will be in a constant ratio to this motion of rotation, which may be measured off on an absolutely fixed dial face immediately over the rotating plane.

The preceding theory, so far as it relates to correlated and contrarelated bodies, is taken from a memoir by Prof. Sylvester in the *Philosophical Transactions* for 1866. He proceeds to investigate in what cases the upper ellipsoid may be reduced to a disc. It appears that there are always two such discs and no more, except in the case of two of the principal moments being equal, when the solution becomes unique. Of these two discs one is correlated and the other contrarelated to the given body, and they will be respectively perpendicular to the axes of greatest and least moments of inertia.

Poinsot has shown that the motion of the body may be constructed by a cone fixed in the body rolling on a plane which turns uniformly round the invariable line. If, as in the preceding theory, we suppose the plane rough, and to be turned by the cone as it rolls on the plane, the angle turned through by the plane will measure the time elapsed.

* As the ellipsoid rolls on the lower plane, a certain geometrical condition must be satisfied that the nucleus may not quit the upper plane or tend to force it upwards. This condition is that the plane containing OI, OI', must contain the invariable line, for then and then only the rotation about OI can be resolved into a component about OI' and a component about the invariable line. That this condition must be satisfied is clear from the reasoning in the text. But it is also clear from the known properties of confocal ellipsoids.

EXAMPLES*.

1. A right cone the base of which is an ellipse is supported at G the centre of gravity, and has a motion communicated to it about an axis through G perpendicular to the line joining G, and the extremity B of the axis minor of the base, and in the plane through B and the axis of the cone. Determine the position of the invariable plane.

Result. The normal to the invariable plane lies in the plane passing through the axis of the cone and the axis of instantaneous rotation, and makes an angle $\tan^{-1}\frac{h}{16b}\cdot\frac{h^2+4a^2}{a^2+b^2}$.

2. A spheroid has a particle of mass m fastened at each extremity of the axis of revolution, and the centre of gravity is fixed. If the body be set rotating about any axis, show that the spheroid will roll on a fixed plane during the motion provided $\frac{m}{M}=\frac{1}{10}\left(1-\frac{a^2}{c^2}\right)$, where M is the mass of the spheroid, a and c are the axes of the generating ellipse, c being the axis of figure.

3. A lamina of any form rotating with an angular velocity α about an axis through its centre of gravity perpendicular to its plane has an angular velocity $\alpha\sqrt{\frac{B+C}{B-C}}$ impressed upon it about its principal axis of least moment, A, B, C being arranged in descending order of magnitude: show that at any time t the angular velocities about the principal axes are respectively

$$\frac{2\alpha}{e^{\alpha t}+e^{-\alpha t}},\quad -\sqrt{\frac{B+C}{B-C}}\,\alpha\,\frac{e^{\alpha t}-e^{-\alpha t}}{e^{\alpha t}+e^{-\alpha t}}\text{ and }\sqrt{\frac{B+C}{B-C}}\,\frac{2\alpha}{e^{\alpha t}+e^{-\alpha t}},$$

and that it will ultimately revolve about the axis of mean moment.

4. A rigid body not acted on by any force is in motion about its centre of gravity: prove that if the instantaneous axis be at any moment situated in the plane of contact of either of the right circular cylinders described about the central ellipsoid, it will be so throughout the motion.

If a, b, c be the semi-axes of the central ellipsoid, arranged in descending order of magnitude, e_1, e_2, e_3 the eccentricities of its principal sections, Ω_1, Ω_2, Ω_3 the initial component angular velocities of the body about its principal axes, prove that the condition that the instantaneous axis should be situated in the plane above described is $\frac{\Omega_1}{e_1}=\frac{ab}{c^2}\frac{\Omega_3}{e_3}$.

5. A rigid lamina not acted on by any forces has one point fixed about which it can turn freely. It is started about a line in the plane of the lamina the moment of inertia about which is Q. Show that the ratio of the greatest to the least angular velocity is $\sqrt{A+B}:\sqrt{B+Q}$, where A, B are the principal moments of inertia about axes in the plane of the lamina.

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* These examples are taken from the Examination Papers which have been set in the University and in the Colleges.

6. If the earth were a rigid body acted on by no force rotating about a diameter which is not a principal axis, show that the latitudes of places would vary and that the values would recur whenever $\sqrt{A-B}\sqrt{A-C}\int\omega_1 dt$ is a multiple to $2\pi\sqrt{BC}$. If a man were to lie down when his latitude is a minimum and to rise when it becomes a maximum, show that he would increase the vis viva, and so cause the pole of the earth to travel from the axis of greatest moment of inertia towards that of least moment of inertia.

7. If $d\theta$ be the angle between two consecutive positions of the instantaneous axis, prove that

$$\omega^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\omega_1}{dt}\right)^2 + \left(\frac{d\omega_2}{dt}\right)^2 + \left(\frac{d\omega_3}{dt}\right)^2 - \left(\frac{d\omega}{dt}\right)^2.$$

8. If n be the angular velocity of the plane through the invariable line and the instantaneous axis about the invariable line and λ the component angular velocity of the body about the invariable line, prove that

$$\left(\frac{1}{2}\frac{dn}{dt}\right)^2 + (n-\lambda)\left(n-\frac{G}{A}\right)\left(n-\frac{G}{B}\right)\left(n-\frac{G}{C}\right) = 0.$$

9. If a body move in any manner, and all the forces pass through the centre of gravity, prove that

$$\frac{d(\omega^2)}{dt} + 2\frac{d}{dt}(\log\omega_1)\frac{d}{dt}(\log\omega_2)\frac{d}{dt}(\log\omega_3) = 0,$$

where ω_1, ω_2, ω_3 are the angular velocities about the principal axes at the centre of gravity, and ω is the resultant angular velocity.

CHAPTER X.

MOTION OF A BODY UNDER ANY FORCES.

557. In this Chapter it is proposed to discuss some cases of the motion of a rigid body in three dimensions as examples of the processes explained in Chapter V. The reader will find it an instructive exercise to attempt their solution by other methods; for example, the equations of Lagrange might be applied with advantage in some cases.

Motion of a Top.

558. *A body two of whose principal moments at the centre of gravity are equal moves about some fixed point* O *in the axis of unequal moment under the action of gravity. Determine the motion.* See Art. 374.

To give distinctness to our ideas we may consider the body to be a top spinning on a perfectly rough horizontal plane.

Let the axis OZ be vertical. Let the axis of unequal moment at the centre of gravity be the axis OC and let this be called the axis of the body. Let h be the distance of the centre of gravity G of the body from the fixed point O and let the mass of the body be taken as unity. Let OA be that principal axis at O which lies in the plane ZOC, OB the principal axis perpendicular to this plane.

If we take moments about the axis OC we have by Euler's equations (Art. 230),

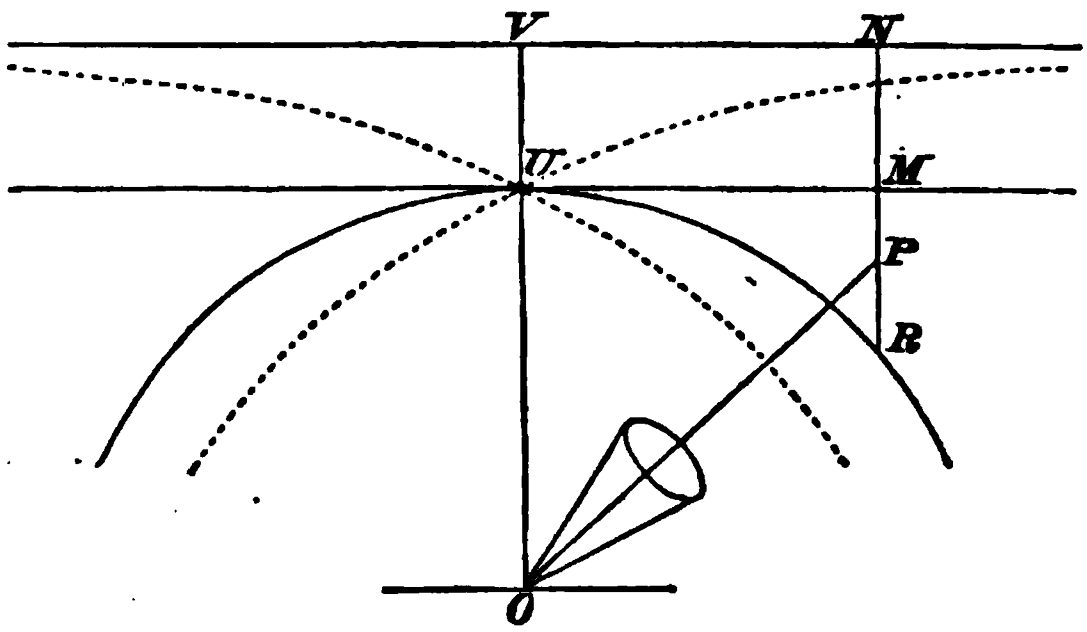
$$C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 = N.$$

But in our case $A = B$, and since the centre of gravity lies in the axis OC, we have $N = 0$. Hence ω_3 is constant and equal to its initial value. Let this be called n.

Let us measure along the axis OC in the direction OG a

length $OP = \frac{A}{h}$. Then, by Art. 92, P is the centre* of oscillation of the body. This length we shall call l. Let θ be the inclination of the axis OC to the vertical, ψ the angle the plane ZOC makes with some plane fixed in space passing through OZ. Then by the same reasoning as in Art. 235 we find that the velocities of P resolved

$$\left.\begin{array}{r} \text{perpendicular to plane } ZOC = -l\omega_1 = l\sin\theta\dfrac{d\psi}{dt} \\ \text{parallel to plane } ZOC = \quad l\omega_2 = l\dfrac{d\theta}{dt} \end{array}\right\}\dots\dots\dots(1).$$



It is clear that the moment of the momentum about OZ will be constant throughout the motion. Since the direction-cosines of OZ referred to OA, OB, OC are $-\sin\theta$, 0 and $\cos\theta$, this principle gives

$$-A\omega_1 \sin\theta + Cn\cos\theta = E\dots\dots\dots\dots\dots\dots(2),$$

where E is some constant depending on the initial conditions, and whose value may be found from this equation by substituting the initial value of ω_1 and θ.

The equation of Vis Viva gives

$$A(\omega_1^2 + \omega_2^2) + Cn^2 = F - 2gh\cos\theta\dots\dots\dots\dots\dots(3),$$

where F is some constant, whose value may be found by substituting in this equation the initial values of ω_1, ω_2, and θ †.

* To avoid confusion in the figure, the body which is represented by a top is drawn smaller than it should be.

† If we eliminate ω_1, ω_2 from equations (1), (2), (3) we have two equations from which θ and ψ may be found by quadratures. These were first obtained by Lagrange in his *Mécanique Analytique*, and were afterwards given by Poisson in his *Traité de Mécanique*. The former passes them over with but slight notice, and proceeds to discuss the small oscillations of a body of any form suspended under the action of gravity from a fixed point. The latter limits the equations to

559. Let us measure along the vertical OZ, in the direction opposite to gravity as the positive direction, two lengths $OU=\frac{El}{Cn}$, $OV=\frac{l(F-Cn^2)}{2gh}$. These lengths we shall write briefly $OU=a$, and $OV=b$. Draw through U and V two horizontal planes, and let the vertical through P intersect these planes in M and N. Then the equations (2) and (3) give by (1),

$$\left.\begin{matrix}\text{horizontal velocity}\\ \text{of } P\end{matrix}\right\} = \frac{Cn}{h}\tan PUM \text{..........................(4).}$$

$$(\text{velocity of } P)^2 = 2gPN \text{..................................(5).}$$

Thus the resultant velocity of P is that due to the depth of P below the horizontal plane through V, and the velocity of P resolved perpendicular to the plane ZOP is proportional to the tangent of the angle PU makes with a horizontal plane.

It appears from this last result that when P is below the horizontal plane through U, the plane POV turns round the vertical in the same direction as the body turns round its axis, i.e. according to the rule in Art. 199, OV and OP are the positive directions of the axes of rotation. When P passes above the horizontal plane through U, the plane POV turns round the vertical in the opposite direction. If P be below both the horizontal planes through O and U these results are still true, but if a top is viewed from above, the axis will appear to turn round the vertical in the direction opposite to the rotation of the top. In all the cases in which P is below the plane UM the lowest point of the rim of the top moves round the vertical in the same direction as the axis of the top.

If we substitute for ω_1, ω_2, E and F in (2) and (3) their values, we easily obtain

$$\left.\begin{aligned} hl\sin^2\theta\frac{d\psi}{dt}+Cn\cos\theta &= Cn\frac{a}{l} \\ l^2\left\{\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\psi}{dt}\right)^2\right\} &= 2g(b-l\cos\theta)\end{aligned}\right\}\text{.....................(6).}$$

These equations give in a convenient analytical form the whole motion. We see from the last equation, what is indeed obvious otherwise, that $b-l\cos\theta$ is always positive. The horizontal plane through V is therefore above the initial position of P and remains above P throughout the whole motion.

Ex. 1. If ω be the resultant angular velocity of the body and v the velocity of P show that $\omega^2=n^2+\left(\frac{v}{l}\right)^2$.

Ex. 2. Show that the cosine of the inclination of the instantaneous axis to the vertical is $\frac{E+(A-C)n\cos\theta}{A\omega}$.

560. As the axis of the body goes round the vertical its inclination to the vertical is continually changing. These changes

the case in which the body has an initial angular velocity only about its axis, and applies them to determine directly the small oscillations of a top (1) when its axis is nearly vertical, and (2) when its axis makes a nearly constant angle with the vertical. His results are necessarily more limited than those given in this treatise.

may be found by eliminating $\frac{d\psi}{dt}$ between the equation (6). We thus obtain

$$\left(l\frac{d\theta}{dt}\right)^2 = 2g(b - l\cos\theta) - \frac{C^2n^2}{h^2}\left(\frac{a - l\cos\theta}{l\sin\theta}\right)^2 \ldots\ldots\ldots\ldots(7).$$

It appears from this equation that θ can never vanish unless $a = l$, for in any other case the right-hand side of this equation would become infinite. This may be proved otherwise. Since $\frac{a}{l}$ is equal to the ratio of the angular momentum about the vertical to that about the axis of the body, it is clear the axis could not become vertical unless the ratio is unity.

Suppose the body to be set in motion in any way with its axis at an inclination i to the vertical. The axis will begin to approach or to fall away from the vertical according as the initial value of $\frac{d\theta}{dt}$ or ω_2 is negative or positive. The axis will then oscillate between two limiting angles given by the equation

$$0 = 2gh^2l^2(b - l\cos\theta)(1 - \cos^2\theta) - C^2n^2(a - l\cos\theta)^2 \ldots\ldots(8).$$

This is a cubic equation to determine $\cos\theta$. It will be necessary to examine its roots. When $\cos\theta = -1$ the right-hand side is *negative;* when $\cos\theta = \cos i$, since the initial value of $\left(\frac{d\theta}{dt}\right)^2$ is essentially positive, the right-hand side is either zero or *positive;* hence the equation has one real root between $\cos\theta = -1$ and $\cos\theta = \cos i$. Again, the right-hand side is *negative* when $\cos\theta = +1$ and *positive* when $\cos\theta = \infty$. Hence there is another real root between $\cos\theta = \cos i$, and $\cos\theta = 1$, and a third root greater than unity. This last root is inadmissible.

561. These limits may be conveniently expressed geometrically. The equation (7) may evidently be written in the form

$$\left(l\frac{d\theta}{dt}\right)^2 = 2g\,.\,PN - \frac{C^2n^2}{h^2}\cdot\left(\frac{PM}{UM}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Describe a parabola with its vertex at U, its axis vertically downwards and its latus rectum equal to $\frac{C^2n^2}{2gh^2}$. Let the vertical PMN cut this parabola in R, we then have

$$\frac{2g}{\left(l\frac{d\theta}{dt}\right)^2 - 2gMN} = \frac{1}{PM} + \frac{1}{PR} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

The point P oscillates between the two positions in which the harmonic mean of PM and PR is equal to $-2\,.\,MN$. In the figure V is drawn above U, and in this case one of the limits of P is above UM, and the other below the parabola. If we take U as origin and UO the axis of x, we have $PM = x$, $UM = y$. Let $2pl$ be the

latus rectum of the parabola, and $UV=c$, then the axis of the body oscillates between the two positions in which P lies on the cubic curve

$$y^2(x+c)=2plx^2 \qquad (11).$$

When c is positive, i.e. when V is above U, the form of the curve is indicated in the figure by the dotted line. The tangents at U cut each other at a finite angle and the tangent of the angle either makes with the vertical is $\left(\frac{2pl}{c}\right)^{\frac{1}{2}}$. When c is negative the curve has two branches, one on each side of the vertical, with a conjugate point at the origin. It is clear from what precedes that the upper branch will lie above, and the lower branch below, the initial position of P, and that P must always lie between the two branches.

562. In the case of a top, the initial motion is generally given by a rotation n about the axis. We have initially $\omega_1=0$, $\omega_2=0$, and therefore by (2) and (3) $E=Cn\cos i$, and $F-Cn^2=2gh\cos i$. This gives $a=b=l\cos i$. Putting $\frac{C^2n^2}{2gh^2}=2pl$, as before, the roots of equation (8) are $\cos\theta=\cos i$, and $\cos\theta=p-\sqrt{1-2p\cos i+p^2}$. The value $\cos\theta=p+\sqrt{1-2p\cos i+p^2}$ is always greater than unity, for it is clearly decreased by putting unity for $\cos i$, and its value is then not less than unity. The axis of the body will therefore oscillate between the values of θ just found.

Since $a=b$, the horizontal planes through U and V coincide, and $c=0$. The cubic curve which determines the limits of oscillation, becomes the parabola UR and the straight line UM. The axis of the body will then oscillate between the two positions in which P lies on the horizontal through U and on the parabola.

Generally the angular velocity n about the axis of figure is very great. In this case p is very great, and if we reject the squares of $\frac{1}{p}$ we see that $\cos\theta$ will vary between the limits $\cos i$ and $\cos i-\frac{1}{2p}\sin^2 i$.

If the initial value of i is zero, we see that the two limits of $\cos i$ are the same. The axis of the body will therefore remain vertical.

563. Ex. 1. When the limiting angles between which θ varies are equal to each other, so that θ is constant throughout the motion and equal to a, show that

$$\tan^2\phi-\tan\phi\tan a+\frac{\cos a}{4p}\tan^2 a=0,$$

where ϕ is the angle PUM.

Ex. 2. A top is set in motion on a smooth horizontal plane with an initial resultant angular velocity about its axis of figure. Show that the path traced out by the apex on the horizontal plane lies between two circles, one of which it touches and the other it cuts at right angles. [*M. Finck, Nouvelles Annales de Mathématiques.* Tom. IX, 1850.]

564. *A body, two of whose principal moments at the centre of gravity G are equal, turns about a fixed point O in the axis of unequal moment under the action of gravity. The axis OG being inclined to the vertical at an angle α, and revolving about it with a uniform angular velocity, find the condition that the motion may be steady, and the time of a small oscillation.*

The equations (2) and (3) of Art. 558 contain the solution of this problem. But if we use the equation of Vis Viva in the form (3) we shall have to take into account the squares of small quantities. It will be found more convenient to replace it by one of the equations of the second order from which it has been derived. The simplest method of obtaining this equation is to use Lagrange's Rule as in Art. 374.

We thus obtain

$$A\frac{d^2\theta}{dt^2} - A\cos\theta\sin\theta\left(\frac{d\psi}{dt}\right)^2 + Cn\sin\theta\frac{d\psi}{dt} = gh\sin\theta \text{ .... (12).}$$

This equation might also have been obtained by differentiating both (2) and (3) and eliminating $\frac{d^2\psi}{dt^2}$.

When the motion is steady both θ and $\frac{d\psi}{dt}$ are constants. Let $\theta = \alpha$, $\frac{d\psi}{dt} = \mu$, then the equation (2) only determines the constant E and (12) becomes

$$\sin\alpha\,(-A\cos\alpha\mu^2 + Cn\mu - gh) = 0 \text{ .... (13).}$$

This indicates two possible states of steady motion, one in which $\alpha = 0$ or π, and the other in which

$$\mu = \frac{Cn \pm \sqrt{C^2n^2 - 4ghA\cos\alpha}}{2A\cos\alpha} \text{ .... (14),}$$

a relation which does not necessarily hold when $\alpha = 0$ or π.

In the former of these two motions the axis of the body will oscillate about the vertical and $\frac{d\psi}{dt}$ will not be small or nearly constant. It will therefore be more convenient to discuss the oscillations about this state of steady motion with other co-ordinates than θ and ψ.

In the latter of these motions, we must have n^2 not less than $\frac{4ghA\cos\alpha}{C^2}$. When α and n are given we can make the body move with either of these two values of μ by giving the proper initial angular velocities to the body. By equations (1) we see that the conditions of steady motion are $\omega_1 = -\mu\sin\alpha$, $\omega_2 = 0$. When a top is set in motion by unwinding a string from the axis, the value of n is very great while the initial values of ω_1 and ω_2 are zero. The steady motion about which the top makes small oscillations will therefore have μ small. Hence the radical in (14) will have the negative sign. We have therefore very nearly $\mu = \frac{gh}{Cn}$.

565. *To find the small oscillation.* Let $\theta = \alpha + \theta'$, and $\frac{d\psi}{dt} = \mu + \frac{d\psi'}{dt}$, where θ' and $\frac{d\psi'}{dt}$ are small quantities whose squares are to be neglected. Let α and μ be such that they contain the whole of the constant parts of θ and $\frac{d\psi}{dt}$, so that θ' and $\frac{d\psi'}{dt}$ contain only trigonometrical terms. Then when we substitute these values in equations (2) and (12), the constant parts must vanish of themselves. The equa-

tions thus obtained determine F and μ, and show that their values are the same as those determined when the motion is steady. The variable parts of the two equations become, after writing for Cn its value obtained from (13),

$$\left.\begin{aligned} A\mu \sin\alpha \frac{d\psi'}{dt} - (gh - A\mu^2\cos\alpha)\,\theta' = 0 \\ A\mu\frac{d^2\theta'}{dt^2} + \sin\alpha\,(gh - A\mu^2\cos\alpha)\frac{d\psi'}{dt} + \mu^2 A\sin^2\alpha\theta' = 0 \end{aligned}\right\}.$$

To solve these, put $\theta' = F\sin(pt+f)$, and $\psi' = G\cos(pt+f)$.

Substituting, we have

$$\left.\begin{aligned} -A\mu\sin\alpha\,.\,pG = (gh - A\mu^2\cos\alpha)\,F \\ (A\mu p^2 - \mu^3 A\ \sin^2\alpha)\,F = -(gh - A\mu^2\cos\alpha)\sin\alpha\,.\,Gp \end{aligned}\right\}.$$

Multiplying these equations together, we have

$$p^2 = \frac{A^2\mu^4 - 2ghA\cos\alpha\,\mu^2 + g^2h^2}{A^2\mu^2},$$

and the required time is $\frac{2\pi}{p}$*. It is evident that p^2 is always positive, and therefore both the values of μ given by (14) correspond to stable motions.

It is to be observed that this investigation does not apply if α be very small, for in that case some of the terms rejected are of the same order of magnitude as those retained. A different mode of investigation is therefore required, this case will be considered in Art. 569.

566. We may also determine the *steady motion* very simply by another process, which will be found useful when we come to consider Precession and Nutation. Let OC be the axis of the body, OI the instantaneous axis of rotation, OZ the vertical. Then when the motion is steady, these three must be in one vertical plane which revolves about OZ with a uniform angular velocity μ. Let ω be the angular velocity about OI, then $\omega\cos IC = n$. Let OB be the horizontal axis about which gravity tends to turn the body, then OB is perpendicular to the plane ZOC.

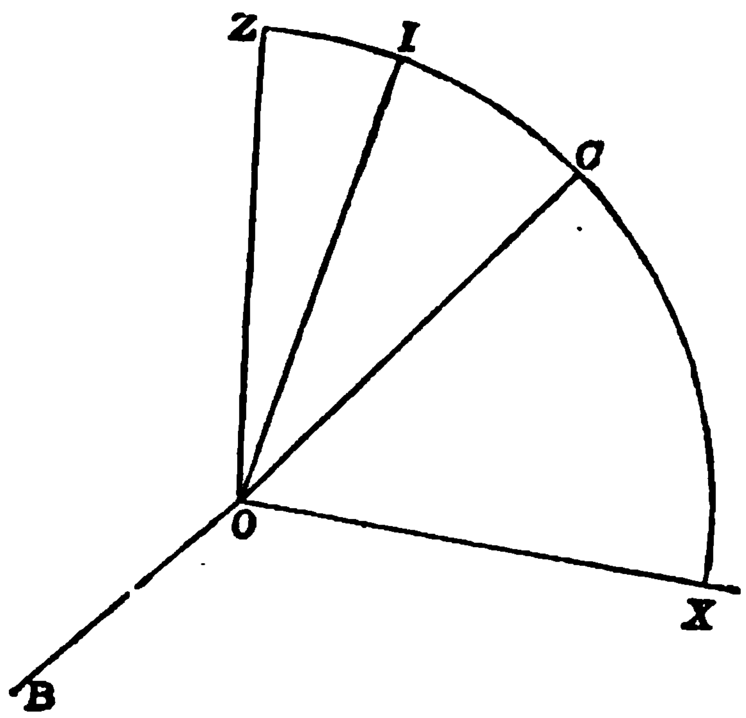
Since gravity generates an angular velocity $\frac{gh\sin\alpha}{A}\,dt$ in the time dt about OB, therefore by the parallelogram of angular velocities, the instantaneous axis OI has moved in the time dt through an angle $\frac{gh\sin\alpha}{A\omega}\,dt$ in a plane perpendicular to the plane ZOI. Hence the angular velocity of I round Z due to the action of the forces is $\frac{d\psi_1}{dt} = \frac{gh\sin\alpha}{A\omega}\cdot\frac{1}{\sin IZ}$.

Also, since the angular velocity of the body about OB is zero, the moments of the centrifugal forces about the axes OA, OC are zero. The moment about OB is $(A - C)\,n\omega\sin IC\,dt$, and this generates an angular velocity $\frac{A-C}{A}\,n\omega\sin IC\,dt$ about OB. Hence the angular velocity of I round Z due to the centrifugal forces of the body is $\frac{d\psi_2}{dt} = \frac{A-C}{A}\,n\,\frac{\sin IC}{\sin IZ}$.

* This expression was given by the Rev. N. M. Ferrers of Gonville and Caius College, as the result of a problem proposed by him for solution in the Mathematical Tripos, 1859.

The whole angular velocity is the sum of these two, i.e.

$$\mu = \left(\frac{gh \sin \alpha}{An} \cot IC + \frac{A-C}{A} n\right) \frac{\sin IC}{\sin IZ}.$$



But when the motion is steady OZ, OI and OC are all in one plane. Now the angular velocity of C round I is ω, and therefore its angular velocity round Z is $\mu = \omega \frac{\sin IC}{\sin ZC}$. But $\omega \cos IC = n$, hence, $\tan IC = \frac{\mu \sin \alpha}{n}$. Substituting this value of $\tan IC$ in the value of μ, we get $\frac{gh}{\mu} = Cn - A\mu \cos \alpha$, the same expression as before.

567. Ex. A top two of whose principal moments at O are equal is set in rotation about its axis of figure viz. OC with an angular velocity n, the point O being fixed. If OC be *horizontal*, and if the proper initial angular velocity be communicated to the top about the vertical through O, prove that the top will not fall down, but that the axis of figure will revolve round the vertical, in steady motion, with an angular velocity $\mu = \frac{gh}{Cn}$, where h is the distance of the centre of gravity of the top from O, and C is the moment of inertia about the axis of figure. Show also that if the top be initially placed with OC nearly horizontal and if a very great angular velocity be communicated to it about OC without any initial angular velocity about OA or OB, then OC will revolve round the vertical remaining very nearly in a horizontal plane with an angular velocity μ given by the same formula as before, and the time of the vertical oscillations of OC about its mean position will be $\frac{2\pi A}{Cn}$.

568. *A body whose principal moments of inertia are not necessarily equal has a point* O *fixed in space and moves about* O *under the action of gravity. It is required to form the general equations of motion.*

Let OA, OB, OC be the principal axes at the fixed point O, and let these be taken as axes of reference. Let h, k, l be the co-ordinates of the centre of gravity G, and let the mass of the body be taken as unity. Let OV be drawn vertically *upwards*

and let p, q, r be the direction-cosines of OV referred to OA, OB, OC. Then we have by Euler's equations

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\omega_3&=-g\,(kr-lq)\\ B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\omega_1&=-g\,(lp-hr)\\ C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2&=-g\,(hq-kp)\end{aligned}\right\}\text{............}(1).$$

Also p, q, r may be regarded as the co-ordinates of a point in OV, distant unity from O. This point is fixed in space, and therefore its velocities as given by Art. 248 are zero. We have

$$\left.\begin{aligned}\frac{dp}{dt}&=\omega_3 q-\omega_2 r\\ \frac{dq}{dt}&=\omega_1 r-\omega_3 p\\ \frac{dr}{dt}&=\omega_2 p-\omega_1 q\end{aligned}\right\}\text{........................}(2).$$

It is obvious that two integrals of these equations are supplied by the principles of Angular Momentum and Vis Viva. These give

$$A\omega_1 p+B\omega_2 q+C\omega_3 r=E,$$
$$A\omega_1^2+B\omega_2^2+C\omega_3^2=F-2g\,(ph+qk+rl),$$

where E and F are two arbitrary constants. The first of these might also have been obtained by multiplying the equations (1) by p, q, r respectively, and (2) by $A\omega_1$, $B\omega_2$, $C\omega_3$, and adding all six results. The second might have been obtained by multiplying the equations (1) by ω_1, ω_2, ω_3 respectively, adding and simplifying the right-hand side by (2).

569. *A body whose principal moments of inertia at the centre of gravity* G *are not necessarily equal, has a point* O *in one of the principal axes at* G *fixed in space and moves about* O *under the action of gravity. Supposing the body to be performing small oscillations about the position in which* OG *is vertical, find the motion.*

Referring to the general equations of Art. 568, we see that in this case $h=0$, $k=0$. Since OC remains always nearly vertical, ω_1 and ω_2 are small quantities, we may therefore reject the product $\omega_1\omega_2$ in the last of equations (1). This gives ω_3 constant. Let this constant value be called n. For the same reason $r=1$ nearly and p, q are both small quantities. Substituting we get the following linear equations,

$$\left.\begin{aligned}A\frac{d\omega_1}{dt}-(B-C)\,n\omega_2&=lgq\\ B\frac{d\omega_2}{dt}-(C-A)\,n\omega_1&=-lgp\end{aligned}\right\}\text{.........}(3),\qquad \left.\begin{aligned}\frac{dp}{dt}&=qn-\omega_2\\ \frac{dq}{dt}&=-pn+\omega_1\end{aligned}\right\}\text{.........}(4).$$

To solve these, assume

$$\left.\begin{array}{l}\omega_1 = F\sin(\lambda t+f)\\ \omega_2 = G\cos(\lambda t+f)\end{array}\right\},\qquad \left.\begin{array}{l}p = P\sin(\lambda t+f)\\ q = Q\cos(\lambda t+f)\end{array}\right\}.$$

Substituting, we get

$$\left.\begin{array}{l}A\lambda F-(B-C)nG=glQ\\ B\lambda G-(A-C)nF=glP\end{array}\right\}\ \ldots\ldots\ldots\ldots(5),\qquad \left.\begin{array}{l}\lambda P=Qn-G\\ \lambda Q=Pn-F\end{array}\right\}\ \ldots\ldots\ldots\ldots(6).$$

Eliminating the ratios $F:G:P:Q$ we have

$$\lambda^2n^2(A+B-C)^2=\{gl+A\lambda^2+(B-C)n^2\}\{gl+B\lambda^2+(A-C)n^2\}.$$

If the values of λ thus found should be real, the body will make small oscillations about the position in which OG is vertical. If C be the greatest moment, and n^2 sufficiently great to make both $gl-(C-A)n^2$ and $gl-(C-B)n^2$ negative, then all the values of λ are real and the body will continue to spin with OG vertical. If G be beneath O, l is negative and it will be sufficient that OC should be the axis of greatest moment.

In order that the values of λ^2 may be real, we must have

$$\{gl(A+B)+n^2(AC+BC-2AB-C^2)\}^2>4\{(B-C)n^2+gl\}\{(A-C)n^2+gl\}AB,$$

and in order that the two values of λ^2 may have the same sign we must have the last term of the quadratic positive;

$$\therefore\ \{(B-C)n^2+gl\}\{(A-C)n^2+gl\}=\text{ a positive quantity},$$

and in order that the values of λ^2 may be both positive, we must have the coefficient of λ^2 in the quadratic negative;

$$\therefore\ gl(A+B)<n^2(B-C)(A-C).$$

In the particular case in which $A=B$, each side of the quadratic becomes a perfect square and we have

$$A\lambda^2\pm(2A-C)n\lambda+(A-C)n^2+gl=0;$$

$$\therefore\ \lambda=\mp\frac{2A-C}{2A}n\pm\frac{\sqrt{C^2n^2-4Agl}}{2A}.$$

With the reservations mentioned in Art. 434, the necessary and sufficient condition of stability is in this case $n>\dfrac{2\sqrt{Agl}}{C}$. By referring to equations (5) and (6) it will be seen that when $A=B$ we have $F=G$ and $P=Q$. If λ_1, λ_2 be the two values of λ found above, we have

$$\left.\begin{array}{l}p=P_1\sin(\lambda_1t+f_1)+P_2\sin(\lambda_2t+f_2)\\ q=P_1\cos(\lambda_1t+f_1)+P_2\cos(\lambda_2t+f_2)\end{array}\right\}.$$

Let θ be the angle OC makes with the vertical, then $r^2=\cos^2\theta=1-\theta^2$, and hence

$$\theta^2=p^2+q^2=P_1^2+P_2^2+2P_1P_2\cos\{(\lambda_1-\lambda_2)t+f_1-f_2\}.$$

Also if, as in Art. 285, we let ϕ be the angle the plane containing OA, OC makes with the plane containing OC and the vertical OV, we have $p = -\sin\theta\cos\phi$, and $q = \sin\theta\sin\phi$, and hence

$$-\tan\phi = \frac{P_1\cos(\lambda_1 t+f_1)+P_2\cos(\lambda_2 t+f_2)}{P_1\sin(\lambda_1 t+f_1)+P_2\sin(\lambda_2 t+f_2)}.$$

Also since θ is very small we have, following the notation of the same Article,

$$\psi = nt + \alpha - \phi,$$

where α is some constant, depending on the position of the arbitrary plane from which ψ is measured*.

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* In order to understand the relation which exists between these results and those of Art. 565, it will be necessary to determine the oscillations by some process which holds both when α is large and very small. This may be done as follows. We have by Vis Viva the equation (see Art. 558)

$$\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{E - Cn\cos\theta}{A\sin\theta}\right)^2 = \frac{F' - 2gh\cos\theta}{A} \quad\ldots\ldots\ldots\ldots(1),$$

where F' has been put for $F - Cn^2$. If we put $z = \cos\theta$, this takes the form

$$A^2\left(\frac{dz}{dt}\right) + (E - Cnz)^2 = A(F' - 2ghz)(1 - z^2)\ldots\ldots\ldots\ldots(2).$$

Let us assume as the solution of this equation

$$z = \cos\alpha + P\cos(\lambda t + f)\ldots\ldots\ldots\ldots(3),$$

where P is so small that on substituting in the above equation we may neglect P^3. Substituting and equating to zero the coefficients of the several powers of $\cos(\lambda t+f)$ we get

$$\left.\begin{aligned} &A^2P^2\lambda^2 + (E - Cn\cos\alpha)^2 = A(F' - 2gh\cos\alpha)(1-\cos^2\alpha) \\ &-(E - Cn\cos\alpha)\,Cn = -ghA - AF'\cos\alpha + 3ghA\cos^2\alpha \\ &-A^2\lambda^2 + C^2n^2 = -AF' + 6ghA\cos\alpha \end{aligned}\right\}\ldots\ldots\ldots(4).$$

Now let us change the constant E into another μ by putting $\dfrac{E - Cn\cos\alpha}{A\sin^2\alpha} = \mu + \gamma P^2$, where γ is to be so chosen as to remove the term $A^2P^2\lambda^2$ in our first equation. Since

$$\frac{d\psi}{dt} = \frac{E - Cn\cos\theta}{A\sin^2\theta}\quad\ldots\ldots\ldots\ldots(5),$$

we see that, when θ is not small, μ differs from the constant part of $\dfrac{d\psi}{dt}$ only by quantities depending on the *squares* of the small oscillation, and which are neglected in the text. Substituting for E and eliminating F' between the first and second equations we get $Cn\mu = A\cos\alpha\mu^2 + gh$.

Eliminating F' between the first and third of equations (4) and substituting for n we get

$$\lambda^2 = \frac{\mu^4A^2 - 2ghA\cos\alpha\mu^2 + g^2h^2}{A^2\mu^2}.$$

This process gives the period of the small oscillation in $\cos\theta$. When θ is finite this is the same as the oscillation in θ, since $\cos\theta = \cos\alpha - \sin\alpha\theta'$. When θ is very small, $\cos\theta = 1 - \dfrac{\theta^2}{2}$ and the time of oscillation in $\cos\theta$ is the same as that in θ^2. With this understanding it will be seen that there is a perfect agreement between the results of Arts. 565 and 569, when α is put equal to zero.

570. *A body whose principal moments at the centre of gravity are not necessarily equal is free to turn about a fixed point* O, *and is in equilibrium under the action of gravity. A small disturbance being given, find the oscillations.*

Referring to the general equations in Art. 568 we see that in this case ω_1, ω_2, ω_3, are small, hence in equations (1) we may omit the terms containing the products $\omega_1\omega_2$, $\omega_2\omega_3$, $\omega_3\omega_1$. Also since in equilibrium OG is vertical, p, q, r are always nearly in the ratio $h:k:l$; hence if $OG=a$, we may write $\frac{h}{a}, \frac{k}{a}, \frac{l}{a}$ for p, q, r on the right-hand sides of equations (2). The six equations are now all linear. To solve these we put

$$\omega_1 = H\sin(\lambda t+\mu) \text{ and } p=\frac{h}{a}+P\cos(\lambda t+\mu) \qquad (3),$$

ω_2, ω_3, q and r being represented by similar expressions with K and L written for H; Q, k and R, l written for P and h. Substituting these in the equations we get six linear equations. Eliminating P, Q, R we have

$$\left.\begin{aligned} \left(\frac{a}{g}A\lambda^2+k^2+l^2\right)H-hkK-lhL&=0\\ -hkH+\left(\frac{a}{g}B\lambda^2+l^2+h^2\right)K-lkL&=0\\ -lhH-lkL+\left(\frac{a}{g}C\lambda^2+h^2+k^2\right)L&=0\end{aligned}\right\} \qquad (4).$$

Eliminating the ratios of H, K, L we have an equation to find λ^2. One root is $\lambda^2=0$, the others are given by the quadratic

$$\lambda^4+\left(\frac{k^2+l^2}{A}+\frac{l^2+h^2}{B}+\frac{h^2+k^2}{C}\right)\frac{g}{a}\lambda^2+g^2\frac{Ah^2+Bk^2+Cl^2}{ABC}=0 \qquad (5).$$

To ascertain if the roots are real we must apply the usual criterion for a quadratic. This requires that

$$\{A(B-C)h^2+B(C-A)k^2-C(A-B)l^2\}^2+4AB(B-C)(A-C)h^2k^2 \qquad (6)$$

should be positive. Since A, B, C can be chosen to be in descending order, we see that the condition is satisfied. See also Art. 448.

If G is above O, a is positive and the values of λ^2 are both negative. The equilibrium is therefore unstable. If G is below O, a is negative and the values of λ^2 are both positive. If the roots are equal, the two positive terms in (6) must be separately zero, this gives $k=0$ and $A(B-C)h^2=C(A-B)l^2$, i.e. the centre of gravity lies in the asymptote to the focal hyperbola of the momental ellipsoid. In this case we find $\lambda^2=-\frac{ag}{B}$. The case in which $k=0$, $l=0$, $B=C$ has been considered in Art. 564.

If the values of λ^2 are written 0, λ_1^2, λ_2^2 we have

$$\omega_1=H_0+H_0't+H_1\sin(\lambda_1t+\mu_1)+H_2\sin(\lambda_2t+\mu_2),$$

with similar expressions for ω_2, ω_3. Equations (2) then give p, q, r. But substituting in (1) we find that all the non-periodic terms which contain t are zero. Remembering that $p^2+q^2+r^2=1$ we have finally

$$\omega_1=\Omega\frac{h}{a}+H_1\sin(\lambda_1t+\mu_1)+H_2\sin(\lambda_2t+\mu_2),$$

ω_2 and ω_3 being represented by similar expressions with k, K and l, L written for h, H. The values of K_1, L_1 and K_2, L_2 are determined by equations (4) in terms of H_1 and H_2 respectively. We also have

$$p=\frac{h}{a}+\frac{lK_1-kL_1}{a\lambda_1}\cos(\lambda_1 t+\mu_1)+\frac{lK_2-kL_2}{a\lambda_2}\cos(\lambda_2 t+\mu_2),$$

with similar expressions for q and r. There remain five constants viz. Ω, H_1, H_2, μ_1, μ_2 to be determined by the initial values of ω_1, ω_2, ω_3, r and q.

When the roots are equal the equations depending on p, r, ω_2 separate from those depending on q, ω_1, ω_3, forming two sets; we find

$$\left.\begin{aligned}\omega_1&=\Omega\frac{h}{a}+H\sin(\lambda t+\mu_1)\\ \omega_3&=\Omega\frac{l}{a}+H\frac{Aa\lambda^2+gl^2}{ghl}\sin(\lambda t+\mu_1)\\ q&=\quad H\frac{A\lambda}{gl}\cos(\lambda t+\mu_1)\end{aligned}\right\},\qquad \left.\begin{aligned}\omega_2&=\quad K\sin(\lambda t+\mu_2)\\ p&=\frac{h}{a}+K\frac{l}{a\lambda}\cos(\lambda t+\mu_2)\\ r&=\frac{l}{a}-K\frac{h}{a\lambda}\cos(\lambda t+\mu_2)\end{aligned}\right\}.$$

A solution of this problem conducted in a totally different manner has been given by Lagrange in his *Mécanique Analytique*. His results do not altogether agree with those given here.

If we substitute the values of ω_1, ω_2, ω_3, p, q, r in the equation of angular momentum of Art. 568 and neglect the squares of small quantities, we evidently obtain

$$(Ah^2+Bk^2+Cl^2)\,\Omega=Ea^2,\qquad AHh+BKk+CLl=0.$$

The first of these equations shows that Ω vanishes when the initial conditions are such that the angular momentum about the vertical is zero. In this case the problem reduces to that considered in Art. 455.

571. *A body whose principal moments of inertia are not necessarily equal has a point* O *fixed in space and moves about* O *under the action of gravity. It is required to find what cases of steady motion are possible in which one principal axis* OC *at* O *describes a right cone round the vertical while the angular velocity of the body about* OC *is constant; and to find the small oscillations.*

Referring to the general equations of Art. 568, we see that r and ω_3 are given to be constants. In this case the first two equations of (1) and (2) form a set of *linear* equations to find the four quantities p, q, ω_1, ω_2. The solution of these equations is therefore of the form

$$\left.\begin{aligned}\omega_1&=F_0+F_1\sin(\lambda t+f)\\ \omega_2&=G_0+G_1\cos(\lambda t+f)\end{aligned}\right\},\qquad \left.\begin{aligned}p&=P_0+P_1\sin(\lambda t+f)\\ q&=Q_0+Q_1\cos(\lambda t+f)\end{aligned}\right\}.$$

But these must also satisfy the last of equations (1). Substituting we see that there will be a term on the left side of the form

$$-\frac{1}{2}(A-B)\,F_1G_1\sin 2(\lambda t+f).$$

But there will be no such term on the right side. Hence we must have either $A=B$, $F_1=0$ or $G_1=0$. The motion in the case in which $A=B$ has already been considered in Art. 564. Again, substituting in the last of equations (2) and equating to zero the coefficient of $\sin 2(\lambda t+f)$ we find

$$P_1G_1-F_1Q_1=0.$$

Substituting in the first two of equations (1) and equating to zero the coefficients of $\cos(\lambda t+f)$ and $\sin(\lambda t+f)$, we find

$$A\lambda F_1-(B-C)\,nG_1=glQ_1$$
$$-B\lambda G_1-(C-A)\,nF_1=-glP_1;$$

from these equations we have F_1, G_1, P_1, Q_1 all equal to zero and therefore ω_1, ω_2, p, q are all constant as well as the given constants ω_3 and r.

In this case the equations (2) give

$$\frac{\omega_1}{p}=\frac{\omega_2}{q}=\frac{\omega_3}{r},$$

so that the axis of revolution must be vertical. Let ω be the angular velocity about the vertical. Then $\omega_1=p\omega$, $\omega_2=q\omega$, $\omega_3=r\omega$. Substituting in equations (1) we get

$$\frac{h}{p}-\frac{A\omega^2}{g}=\frac{k}{q}-\frac{B\omega^2}{g}=\frac{l}{r}-\frac{C\omega^2}{g}\qquad\qquad(3).$$

Unless, therefore, two of the principal moments are equal, it is necessary for steady motion that the axis of rotation should be vertical and the centre of gravity (hkl) must lie in the vertical straight line whose equations are (3).

This straight line may be constructed geometrically in the following manner. Measure along the vertical a length $OV=\frac{g}{\omega^2}$ and draw a plane through V perpendicular to OV to touch an ellipsoid confocal with the ellipsoid of gyration. The centre of gravity must lie on the normal at the point of contact.

To find the small oscillations about the steady motion, i.e. to determine whether this motion be stable or not, we must put

$$p=\cos\alpha+P_0\sin\lambda t+P_1\cos\lambda t,$$

with similar expressions for q, r, ω_1, ω_2, ω_3. Substituting we shall get twelve linear equations to determine eleven ratios. Eliminating these we have an equation to find λ. It is sufficient for stability that all the roots of this equation should be real.

Motion of a Sphere.

572. *To determine the motion of a sphere on any perfectly rough surface under the action of any forces whose resultant passes through the centre of the sphere.*

Let G be the centre of gravity of the body and let the moving axes GC, GA, GB be respectively a normal to the surface and some two lines at right angles to be afterwards chosen at our convenience. Let the motions of these axes be determined by the angular velocities θ_1, θ_2, θ_3 about their instantaneous positions in the manner explained in Art. 243. Let u, v, w be the velocities of G resolved parallel to the axes so that $w=0$, and ω_1, ω_2, ω_3 the angular velocities of the body about these axes. Let F, F' be the resolved parts of the friction of the perfectly rough surface on the sphere parallel to the axes, GA, GB, and let R be the normal reaction. Let X, Y, Z be the resolved parts of the impressed forces on the centre of gravity. Let k be the radius of gyration of the sphere about a diameter, a its radius, and let its mass be unity. The equations of motion of the sphere are by Arts. 254 and 245,

$$\left.\begin{aligned} &\frac{d\omega_1}{dt}-\theta_3\omega_2+\theta_2\omega_3=\ \ \frac{F'a}{k^2} \\ &\frac{d\omega_2}{dt}-\theta_1\omega_3+\theta_3\omega_1=-\frac{Fa}{k^2} \\ &\frac{d\omega_3}{dt}-\theta_2\omega_1+\theta_1\omega_2=0 \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{aligned} &\frac{du}{dt}-\theta_3 v &&=X+F \\ &\frac{dv}{dt}+\theta_3 u &&=Y+F' \\ &-\theta_2 u+\theta_1 v &&=Z+R \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and since the point of contact of the sphere and surface is at rest, we have

$$\left.\begin{aligned} u-a\omega_2=0 \\ v+a\omega_1=0 \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Eliminating F, F', ω_1, ω_2 from these equations, we get

$$\left.\begin{aligned} \frac{du}{dt}-\theta_3 v=\frac{a^2}{a^2+k^2}X+\frac{k^2}{a^2+k^2}\theta_1 a\omega_3 \\ \frac{dv}{dt}+\theta_3 u=\frac{a^2}{a^2+k^2}Y+\frac{k^2}{a^2+k^2}\theta_2 a\omega_3 \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

573. The meaning of these equations may be found as follows. They are the two equations of motion of the centre of gravity of the sphere, which we should have obtained if the given surface had been smooth and the centre of gravity had been acted on by accelerating forces $\frac{k^2}{a^2+k^2}\theta_1 a\omega_3$ and $\frac{k^2}{a^2+k^2}\theta_2 a\omega_3$ along the axes GA, GB, and by the same impressed forces as before reduced in the ratio $\frac{a^2}{a^2+k^2}$. The motion therefore of the centre of gravity in these two cases with the same initial conditions will be the same. More convenient expressions for these two additional forces may be found thus. The centre of gravity moves along a surface formed by producing all the normals to the given surface a constant length equal to the radius of the sphere. Let us take the axes GA, GB to be tangents to the lines of curvature of this surface and let ρ_1, ρ_2 be the radii of curvature of the normal sections through these tangents respectively. Then

$$\theta_1=-\frac{v}{\rho_2},\quad \theta_2=\frac{u}{\rho_1}\ \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If G be the position of the centre of gravity at the time t, the quantity $\theta_3 dt$ is the angle between the projections of two successive positions of GA on the tangent plane at G. Let χ_1, χ_2 be the angles the radii of the curvature of the lines of curvature at G make with the normal. The centre of the sphere may be brought from G to any neighbouring position G' by moving it first from G to H along one line of curvature and then from H to G' along the other. As the sphere moves from G to H, the angle turned round by GA is the product of the arc GH into the resolved curvature of GH in the tangent plane. By Meunier's theorem, the curvature is $\frac{1}{\rho_1\cos\chi_1}$, multiplying this by $\sin\chi_1$ to resolve it into the tangent plane we find that the part of θ_3 due to the motion along GH is $\frac{u}{\rho_1}\tan\chi_1$. Treating the

arc HG' in the same way, we have

$$\theta_2 = \frac{u}{\rho_1}\tan\chi_1 + \frac{v}{\rho_2}\tan\chi_2 \quad\text{.................................(6).}$$

We have also an expression for ω_3 given by equations (1). Substituting for ω_1, ω_2 from the geometrical equations (3) we get

$$a\frac{d\omega_3}{dt} = uv\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \quad\text{.................................(7).}$$

The solution of the equations may be conducted as follows. Let (x, y, z) be the co-ordinates of the centre of the sphere. Then u, v may be found from the equation to the surface in terms of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ by resolving parallel to the axes of reference. If we eliminate u, v, θ_1, θ_2, θ_3 by means of (4), (5), and (6), we shall get three equations containing x, y, z, ω_3, and their differential coefficients with respect to t. These together with the equation to the surface will be sufficient to determine the motion at any time. One integral can always be found by the principle of Vis Viva. Since the sphere is turning about the point of contact as an instantaneously fixed point we have

$$(a^2+k^2)(\omega_1^2+\omega_2^2)+k^2\omega_3^2 = 2\phi,$$

where ϕ is the force function of the impressed forces. This is the same as

$$u^2+v^2+\frac{k^2a^2}{a^2+k^2}\omega_3^2 = 2\frac{a^2}{a^2+k^2}\phi \quad\text{..........................(8),}$$

and the right-hand side of this equation is twice the force function of the altered impressed forces.

574. It will sometimes be more convenient to take the axis GA to be a tangent to the path. Then $v=0$ and therefore $\omega_1=0$. If U be the resultant velocity of the centre of the sphere we have $u=U$. Also if R be the radius of torsion of a geodesic touching the path at G and ρ the radius of curvature of the normal section at G through a tangent to the path, we have $\theta_1 = \frac{U}{R}$ and $\theta_2 = \frac{U}{\rho}$. In these expressions, as elsewhere, R is estimated positive when the torsion round GA is from the positive direction of GB to the positive direction of GC. If χ be the angle the radius of curvature of the path makes with the normal, we have as before $\theta_3 = \frac{U}{\rho}\tan\chi$. The equations (4) become

$$\left.\begin{aligned}\frac{dU}{dt} &= \frac{a^2}{a^2+k^2}X + \frac{k^2}{a^2+k^2}\frac{U}{R}a\omega_3\\ \frac{U^2}{\rho}\tan\chi &= \frac{a^2}{a^2+k^2}Y + \frac{k^2}{a^2+k^2}\frac{U}{\rho}a\omega_3\end{aligned}\right\} \quad\text{..................... (IV).}$$

The expression for ω_3 given by equations (1) now takes the form

$$a\frac{d\omega_3}{dt} = -\frac{U^2}{R} \quad\text{...................................(VII).}$$

It may be shown by geometrical considerations that this form is identical with that given in (7).

575. *To find the pressure on the surface* we use the last of equations (2). This may be written in either of the forms

$$\frac{U^2}{\rho} = \frac{u^2}{\rho_1} + \frac{v^2}{\rho_2} = -Z - R \quad\text{...............................(9).}$$

The sphere will leave the surface when R changes sign. This will generally occur when the velocity of the centre of the sphere is that due to one half of the projection of the radius of curvature of the normal section on the direction of the resultant force.

576. Ex. 1. Show that the angular velocity of the sphere about a normal to the surface, viz. ω_3, is constant when the direction of motion of the centre of gravity is a tangent to a line of curvature, and only then.

Ex. 2. A sphere is projected without initial angular velocity about the radius normal to the surface, so that its centre begins to move along a line of curvature. Show that it will continue to describe that line of curvature if the force transverse to the line of curvature and tangential to the surface is equal to seven-fifths of the centrifugal force of the whole mass collected into the centre, resolved in the tangent plane to the surface.

Ex. 3. If the sphere be homogeneous and be not acted on by any forces, show that

$$U^2\left(\tan^2\chi+\frac{2}{7}\right)=\text{constant}, \qquad a\omega_3=\frac{7}{2}\,U\tan\chi,$$

$$\frac{d}{ds}\log\left(\tan^2\chi+\frac{2}{7}\right)=-\frac{2}{R}\tan\chi.$$

Show also that the path will not be a geodesic unless the path is a plane curve.

577. *If the given surface on which the sphere rolls be a plane*, we have ρ_1 and ρ_2 both infinite, hence θ_1, θ_2 are both zero. If therefore a homogeneons sphere roll on a perfectly rough plane under the action of any forces whatever of which the resultant passes through the centre of the sphere, the motion of the centre of gravity is the same as if the plane were smooth, and all the forces were reduced in ratio $\frac{5}{7}$. And it is also clear that the plane is the only surface which possesses this property for all initial conditions.

Ex. A homogeneous sphere is placed upon an inclined plane sufficiently rough to prevent sliding and a velocity in any direction is communicated to it. Show that the path of its centre will be a parabola, and if V be the initial horizontal velocity of the centre of gravity, α the inclination of the plane to the horizon, the latus rectum will be $\frac{14}{5}\,\frac{V^2}{g\sin\alpha}$.

578. *If the given surface on which the sphere rolls be another sphere* of radius $b-a$, we have $\rho_1=\rho_2=b$. Hence ω_3 is constant; let this constant value be called n, and let U be the velocity of the centre of gravity. Since every normal section is a principal section, let us take GA a tangent to the path. Hence the motion of the centre of gravity is the same as if the whole mass collected at that point were acted on by an accelerating force $\frac{k^2}{a^2+k^2}\,\frac{anU}{b}$ in a direction perpendicular to the path, and all the impressed forces were reduced in the ratio $\frac{a^2}{a^2+k^2}$. According to the usual convention as to the relative positions of the axes GA, GB, GC it is clear that if the positive direction of GA be in the direction of motion, the angular velocity n should be estimated positive when the part of the sphere in front is moving to the right of GA and the additional force when positive will also act

toward the right-hand side of the tangent. Since this additional force acts perpendicular to the path, it will not appear in the equation of Vis Viva. Hence the velocity of the centre of gravity in any position is the same as if it had arrived there simply under the action of the reduced forces. Let O be the centre of the fixed sphere, θ the angle OG makes with the vertical OZ, and ψ the angle the plane ZOG makes with any fixed plane passing through OZ. Then by Vis Viva we have

$$\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = F - \frac{2g}{b}\frac{a^2}{a^2+k^2}\cos\theta,$$

where F is some constant to be determined from the initial conditions. This also follows from equation (8).

Also taking moments about OZ, we have

$$\frac{b}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta\frac{d\psi}{dt}\right) = \frac{k^2}{a^2+k^2}an\frac{d\theta}{dt},$$

an equation which will be found to be a transformation of the second of equations (4). Integrating this equation we have

$$\sin^2\theta\frac{d\psi}{dt} = E - \frac{k^2}{a^2+k^2}\frac{an}{b}\cos\theta,$$

where E is some constant. These two equations will suffice to determine $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}$ under any given initial conditions.

If the sphere have no initial angular velocity about the normal to the surface it is clear that $n=0$ and the additional impressed force is zero. In this special case the motion of the sphere may be very simply found by treating it as a particle acted on by the reduced impressed forces.

Ex. A homogeneous sphere rolls under the action of gravity in any manner on a perfectly rough fixed sphere whose centre is O. Prove that throughout the motion (1) the velocity of the centre G of the moving sphere is that due to $\frac{5}{7}$ ths of its depth below a fixed horizontal plane; (2) the moving sphere will leave the fixed sphere when the altitude of its centre above O is $\frac{10}{17}$ ths of the altitude of the fixed plane above the same point; (3) the horizontal velocity of G is proportional to the tangent of the angle GU makes with the horizon, where U is a fixed point on a vertical through O.

579. *If the surface on which the sphere rolls be a cylinder* the lines of curvature are the generators and the transverse sections. Let the axis GA be directed parallel to the generators, then ρ_1 is infinite and $\rho_2 - a$ is the radius of curvature of the transverse section. We have $\theta_1 = -\frac{v}{\rho_2}$, $\theta_2 = 0$, and since $\chi_2 = 0$, $\theta_3 = 0$. The equations (4) and (7) therefore become

$$\left.\begin{aligned}\frac{du}{dt} &= \frac{a^2}{a^2+k^2}X - \frac{k^2}{a^2+k^2}\frac{v}{\rho_2}a\omega_3\\ \frac{dv}{dt} &= \frac{a^2}{a^2+k^2}Y\\ \frac{d(a\omega_3)}{dt} &= \frac{uv}{\rho_2}\end{aligned}\right\}.$$

From these equations the motion may be found.

The second of these gives the motion transverse to the generators of the cylinder, and if Y be the same for all positions of the sphere on the same generator, this equation may be solved independently of the other two. The transverse motion of the centre of the sphere is therefore the same under the same initial circumstances as that of a smooth sphere constrained to slide in a plane perpendicular to the generators on the transverse section of the cylinder and acted on by the same impressed forces but reduced in the ratio $\frac{a^2}{a^2+k^2}$.

Having found v we may proceed thus; let ϕ be the angle the normal plane to the cylinder through a generator and through the centre of the sphere makes with some fixed plane passing through a generator, then $v=\rho_2 \frac{d\phi}{dt}$. If $\frac{d\phi}{dt}$ be not zero, the first and third equations then become

$$\left.\begin{aligned} \frac{du}{d\phi}+\frac{k^2}{a^2+k^2}a\omega_3 &= \frac{a^2}{a^2+k^2}\frac{\rho_2}{v}X \\ u &= \frac{d(a\omega_3)}{d\phi} \end{aligned}\right\}.$$

If X be the same for all positions of the sphere on the same generator these equations can be solved without difficulty. For v and ρ_2 being known in terms of ϕ, we have in this case two linear equations to find u and $a\omega_3$. If X be zero, and $k^2=\frac{2}{5}$, we find

$$a\omega_3=A\sin\left(\sqrt{\frac{2}{7}}\phi+B\right),\quad u=A\sqrt{\frac{2}{7}}\cos\left(\sqrt{\frac{2}{7}}\phi+B\right),$$

where A and B are two arbitrary constants to be determined by the initial values of u and ω_3.

If X be not the same for all positions of the sphere on the same generator, let ξ be the space traversed by the sphere measured along a generator. Then

$$u=\frac{d\xi}{dt}=\frac{d\xi}{d\phi}\frac{v}{\rho_2}.$$

Substituting this value of u, we have two equations to find ξ and $a\omega_3$ in terms of ϕ. One integral of these is equation (8) of Art. 578 which was obtained by the principle of Vis Viva.

Ex. A sphere rolls under the action of gravity on a perfectly rough cylindrical surface with its axis inclined at an angle α to the horizon. The section of the cylinder is such that when the sphere rolls on it, the centre describes a cycloid with its cusps on the same horizontal line. If the sphere start from rest with its centre at a cusp, find the motion.

Let the position of the sphere be defined by ξ the space described along a generator and s the arc of the cycloid measured from the vertex. If $4b$ be the radius of curvature of the cycloid at its vertex, we have

$$s=4b\cos\sqrt{\frac{5g\cos\alpha}{28b}}\,t.$$

Since $v=\frac{ds}{dt}$ and $\rho_2^2+s^2=16b^2$ we find that $\frac{v}{\rho_2}$ is constant. This gives without difficulty

$$\omega_3 = -\frac{\sin\alpha}{a}\sqrt{\frac{35bg}{\cos\alpha}}\left\{1-\cos\frac{1}{7}\sqrt{\frac{5g\cos\alpha}{2b}}\,t\right\},$$

$$u = \sin\alpha\sqrt{\frac{10bg}{\cos\alpha}}\sin\frac{1}{7}\sqrt{\frac{5g\cos\alpha}{2b}}\,t.$$

The relation, $\frac{v}{\rho_2}$ = constant, holds whenever (1) the forces acting at the centre of the sphere, and the form of the section of the cylinder, are so related that the tangential component bears a constant ratio to $\rho_2\frac{d\rho_2}{ds}$, and (2) the sphere starts from rest at a point where ρ_2 is zero. In such a case, the normal plane to the section through the centre of the sphere has a constant angular velocity in space and the resolved motion of the sphere perpendicular to the generators is independent of that along the generators.

Ex. A sphere rolls on a perfectly rough right circular cylinder whose radius is c under the action of no forces, show that the path traced out by the point of contact becomes the curve $x = A\sin\sqrt{\frac{2}{7}}\frac{y}{c}$ when the cylinder is developed on a plane.

This result shows that the sphere cannot be made to travel continually in one direction along the length of the cylinder except when the point of contact describes a generator.

580. *If the surface on which the sphere rolls be a cone*, the lines of curvature are the generators and their orthogonal trajectories. Let the axis GA be directed parallel to the generator, then ρ_1 is infinite and $\rho_2 - a$ is the radius of curvature of a normal section perpendicular to the generators. Also $\theta_1 = -\frac{v}{\rho_2}$, $\theta_2 = 0$. Let the position of the sphere be defined by the distance r of its centre from the vertex 0 of the cone on which the centre always lies and by an angle ϕ such that $d\phi$ is the angle between two consecutive positions of the distance r, $d\phi$ being taken as positive when the centre moves in the positive direction of GB. If the cone were developed on a plane it is clear that r and ϕ would be the ordinary polar co-ordinates of a point G. We have

$$\theta_3 = \frac{d\phi}{dt}, \qquad u = \frac{dr}{dt}, \qquad v = r\frac{d\phi}{dt}.$$

The equations (4) and (7) become therefore

$$\left.\begin{aligned}\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 &= \frac{a^2}{a^2+k^2}X - \frac{k^2}{a^2+k^2}\frac{r}{\rho_2}a\omega_3\frac{d\phi}{dt}\\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) &= \frac{a^2}{a^2+k^2}Y\\ \frac{d(a\omega_3)}{dt} &= \frac{r}{\rho_2}\frac{d\phi}{dt}\frac{dr}{dt}\end{aligned}\right\}.$$

If the impressed forces have no component perpendicular to the normal plane through a generator, $Y=0$, and we have $r^2\frac{d\phi}{dt} = h$, where h is some constant depending on the initial values of r and v.

If also the component X of the forces along a generator be a function of r only, another integral can be found by the principle of Vis Viva, viz.

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\phi}{dt}\right)^2+\frac{k^2}{a^2+k^2}a^2\omega_3^2=\frac{2a^2}{a^2+k^2}\int Xdr+h',$$

where h' is another constant depending on the initial values of u, v and r.

If, further, the cone be a right cone, $\rho_2=r\tan\alpha$ where α is the semi-angle, and we have

$$a\omega_3=-\frac{h\cot\alpha}{r}+h'',$$

where h'' is a third constant depending on the initial values of ω_3 and r. The equations of the motion of the centre of the sphere resemble those of a particle in central forces. Hence r and ϕ will be found as functions of the time if we regard them as the co-ordinates of a free particle moving in a plane under the action of a central force represented by

$$\frac{a^2}{a^2+k^2}\left\{X-k^2\omega_3\frac{d\omega_3}{dr}\right\},$$

where ω_3 has the value just found.

Ex. A sphere rolls on a perfectly rough cone such that the equation to the cone on which the centre G always lies is $\frac{r}{\rho_2}=F(\phi)$. If the centre is acted on by a force tending to the vertex, find the law of force that any given path may be described. If the equation to the path be $\frac{1}{r}=f(\phi)$, prove that the force X is

$$X=k^2\omega_3\frac{d\omega_3}{dr}+\frac{a^2+k^2}{a^2}h^2f^2\left(f+\frac{d^2f}{d\phi^2}\right),$$

where ω_3 is given by

$$\frac{d\omega_3}{d\phi}=-\frac{h}{a}F\frac{df}{d\phi}.$$

581. *Let the given rough surface be any surface of revolution placed with its axis of figure vertical and vertex upwards, and let gravity be the only impressed force.* In this case the meridians and parallels are the lines of curvature. Let the axis of figure be the axis of Z. Let θ be the angle the axis GC makes with the axis of Z, ψ the angle the plane containing Z and GC makes with any fixed vertical plane.

Then $\qquad \theta_1=-\sin\theta\frac{d\psi}{dt}, \qquad \theta_2=\frac{d\theta}{dt}, \qquad \theta_3=\cos\theta\frac{d\psi}{dt}.$

Hence the equations (4) become

$$\frac{du}{dt}-\cos\theta\frac{d\psi}{dt}v=\frac{a^2}{a^2+k^2}g\sin\theta-\frac{k^2}{a^2+k^2}a\omega_3\sin\theta\frac{d\psi}{dt} \quad\text{.........(i)},$$

$$\frac{dv}{dt}+\cos\theta\frac{d\psi}{dt}u=\frac{k^2}{a^2+k^2}a\omega_3\frac{d\theta}{dt} \quad\text{.........(ii)},$$

and equation (8) becomes

$$u^2+v^2+\frac{k^2}{a^2+k^2}a^2\omega_3^2=E+2g\frac{a^2}{a^2+k^2}\int\rho\sin\theta\,d\theta \quad\text{.........(iii)},$$

where E is some constant, and ρ is the radius of curvature of the meridian. Also we have by (7)

$$\frac{d\omega_3}{dt}=-\frac{uv}{a}\left(\frac{1}{\rho}-\frac{\sin\theta}{r}\right) \quad\text{.........(iv)},$$

where r is the distance of the centre of the sphere from the axis of z. The geometrical equations (5) become

$$u = \rho \frac{d\theta}{dt}, \qquad v = r\frac{d\psi}{dt} \quad \text{.............................(v).}$$

To solve these, we may put (ii) into the form

$$\frac{dv}{d\theta} + \cos\theta \frac{d\psi}{d\theta} u = \frac{k^2}{a^2+k^2} a\omega_3,$$

which by (v) becomes

$$\frac{dv}{d\theta} + \frac{\rho\cos\theta}{r} v = \frac{k^2}{a^2+k^2} a\omega_3;$$

differentiating this, we have by (iv),

$$\frac{d^2v}{d\theta^2} + \frac{\rho\cos\theta}{r}\frac{dv}{d\theta} + Pv = 0 \quad \text{.............................(vi),}$$

where $$P = \frac{d}{d\theta}\left(\frac{\rho\cos\theta}{r}\right) + \frac{k^2}{k^2+a^2}\left(1 - \frac{\rho\sin\theta}{r}\right).$$

Now ρ and r may be found from the equation to the meridian curve as functions of θ. Hence P is a known function of θ. Solving this linear equation we have v found as a function of θ. Then by (iv) we have

$$\frac{d\omega_3}{d\theta} = -\frac{v}{a}\left(1 - \frac{\rho\sin\theta}{r}\right),$$

and thence having found ω_3 we have u by equation (iii). Knowing u and v; θ and ψ may be found by equations (v).

582. *A heavy sphere rotating about a vertical axis is placed in equilibrium on the highest point of a surface of any form and being slightly disturbed makes small oscillations, find the motion.*

Let O be the highest point of the surface on which the centre of gravity G always lies. Let the tangents to the lines of curvature at O be taken as the axes of x and y, and let (x, y, z) be the co-ordinates of G. We shall assume that O is not a singular point on the surface. In order to simplify the general equations of motion (4) we shall take as the axes GA and GB the tangents to the lines of curvature at G. But since G always remains very near O, the tangents to the lines of curvature at G will be nearly parallel to those at O. So that to the first order of small quantities we have

$$\theta_1 = -\frac{1}{\rho_2}\frac{dy}{dt}, \qquad \theta_2 = \frac{1}{\rho_1}\frac{dx}{dt}, \qquad u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt},$$

and θ_3 will be a small quantity of at least the first order. Also since the sphere is supposed not to deviate far from the highest point of the surface, we have ω_3 constant, let this constant be called n.

The equation to the surface on which G moves, in the neighbourhood of the highest point, is $z = -\frac{1}{2}\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right)$. The equation to the normal at x, y, z is $\dfrac{\xi - x}{-\frac{x}{\rho_1}} = \dfrac{\eta - y}{-\frac{y}{\rho_2}} = \dfrac{\zeta - z}{-1}$. Hence the resolved parts parallel to the axes of the normal pressure R on the sphere are $R\frac{x}{\rho_1}$, $R\frac{y}{\rho_2}$ and R. The equations of motion (4)

therefore become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{a^2}{a^2+k^2} R \frac{x}{\rho_1} - \frac{k^2}{a^2+k^2} \frac{dy}{dt} \frac{an}{\rho_2} \\ \frac{d^2y}{dt^2} &= \frac{a^2}{a^2+k^2} R \frac{y}{\rho_2} + \frac{k^2}{a^2+k^2} \frac{dx}{dt} \frac{an}{\rho_1} \\ \frac{d^2z}{dt^2} &= R - g \end{aligned}\right\} \text{.......................} \text{(iv)}.$$

But z is a small quantity of the second order, hence the last equation gives $R=g$. To solve these equations, we put

$$x = F \cos(\lambda t + f), \quad y = G \sin(\lambda t + f).$$

These give

$$\left.\begin{aligned} \left(\lambda^2 + \frac{a^2}{a^2+k^2} \frac{g}{\rho_1}\right) F &= \frac{k^2}{a^2+k^2} \frac{a\lambda n}{\rho_2} G \\ \left(\lambda^2 + \frac{a^2}{a^2+k^2} \frac{g}{\rho_2}\right) G &= \frac{k^2}{a^2+k^2} \frac{a\lambda n}{\rho_1} F \end{aligned}\right\}.$$

The equation to find λ is therefore

$$\left(\lambda^2 + \frac{a^2}{a^2+k^2} \frac{g}{\rho_1}\right)\left(\lambda^2 + \frac{a^2}{a^2+k^2} \frac{g}{\rho_2}\right) = \frac{k^4}{(a^2+k^2)^2} \frac{a^2\lambda^2 n^2}{\rho_1\rho_2}.$$

This is a quadratic equation to determine λ^2. In order that the motion may be oscillatory it is necessary and sufficient that the roots should be both positive. If ρ_1, ρ_2 be both negative, so that the sphere is placed like a ball inside a cup, the roots of the quadratic are positive for all values of n. If ρ_1, ρ_2 have opposite signs the roots cannot be both positive. If ρ_1, ρ_2 be both positive the two conditions of stability will be found to reduce to

$$n^2 > \frac{a^2+k^2}{k^4} g \left(\sqrt{\rho_1} + \sqrt{\rho_2}\right)^2.$$

If ρ_1 be infinite, it is necessary that ρ_2 should be negative, and in that case the two values of λ^2 are $-\dfrac{a^2}{a^2+k^2}\dfrac{g}{\rho_2}$ and zero, which are both independent of n. If $\rho_1 = \rho_2$, we have $F=G$. In this case if θ be the inclination of the normal to the vertical, we have $\theta^2 = \dfrac{x^2+y^2}{\rho^2}$ and, as in Art. 569, we find

$$\theta^2 = F_1^2 + F_2^2 + 2F_1F_2 \cos\{(\lambda_1 - \lambda_2)t + f_1 - f_2\},$$

where λ_1, λ_2 are the roots of the quadratic

$$\lambda^2 \pm \frac{k^2}{a^2+k^2} \frac{an}{\rho} \lambda + \frac{a^2}{a^2+k^2} \frac{g}{\rho} = 0.$$

This problem may also be solved by Lagrange's method in the manner explained in Art. 388. Let the axes of reference Ox, Oy, Oz be the same as before. Let GC be that diameter which is vertical when the sphere is in equilibrium on the summit. Let GA, GB be two other diameters forming with GC a system of rectangular axes fixed on the sphere. Let the position of these with reference to the axes fixed in space be defined by the angular co-ordinates θ, ϕ, ψ in the manner explained in Art. 235. The Vis Viva of the sphere may then be found as in Art. 349, Ex. 1. If

we put $\sin\theta\cos\psi=\xi$, $\sin\theta\sin\psi=\eta$, $\phi+\psi=\chi$, and reject all small quantities above the second order, we find that the Lagrangian function is

$$L=\frac{1}{2}(x'^2+y'^2)+\frac{1}{2}k^2\{\chi'^2-\chi'(\xi\eta'-\xi'\eta)+\xi'^2+\eta'^2\}+\frac{1}{2}g\left(\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}\right).$$

It is easy to see by reference to the figure in Art. 235 that ξ and η are the cosines of the angles the diameter GC makes with the axes Ox, Oy.

If ω_x, ω_y, ω_z are the angular velocities of the sphere about parallels to the axes fixed in space, the geometrical equations are

$$\left.\begin{aligned} x'-a\left(\omega_y-\omega_z\frac{y}{\rho_2}\right)&=0\\ y'+a\left(\omega_x-\omega_z\frac{x}{\rho_1}\right)&=0\end{aligned}\right\}.$$

These are found by making the resolved velocities of the point of contact in the directions of the axes of x and y equal to zero; see Art. 219. The angular velocities ω_x, ω_y, ω_z may be expressed in terms of θ, ϕ, ψ by formulæ analogous to those in Art. 235. See also the note. Thus

$$\left.\begin{aligned} \omega_x&=-\theta'\sin\psi+\phi'\sin\theta\cos\psi\\ \omega_y&=\theta'\cos\psi+\phi'\sin\theta\sin\psi\\ \omega_z&=\phi'\cos\theta+\psi'\end{aligned}\right\}.$$

Substituting and expressing the result in terms of the new co-ordinates ξ, η, χ, the geometrical equations become

$$\left.\begin{aligned} L_1&=-\frac{x'}{a}+\chi'\eta+\xi'-\chi'\frac{y}{\rho_2}=0\\ L_2&=\frac{y'}{a}+\chi'\xi-\eta'-\chi'\frac{x}{\rho_1}=0\end{aligned}\right\}.$$

The equations of motion are given by

$$\frac{d}{dt}\frac{dL}{dq'}-\frac{dL}{dq}=\lambda\frac{dL_1}{dq'}+\mu\frac{dL_2}{dq'},$$

where q stands for any one of the five co-ordinates x, y, ξ, η, χ. The steady motion is given by x, y, ξ, η all zero and $\chi'=n$. Taking $q=x$ and $q=y$ and giving the several co-ordinates their values in the steady motion, we find that λ and μ are both zero in the steady motion.

To find the oscillations, we write for q in turn x, y, χ, ξ and η, and retain the first powers of the small quantities. Remembering that λ and μ are small quantities (Art. 461), we find

$$\left.\begin{aligned} x''-g\frac{x}{\rho_1}+\frac{\lambda}{a}&=0\\ y''-g\frac{y}{\rho_2}-\frac{\mu}{a}&=0\\ k^2\chi''&=0\end{aligned}\right\},\qquad \left.\begin{aligned} k^2(\xi''+\chi'\eta')-\lambda&=0\\ k^2(\eta''-\chi'\xi')+\mu&=0\end{aligned}\right\}.$$

These and the two geometrical equations L_1 and L_2 are all linear, and may be solved in the manner explained in Art. 432. If we put $\chi'=n$ and eliminate first λ and μ and then ξ and η we get two equations to find x and y, which are the same as those marked (iv) in the first solution.

Ex. A perfectly rough sphere is placed on a perfectly rough fixed sphere near the highest point. The upper sphere has an angular velocity n about the diameter through the point of contact; prove that its equilibrium will be stable if $n^2 > \dfrac{35g(a+b)}{a^2}$, where b is the radius of the fixed sphere, and a the radius of the moving sphere.

583. *A perfectly rough surface of revolution is placed with its axis vertical. Determine the circumstances of motion that a heavy sphere may roll on it so that its centre describes a horizontal circle. And this state of steady motion being disturbed, find the small oscillations.*

In this case we must recur to the equations of Art. 581, and let us adopt the notation of that article, except that to shorten the expressions we shall put for k^2 its value $\frac{2}{5}a^2$.

To find the steady motion. We must put u, v, ω_3, θ, $\frac{d\psi}{dt}$ all constant. Let α, μ and n be the constant values of θ, $\frac{d\psi}{dt}$ and ω_3. Then we have $u=0$, $v=b\mu$, where b is the constant value of r. The equation (i) becomes

$$-b\cos\alpha\,\mu^2 = \frac{5}{7}g\sin\alpha - \frac{2}{7}an\sin\alpha\mu.$$

The other dynamical equations are satisfied without giving any relation between the constants. If the motion be steady, we have therefore

$$n = \frac{5}{2}\frac{g}{a\mu} + \frac{7}{2}\frac{b}{a}\mu\cot\alpha;$$

thus for the same value of n we have two values of μ, which correspond to different initial values of v.

We have the geometrical relation $a\omega_1 = -v$, so that ω_1 and n have opposite signs. Hence the axis of rotation which necessarily passes through the point of contact of the sphere and the rough surface makes an angle with the vertical less than that made by the normal at the point of contact.

By inspecting the expression for n, it will be seen that it is a minimum when

$$\frac{5}{2}\frac{g}{a\mu} = \frac{7}{2}\frac{b\mu}{a}\cot\alpha,$$

and therefore $$n^2 = 35\frac{bg}{a^2}\cot\alpha, \quad \mu^2 = \frac{5}{7}\frac{g}{b}\tan\alpha.$$

To find the small oscillation.

Put $\theta = \alpha + \theta'$, $\frac{d\psi}{dt} = \mu + \frac{d\psi'}{dt}$, where α and μ are supposed to contain *all* the constant parts of θ and $\frac{d\psi}{dt}$, so that θ' and $\frac{d\psi'}{dt}$ only contain trigonometrical terms. Let $c-a$ be the radius of curvature of the surface of revolution at the point of contact of the sphere in steady motion, so that ρ differs from c only by small quantities, and may be put equal to c in the small terms. Also we have $r = b + c\cos\alpha\,.\,\theta'$.

Now by equations (iv) and (v) of Art. 581 we have

$$\frac{d\omega_3}{dt}=\frac{d\theta}{dt}\frac{d\psi}{dt}\frac{\rho\sin\theta-r}{a}=\frac{d\theta'}{dt}\mu\frac{c\sin\alpha-b}{a};$$

$$\therefore\ \omega_3=\mu\frac{c\sin\alpha-b}{a}\theta'+n,$$

where n is the whole of the constant part of ω_3.

Again, from equation (ii), we have

$$-\frac{1}{a}\frac{d}{dt}\left(r\frac{d\psi}{dt}\right)-\frac{\rho}{a}\frac{d\theta}{dt}\cos\theta\frac{d\psi}{dt}+\frac{k^2}{a^2+k^2}\omega_3\frac{d\theta}{dt}=0;$$

$$\therefore\ -\frac{\mu}{a}c\cos\alpha\frac{d\theta'}{dt}-\frac{b}{a}\frac{d^2\psi'}{dt^2}-\frac{c\cos\alpha\,\mu}{a}\frac{d\theta'}{dt}+\frac{2}{7}n\frac{d\theta'}{dt}=0;$$

integrating we have

$$\left(\frac{2}{7}n-\frac{2\mu c\cos\alpha}{a}\right)\theta'=\frac{b}{a}\frac{d\psi'}{dt},$$

the constant being put zero because θ' and ψ' only contain trigonometrical terms.

Thirdly, from equation (i), we have

$$\frac{1}{a}\frac{d}{dt}\left(\rho\frac{d\theta}{dt}\right)-\frac{r}{a}\left(\frac{d\psi}{dt}\right)^2\cos\theta+\frac{2}{7}\omega_3\sin\theta\frac{d\psi}{dt}=\frac{5}{7}\frac{g}{a}\sin\theta;$$

$$\therefore\ \frac{c}{a}\frac{d^2\theta'}{dt^2}-\frac{b+c\cos\alpha\theta'}{a}(\cos\alpha-\sin\alpha\theta')\left(\mu^2+2\mu.\frac{d\psi'}{dt}\right)$$

$$+\frac{2}{7}(\sin\alpha+\cos\alpha\theta')\left(\mu+\frac{d\psi'}{dt}\right)\left(n+\mu\frac{c\sin\alpha-b}{a}\theta'\right)=\frac{5}{7}\frac{g}{a}(\sin\alpha+\cos\alpha\theta').$$

This expression must be expanded and expressed in the form

$$\frac{d^2\theta'}{dt^2}+A\theta'=B.$$

In this case, since θ' contains only trigonometrical expressions, we must have $B=0$. Putting $\theta'=0$ in the above expression, we find the same value for n as in steady motion. After expanding the preceding equation we find

$$A=\mu^2\left(-\cos^2\alpha+\frac{2}{7}\sin^2\alpha\right)+\mu^2\frac{b}{c\sin\alpha}\left(2\cos^2\alpha+\frac{5}{7}\sin^2\alpha\right)$$

$$+\frac{25}{49}\frac{g^2\sin\alpha}{\mu^2bc}-\frac{10}{7}\frac{g}{b}\sin\alpha\cos\alpha+\frac{10}{7}\frac{g}{c}\cos\alpha.$$

In order that the motion may be steady, it is sufficient and necessary that this value of A should be positive. And the time of oscillation is then $\frac{2\pi}{\sqrt{A}}$.

It is to be observed that this investigation does not apply if α and therefore b be small, for some terms which have been rejected have b in their denominators, and may become important.

584. The general equations of the motion of a sphere on an imperfectly rough surface may be obtained on principles similar to those adopted in Art. 306. The difference in the theory will be made clear by the following example, in which a method of proceeding is explained which is generally applicable, whenever the integrations can be effected.

585. *A homogeneous sphere moves on an imperfectly rough inclined plane with any initial conditions, find the direction of the motion and the velocity of its centre at any time.*

Let G be the centre of gravity of the sphere. Let the axes of reference GA, GB, GC have their directions fixed in space, the first being directed down the inclined plane and the last normal to the plane. Let u, v, w be the velocities of G resolved parallel to these axes, and ω_1, ω_2, ω_3 the angular velocities of the body about these axes. Let F, F' be the resolved parts of the frictions of the plane on the sphere parallel to the axes GA, GB, but taken *negatively* in those directions. Let k be the radius of gyration of the sphere about a diameter, a its radius, and let the mass be unity.

Let α be the inclination of the plane to the horizon. The equations of motion will then be

$$\left.\begin{aligned} k^2\frac{d\omega_1}{dt} &= -F'a \\ k^2\frac{d\omega_2}{dt} &= Fa \end{aligned}\right\}\text{..................(1)},\qquad \left.\begin{aligned} \frac{du}{dt} &= -F + g\sin\alpha \\ \frac{dv}{dt} &= -F' \end{aligned}\right\}\text{..................(2)}.$$

Eliminating F and F' from these equations and integrating we have

$$\left.\begin{aligned} u + \frac{k^2}{a^2}a\omega_2 &= U_0 + g\sin\alpha t \\ v - \frac{k^2}{a^2}a\omega_1 &= V_0 \end{aligned}\right\}\text{..............................(3)},$$

where U_0 and V_0 are two constants determined by the initial values of u, v, ω_1, ω_2.

The meaning of these equations may be found as follows. Let P be the point of contact of the sphere and plane, let Q be a point within the sphere on the normal at P so that $PQ = \dfrac{a^2+k^2}{a}$, so that Q is the centre of oscillation of the sphere when suspended from P. It is clear that the left-hand sides of the equations (3) express the components of the velocity of Q parallel to the axes. The equations assert that the frictional impulses at P cannot affect the motion of Q, and this readily follows from Art. 119, because Q is in the axis of spontaneous rotation for a blow at P.

586. The friction at the point of contact P always acts opposite to the direction of sliding and tends to reduce this point to rest. When sliding ceases the friction (see Art. 148) also ceases to be limiting friction and becomes only of sufficient magnitude to keep the point of contact at rest. If sliding ever does cease, we then have

$$u - a\omega_2 = 0, \qquad v + a\omega_1 = 0\text{..............................(4)}.$$

The equations (3) and (4) suffice to determine these final values of u, v, ω_1 and ω_2. Thus the direction of the motion and the velocity of the centre of gravity after sliding has ceased have been found in terms of the time. It appears that both these elements are independent of the friction.

If the equations (4) hold initially the sphere will begin to move without sliding if the friction found from the equations (1), (2) and (4) is less than the limiting friction. As in Art. 147, this requires that the coefficient of friction $\mu > \dfrac{k^2}{a^2+k^2}\tan\alpha$. Supposing this inequality to hold, the friction called into play will be always less than the limiting friction and therefore equations (3) and (4) give the whole motion.

687. If the equations (4) do not hold initially or if the inequality just mentioned is not satisfied, let S be the velocity of sliding and let θ be the angle the direction of sliding makes with GA. To fix the signs we shall take S to be positive while θ may have any value from $-\pi$ to π. Then

$$S\cos\theta = u - a\omega_2, \qquad S\sin\theta = v + a\omega_1 \quad \text{.........................(5).}$$

The friction is equal to $\mu g\cos\alpha$ and acts in the direction opposite to sliding, hence

$$F = \mu g\cos\alpha\cos\theta, \qquad F' = \mu g\cos\alpha\sin\theta.$$

The equations (1), (2) and (5) therefore give

$$\left.\begin{aligned} \frac{d(S\cos\theta)}{dt} &= -\left(1+\frac{a^2}{k^2}\right)\mu g\cos\alpha\cos\theta + g\sin\alpha \\ \frac{d(S\sin\theta)}{dt} &= -\left(1+\frac{a^2}{k^2}\right)\mu g\cos\alpha\sin\theta \end{aligned}\right\} \text{..................(6).}$$

Expanding we find

$$\left.\begin{aligned} \frac{dS}{dt} &= -\left(1+\frac{a^2}{k^2}\right)\mu g\cos\alpha + g\sin\alpha\cos\theta \\ S\frac{d\theta}{dt} &= -g\sin\alpha\sin\theta \end{aligned}\right\} \text{......................(7).}$$

If θ be not constant, we may eliminate t and integrate with regard to θ, this gives

$$S\sin\theta = 2A\left(\tan\frac{\theta}{2}\right)^n \text{.....................................(8),}$$

where $n = \left(1+\frac{a^2}{k^2}\right)\mu\cot\alpha$, and A is the constant of integration. If S_0 and θ_0 be the initial values of S and θ determined by equations (5), we have

$$2A = S_0\sin\theta_0\left(\cot\frac{\theta_0}{2}\right)^n \text{..................................(9).}$$

Substituting the value of S given by (8) in the second of equations (7) and integrating we find

$$\frac{\left(\tan\frac{\theta}{2}\right)^{n-1}}{n-1} + \frac{\left(\tan\frac{\theta}{2}\right)^{n+1}}{n+1} = \frac{\left(\tan\frac{\theta_0}{2}\right)^{n-1}}{n-1} + \frac{\left(\tan\frac{\theta_0}{2}\right)^{n+1}}{n+1} - \frac{g\sin\alpha}{A}t \text{......(10),}$$

the constant of integration being determined from the condition that $\theta = \theta_0$ when $t=0$. The equations (8), (9) and (10) give S and θ in terms of t. The equations (6) and (5) then give u, v, ω_1 and ω_2 in terms of t.

The second of equations (7) shows that $\frac{d\theta}{dt}$ has an opposite sign to θ, hence θ beginning at any initial value except $\pm\pi$ continually approaches zero. It follows that, unless α is zero, θ will be constant only when $\theta_0 = 0$ or $\pm\pi$.

If $n > 1$, i.e. $\mu > \frac{k^2}{a^2+k^2}\tan\alpha$, we see from (8) that sliding will ceas
vanishes. This, by (10) will occur when

$$t = \frac{S_0}{g\sin\alpha}\left(\frac{\cos^2\frac{\theta_0}{2}}{n-1} + \frac{\sin^2\frac{\theta_0}{2}}{n+1}\right).$$

The subsequent motion has already been found.

If $n<1$ we see by (8) that S increases as θ decreases, so that sliding will never cease. It also follows from (10) that θ vanishes only at the end of an infinite time.

If $S_0=0$, sliding will never begin if $n>1$, but will immediately begin and never cease if $n<1$.

588. The theory of the motion of a sphere on an imperfectly rough *horizontal* plane is so much simpler than when the plane is inclined or when the sphere rolls on any other surface, that it seems unnecessary to consider this case in detail. At the same time the game of billiards supplies many problems which it would be unsatisfactory to pass over in silence. The following examples have been arranged so as both to indicate the mode of proof to be adopted and to supply some results which may be submitted to experiment.

The result given in Ex. 1, was first obtained by J. A. Euler the son of the celebrated Euler, and published in the *Mém. de. l'Acad. de Berlin*, 1758. Most, possibly all, of the other results may be found in the *Jeu de Billard par G. Coriolis*, published at Paris in 1835.

Ex. 1. A billiard-ball is set in motion on an imperfectly rough horizontal plane, show that the direction and magnitude of the friction are constant throughout the motion. The path of the centre of gravity is therefore an arc of a parabola while sliding continues, and finally a straight line. The parabola is described with the given initial motion of the centre of gravity under an acceleration equal to μg tending in a direction opposite to the initial direction of sliding.

Ex 2. If S_0 be the initial velocity of sliding prove that the parabolic path lasts for a time $\frac{2}{7}\frac{S_0}{\mu g}$. From some experiments of Coriolis it appears that $\mu=\frac{1}{5}$ nearly. If the initial velocity of sliding be one foot per second, the parabolic path lasts therefore less than a twentieth part of a second.

Ex. 3. If P be the point of contact in any position and Q the centre of oscillation with regard to P, prove that the velocity of Q is always the same in direction and magnitude. Thence show that the final rectilinear path of the centre of gravity is parallel to the initial direction of the motion of Q and the final velocity of the centre of gravity is $\frac{5}{7}$ of the initial velocity of Q. If PP' be the initial direction of motion and V the initial velocity of the centre of gravity and t the time given by Ex. 2, prove that the final rectilinear path of the centre of gravity intersects PP' in a point P' so that $PP'=\frac{1}{2}Vt$.

Ex. 4. A billiard-ball, at rest on an imperfectly rough horizontal table, is struck by a cue in a horizontal direction at any point whose altitude above the table is h, and the cue is withdrawn as soon as it has delivered its blow. Supposing the cue to be sufficiently rough to prevent sliding, show that the centre of the ball will move in the direction of the blow and that its velocity will become uniform and equal to $\frac{5}{7}\frac{h}{a}B$ after a time $\frac{5h\sim 7a}{7a}\frac{B}{\mu g}$ where B is the ratio of the blow to the mass of the sphere and a is the radius.

In order that there should be no sliding the distance of the cue from the centre of the ball must be less than $a\sin\epsilon$ where $\tan\epsilon$ is the coefficient of friction between the cue and ball.

Ex. 5. A billiard-ball, initially at rest and touching the table at a point P, is struck by a cue making an angle β with the horizon. Show that the final rectilinear motion of the centre of gravity is parallel to the straight line PS joining P to the point S where the direction of the blow meets the table, and the final velocity of the centre of gravity is $\frac{5}{7}\frac{PS}{a}B\sin\beta$ in the direction of the projection of the blow on the horizon.

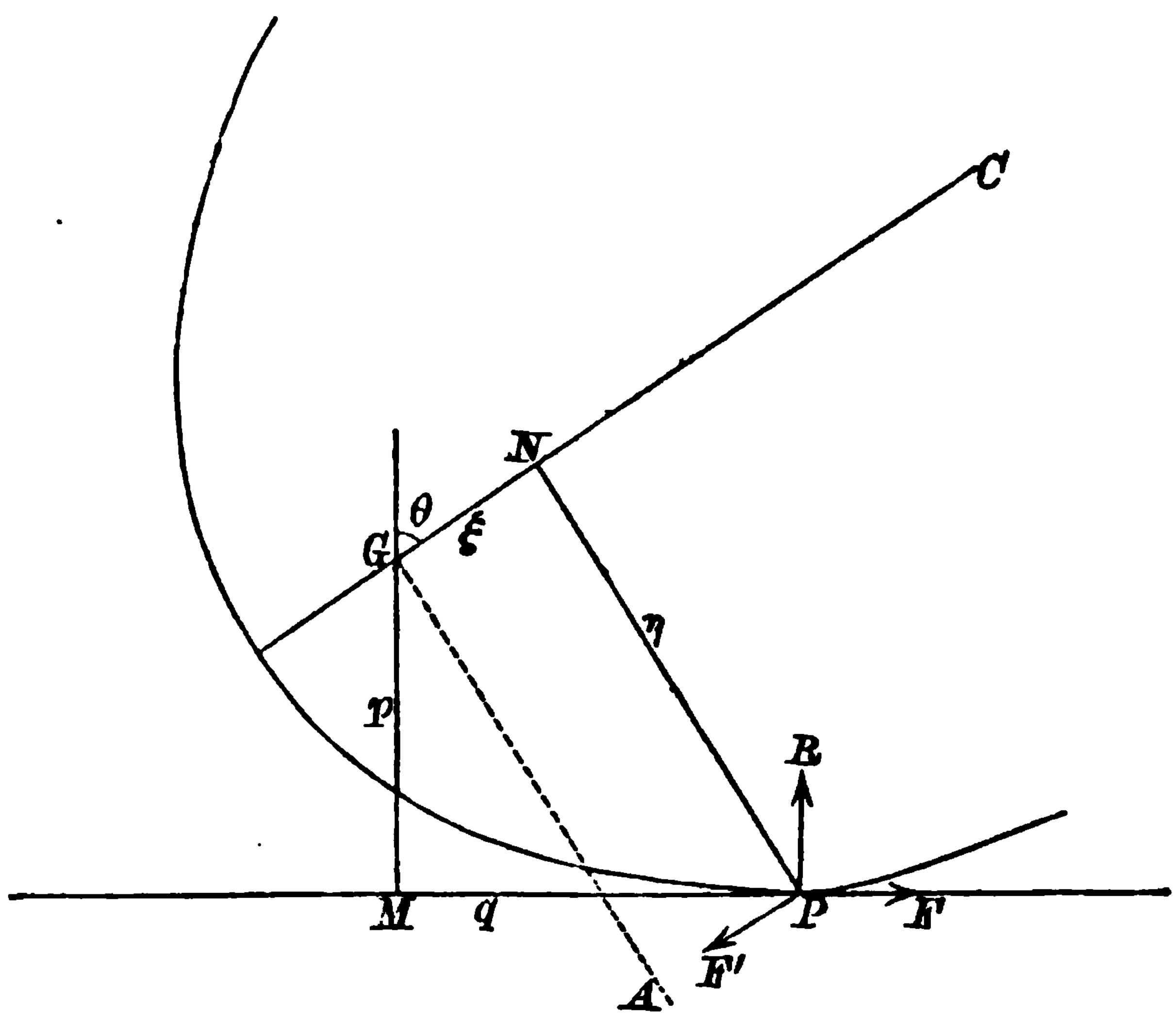
It will be noticed that these results are independent of the friction.

Ex. 6. Measure $ST=\frac{7}{5}a\cot\beta$ along the projection of the blow on the horizontal table, then TS measures the horizontal component of the blow referred to a unit of mass, on the same scale that PS measures the final velocity of the centre of gravity. Prove that during the impact and the whole of the subsequent motion the friction acts along PT and that the whole friction called into play will be measured by PT on the scale just mentioned. Thence show that unless $\mu<\frac{5}{7}\frac{PT}{a}$ the parabolic arc of the path will be suppressed. Show also that PT is the direction in which the lowest point of the ball would begin to move if the horizontal plane were smooth and the ball were acted on by the same blow as before.

Motion of a Solid Body on a plane.

589. *A solid of revolution rolls on a perfectly rough horizontal plane under the action of gravity. To find the steady motion and the small oscillations.*

Let G be the centre of gravity of the body, GC the axis of figure, P the point of contact. Let GA be that principal axis which lies in the plane PGC and GB the axis at right angles to GA, GC. Let GM be a perpendicular from G on the hori-



zontal plane, and PN a perpendicular from P on GC. Let θ be the angle GC makes with the vertical, and ψ the angle MP makes with any fixed line in the horizontal plane. Let R be the normal reaction at P; F, F' the resolved parts of the frictions respectively in and perpendicular to the plane PGC. Let the mass of the body be unity.

Let us take moments about the moving axes GA, GB, GC according to Art. 253. As in the second case of Art. 254, we put $\theta_1=\omega_1$, $\theta_2=\omega_2$ and $\theta_3=\frac{d\psi}{dt}\cos\theta$. Remembering that $h_1'=A\omega_1$, $h_2'=A\omega_2$, $h_3'=C\omega_3$ we have

$$A\frac{d\omega_1}{dt}-A\omega_2\frac{d\psi}{dt}\cos\theta+C\omega_3\omega_2=-F'.GN \text{..........(1)}.$$

$$A\frac{d\omega_2}{dt}-C\omega_3\omega_1+A\omega_1\frac{d\psi}{dt}\cos\theta=-F.GM-R.MP \text{..........(2)}.$$

$$C\frac{d\omega_3}{dt}=F'.PN \text{..........(3)}.$$

The geometrical equations are

$$\frac{d\theta}{dt}=\omega_2 \text{..........(4)}. \qquad \sin\theta\frac{d\psi}{dt}=-\omega_1 \text{..........(5)}.$$

Let u and v be the velocities of the centre of gravity respectively along and perpendicular to MP, both being parallel to the horizontal plane. The accelerations of the centre of gravity along these moving axes will be

$$\frac{du}{dt}-v\frac{d\psi}{dt}=F \text{..........(6)},$$

$$\frac{dv}{dt}+u\frac{d\psi}{dt}=F' \text{..........(7)}.$$

And if z be the altitude of G above the horizontal plane, we have

$$\frac{d^2z}{dt^2}=-g+R \text{..........(8)}.$$

Also since the point P is at rest, we have

$$u-GM\omega_2=0 \text{..........(9)},$$

$$v+PN\omega_3-GN\omega_1=0 \text{..........(10)},$$

$$z=-GN\cos\theta+PN\sin\theta \text{..........(11)}.$$

These are the general equations of motion of a solid of revolution moving on a perfectly rough horizontal plane. If the plane is not perfectly rough the first eight equations will still hold, but the remaining three must be modified in the manner explained in the next proposition.

When the motion is steady, we have the surface of revolution rolling on the plane so that its axis makes a constant angle with the vertical. In this state of motion, let $\theta=\alpha$, $\frac{d\psi}{dt}=\mu$, $\omega_3=n$, $GM=p$, $MP=q$, $GN=\xi$, $NP=\eta$, and let ρ be the radius of curvature of the rolling body at P. Then the relations between these quantities may be found by substitution in the above equations.

Suppose it were required to find the conditions that the surface may roll with a given angular velocity n with its axis of figure making a given angle with the vertical. Here n and α are given, and p, q, ξ, η, ρ may be found from the equations to

the surface. We have to find μ, ω_1, ω_2, u, v and the radius of the circle described by G in space. Then eliminating F, F', R, we get

$$\mu^2 \sin\alpha\,(A\cos\alpha - p\xi) - n\mu\,(C\sin\alpha + p\eta) - gq = 0,$$
$$\omega_1 = -\mu\sin\alpha, \quad \omega_2 = 0,$$
$$u = 0, \qquad v = -n\eta - \xi\mu\sin\alpha.$$

Let r be the radius of the circle described by G as the surface rolls on the plane. Since G describes its circle with angular velocity μ, we have $r\mu = v$, and hence

$$r = -\frac{\eta n}{\mu} - \xi\sin\alpha.$$

Eliminating n we have

$$\mu^2\{A\eta\sin\alpha\cos\alpha + C\xi\sin^2\alpha + r\,(C\sin\alpha + p\eta)\} = gq\eta.$$

For every value of n and α there are two values of μ, which however correspond to different initial conditions. In order that a steady motion may be possible, it is necessary that the roots of this quadratic should be real. This gives

$$(C\sin\alpha + p\eta)^2 n^2 + 4gq\sin\alpha\,(A\cos\alpha - p\xi) = \text{a positive quantity}.$$

If the angular velocity n be very great, one of these values of μ is very great and the other small. If the angular velocity be communicated to the body by unwinding a string, as in a top, the initial value of ω_1 will be small. In this case the body will assume the smaller value of μ_1, and we have approximately

$$\mu = -\frac{gq}{n\,(C\sin\alpha + p\eta)}.$$

To find the small oscillation, we put $\theta = \alpha + \theta'$, $\dfrac{d\psi}{dt} = \mu + \dfrac{d\psi'}{dt}$, $\omega_3 = n + \omega_3'$. Then we have by geometry,

$$z = GM = p + q\theta', \qquad PM = q + (\rho - p)\,\theta',$$
$$GN = \xi + \rho\theta'\sin\alpha, \qquad PN = \eta + \rho\theta'\cos\alpha,$$

and substituting in (5), (9), (10), (6), (7) respectively, we find

$$\omega_1 = -\mu\sin\alpha - \mu\cos\alpha\theta' - \sin\alpha\frac{d\psi'}{dt},$$

$$u = p\frac{d\theta'}{dt},$$

$$v = -\mu\sin\alpha\xi - n\eta - (\mu\cos\alpha\xi + \mu\rho\sin^2\alpha + n\rho\cos\alpha)\,\theta' - \sin\alpha\xi\frac{d\psi'}{dt} - \eta\omega_3',$$

$$F = p\frac{d^2\theta'}{dt^2} + \mu^2\sin\alpha\xi + n\mu\eta + 2\sin\alpha\mu\xi\frac{d\psi'}{dt} + \eta n\frac{d\psi'}{dt}$$
$$+ \mu\,(\mu\cos\alpha\xi + \mu\rho\sin^2\alpha + n\rho\cos\alpha)\,\theta' + \eta\mu\omega_3',$$

$$F' = -(\mu\cos\alpha\xi - p\mu + \mu\rho\sin^2\alpha + n\rho\cos\alpha)\frac{d\theta'}{dt} - \sin\alpha\xi\frac{d^2\psi'}{dt^2} - \eta\frac{d\omega_3'}{dt}.$$

Substituting these in equation (3) and integrating, we have

$$(C + \eta^2)\omega'_3 = (p\mu - \mu\xi\cos\alpha - \mu\rho\sin^2\alpha - n\rho\cos\alpha)\,\eta\theta' - \eta\sin\alpha\xi\frac{d\psi'}{dt} \quad \text{.......(A)},$$

the constant being omitted because n, α and μ are supposed to contain all the constant parts of ω_3, θ, and $\dfrac{d\psi}{dt}$.

Again substituting in (1) and integrating, we have

$$\{Cn - 2A\mu\cos\alpha + \xi(p\mu - \mu\cos\alpha\xi - \mu\sin^2\alpha\rho - n\rho\cos\alpha)\}\,\theta' - (A+\xi^2)\sin\alpha\,\frac{d\psi'}{dt} = \xi\eta\omega_3' \text{ (B)}.$$

Also substituting in (2), we have

$$\left.\begin{aligned}&(A+p^2+q^2)\frac{d^2\theta'}{dt^2} + \theta'\{A\mu^2(\sin^2\alpha - \cos^2\alpha) + Cn\mu\cos\alpha + (\rho - p)g\\ &\quad + \mu^2\sin\alpha\,\xi q + n\mu\eta q + \mu^2\cos\alpha\xi p + n\mu\rho p\cos\alpha + \mu^2\sin^2\alpha\rho p\}\\ &+ \frac{d\psi'}{dt}\{-2A\mu\sin\alpha\cos\alpha + Cn\sin\alpha + 2\xi p\mu\sin\alpha + np\eta\}\\ &+ \omega_3'\{C\mu\sin\alpha + \mu p\eta\}\\ &+ \{-A\sin\alpha\cos\alpha\mu^2 + Cn\mu\sin\alpha + gq + \sin\alpha\mu^2p\xi + n\mu p\eta\}\end{aligned}\right\} = 0 \ldots \text{(C)}.$$

The last term of this equation must vanish since θ', $\frac{d\psi'}{dt}$, ω_3' only contain periodic terms. It is the equation thus formed which determines the steady motion and gives us the value of μ.

To solve these equations we may put

$$\theta' = L\sin(\lambda t + f), \qquad \frac{d\psi'}{dt} = M\sin(\lambda t + f), \qquad \omega_3' = N\sin(\lambda t + f).$$

If we substitute these in (A), (B), (C) we shall get three equations to eliminate the ratios $L : M : N$. Before substitution it will be found convenient to simplify the equations first by multiplying (A) by ξ and (B) by η and subtracting the latter result from the former, and secondly by multiplying (A) by $\frac{\mu p}{\eta}$ and adding the result to (C). We then obtain the following determinant,

$$\begin{vmatrix} -(A+p^2+q^2)\lambda^2 + (\rho - p)g + \mu^2(p^2 - A\cos 2\alpha - qr) + n\mu C\cos\alpha & A\mu\sin\alpha\cos\alpha + \frac{qg}{\mu} & C\mu(\eta\sin\alpha - p) \\ Cn - 2A\mu\cos\alpha & A\sin\alpha & C\xi \\ (p - \xi\cos\alpha - \rho\sin^2\alpha)\mu - \rho n\cos\alpha & \xi\sin\alpha & -(C+\eta^2) \end{vmatrix} = 0.$$

590. Ex. 1. *To find the least angular velocity which will make a hoop roll in a straight line.*

In this case r is infinite and therefore μ must be zero. It follows from the equation of steady motion that $q=0$, or the hoop must be upright. We have $p=a$, $q=0$, $\xi=0$, $\eta=a$, $\mu=0$, and $C=2A$. The determinant becomes

$$\lambda^2 = \frac{2n^2(2A+a^2) - ag}{A+a^2},$$

so that the least angular velocity which will make λ a real quantity is given by

$$n^2 = \frac{ag}{2(C+a^2)}.$$

Let the hoop be an arc, we have $C=a^2$, and if V be the least velocity of the centre of gravity, this equation gives $V>\frac{1}{2}\sqrt{ag}$. Let the hoop be a disc, then $C=\frac{a^2}{2}$, and we have $V>\sqrt{\frac{ag}{3}}$.

Ex. 2. A circular disc is placed with its rim resting on a perfectly rough horizontal table and is spun with an angular velocity Ω about the diameter through the point of contact. Prove that in steady motion the centre is at rest at an altitude $\frac{k^2\Omega^2}{g}$ above the horizontal plane, where k is the radius of gyration about a diameter; and, if α be the inclination of the plane to the horizon, the point of contact has made a complete circuit in the time $\frac{2\pi}{\Omega}\sin\alpha$. If the disc be slightly disturbed from this state of steady motion, show that the time of a small oscillation is $2\pi\left\{\frac{k^2}{ga}\frac{(k^2+a^2)\sin\alpha}{3k^2\cos^2\alpha+a^2\sin^2\alpha}\right\}^{\frac{1}{2}}$.

Ex. 3. An infinitely thin circular disc moves on a perfectly rough horizontal plane in such a manner as to preserve a constant inclination α to the horizon. Find the condition that the motion may be steady and the time of a small oscillation.

Let the radius of the disc be a, and the radius of gyration about a diameter k. Let ω_3 be the angular velocity about the axis, μ the angular velocity of the centre of gravity about the centre of the circle described by it, r the radius of this circle, then in steady motion

$$(2k^2+a^2)\,\omega_3=k^2\mu\cos\alpha-\frac{ga}{\mu}\cot\alpha,\quad (2k^2+a^2)\,r=-k^2a\cos\alpha+\frac{ga^2}{\mu^2}\cot\alpha.$$

If T be the time of a small oscillation

$$\left(\frac{2\pi}{T}\right)^2(k^2+a^2)=\mu^2\{k^2(1+2\cos^2\alpha)+a^2\sin^2\alpha\}-n\mu\cos\alpha(6k^2+a^2)+2n^2(2k^2+a^2)-ga\sin\alpha.$$

Ex. 4. A heavy body is attached to the plane face of a hemisphere so as to form a solid of revolution, the radius of the hemisphere being a and the distance of the centre of gravity of the whole body from the centre of the hemisphere being h. The body is placed with its spherical surface resting on a horizontal plane, and is set in motion in any manner. Show that one integral of the equations of motion is $A\sin^2\theta\frac{d\psi}{dt}+C\omega_3\left(\cos\theta+\frac{h}{a}\right)=\text{constant}$ whether the plane be smooth, imperfectly rough, or perfectly rough.

It is clear that the first two terms on the left-hand side of this equation is the angular momentum about the vertical through G. Let this be called I. Since we may take moments about any axis through G as if G were fixed in space, we have $\frac{dI}{dt}=F'.PM$. But $PM=-PN.\frac{h}{a}$, hence eliminating F' by equation (3) and integrating, we get the required result.

Ex. 5. A surface of revolution rolls on another perfectly rough surface of revolution with its axis vertical. The centre of gravity of the rolling surface lies in its axis. Find the cases of steady motion in which it is possible for the axes of both the surfaces to lie in a vertical plane throughout the motion.

Let θ be the inclination of the axes of the two surfaces, P the point of contact, GM a perpendicular on the tangent plane at P, PN a perpendicular on the axis GC of the rolling body; F the friction, R the reaction at P;

n the angular velocity of the rolling body about its axis GC, μ the angular rate at which G describes its circular path in space, r the radius of this circle. Then in steady motion

$$\begin{aligned} M\mu \sin\theta (Cn - A\mu\cos\theta) &= -F \,.\, GM - R \,.\, MP, \\ R &= -Mr\mu^2 \sin\alpha + Mg\cos\alpha, \\ F &= -Mr\mu^2 \cos\alpha - Mg\sin\alpha, \\ n \,.\, PN + \mu\sin\theta \,.\, GN &= -r\mu, \end{aligned}$$

where M is the mass of the body.

591. *A surface of any form rolls on a fixed horizontal plane under the action of gravity. To form the equations of motion**.

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* The motion of a heavy body of any form on a horizontal plane seems to have been studied first by Poisson. The body is supposed to be either bounded by a continuous surface which touches the plane in a single point or to be terminated by an apex as in a top, while the plane is regarded as perfectly smooth. Poisson uses Euler's equations to find the rotations about the principal axes, and refers these axes to others fixed in space by means of the formulæ of Art. 235. He finds one integral by the principal of vis viva and another by that of angular momentum about the vertical straight line through the centre of gravity. These equations are then applied to find how the motion of a vertical top is disturbed by a slow movement of the smooth plane on which it rests. See the *Traité de Mécanique*.

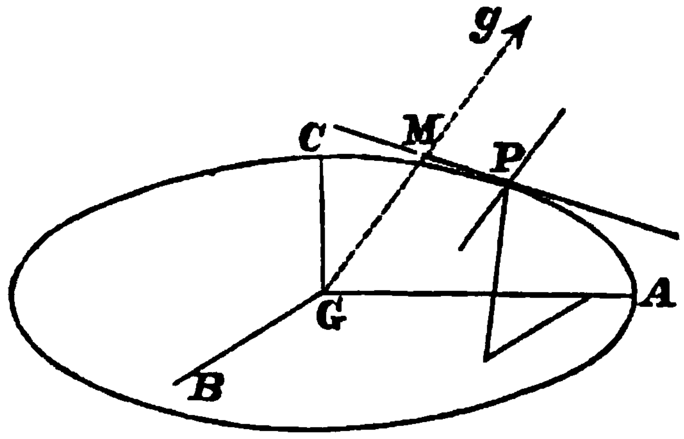
In three papers in the fifth and eighth volumes of *Crelle's Journal* (1830 and 1832) M. Cournot repeated Poisson's equations, and expressed the corresponding geometrical conditions when the body rests on more than one point or rolls on an edge such as the base of a cylinder. He also considers the two cases in which the plane is (1) perfectly rough, and (2) imperfectly rough. He proceeds on the same general plan as Poisson, having two sets of rectangular axes, one fixed in the body and the other in space connected together by the formulæ usually given for transformation of co-ordinates. As may be supposed, the equations obtained are extremely complicated. M. Cournot also forms the corresponding equations for impulsive forces. Those however which include the effects of friction do not agree with the equations given in this treatise.

In the thirteenth and seventeenth volumes of *Liouville's Journal* (1848 and 1852) there will be found two papers by M. Puiseux. In the first he repeats Poisson's equations and applies them to the case of a solid of revolution on a smooth plane. He shows that whatever angle the axis initially makes with the vertical, this angle will remain very nearly constant if a sufficiently great angular velocity be communicated to the body about the axis. An inferior limit to this angular velocity is found only in the case in which the axis is vertical. In the second memoir he applies Poisson's equations to determine the conditions of stability of a solid of any form placed on a *smooth* plane with a principal axis at its centre of gravity vertical and rotating about that axis. He also determines the small oscillations of a body resting on a smooth plane about a position of equilibrium.

In the fourth volume of the *Quarterly Journal of Mathematics*, 1861, Mr G. M. Slesser forms the equations of motion of a body on a perfectly rough horizontal plane and applies them to the problem considered at the end of Art. 597. He uses moving axes, and his analysis is almost exactly the same as that which the author had adopted.

Let GA, GB, GC, the principal axes at the centre of gravity, be the axes of reference and let the mass be unity. Let $\phi(\xi, \eta, \zeta)=0$ be the equation to the bounding surface, (ξ, η, ζ) the co-ordinates of the point P of contact. Let (p, q, r) be the direction-cosines of the *outward* direction of the normal to the surface at the point ξ, η, ζ, then

$$\frac{p}{\frac{d\phi}{d\xi}}=\frac{q}{\frac{d\phi}{d\eta}}=\frac{r}{\frac{d\phi}{d\zeta}}.$$



Firstly, let the plane be perfectly rough. Let X, Y, Z be the resolved parts along the axes of the normal reaction and the two frictions at the point ξ, η, ζ, and let the mass of the body be unity. By Euler's equations we have

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\omega_3&=\eta Z-\zeta Y\\ B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\omega_1&=\zeta X-\xi Z\\ C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2&=\xi Y-\eta X \end{aligned}\right\}\text{..........................(1)},$$

Also the equations of motion of the centre of gravity are by Art. 245,

$$\left.\begin{aligned} \frac{du}{dt}-v\omega_3+w\omega_2&=gp+X\\ \frac{dv}{dt}-w\omega_1+u\omega_3&=gq+Y\\ \frac{dw}{dt}-u\omega_2+v\omega_1&=gr+Z \end{aligned}\right\}\text{..............................(2)}.$$

Also since the line (p, q, r) remains always vertical,

$$\left.\begin{aligned} \frac{dp}{dt}&=q\omega_3-r\omega_2\\ \frac{dq}{dt}&=r\omega_1-p\omega_3\\ \frac{dr}{dt}&=p\omega_2-q\omega_1 \end{aligned}\right\}\text{..................................(3)},$$

and since the point (ξ, η, ζ) is at rest we have

$$\left.\begin{aligned} U&=u-\eta\omega_3+\zeta\omega_2=0\\ V&=v-\zeta\omega_1+\xi\omega_3=0\\ W&=w-\xi\omega_2+\eta\omega_1=0 \end{aligned}\right\}\text{..................................(4)},$$

where U, V, W are the resolved parts of the velocity of the point of contact P in the positive directions of the axes.

592. *Secondly, let the plane be perfectly smooth.* The equations (1), (2), (3), apply equally to this case, but equations (4) are not true. Since the resultant of X, Y, Z is a reaction R normal to the fixed plane, we have

$$X = -pR, \quad Y = -qR, \quad Z = -rR \quad \text{.......................(5)}.$$

The negative sign is prefixed to R because (p, q, r) are the direction-cosines of the outward direction of the normal, and it is clear that when these are taken positively, the components of R are all negative. If at any moment R vanishes and changes sign the body will leave the plane.

Since the velocity of G parallel to the fixed plane is constant in direction and magnitude, it will usually be more convenient to replace the equations (2) by the following single equation. Let GM be the perpendicular on the fixed plane and let $MG = z$, then

$$\frac{d^2z}{dt^2} = -g + R \quad \text{.......................(6)}.$$

It is necessary that the velocity of the point of contact resolved normal to the plane should be zero, this condition may be written in either of the equivalent forms

$$\left.\begin{aligned} Up + Vq + Wr = 0 \\ \frac{dz}{dt} + (\eta\omega_3 - \zeta\omega_2)\,p + (\zeta\omega_1 - \xi\omega_3)\,q + (\xi\omega_2 - \eta\omega_1)\,r = 0 \end{aligned}\right\} \quad \text{...............(7)}.$$

593. *Thirdly, let the body slide on an imperfectly rough plane.* The equations (1), (2), (3) and (7) hold as before. If μ be the coefficient of friction the resultant of the forces X, Y, Z must make an angle $\tan^{-1}\mu$ with the normal at the point of contact, hence

$$\frac{(Xp + Yq + Zr)^2}{X^2 + Y^2 + Z^2} = \frac{1}{1+\mu^2} \quad \text{.......................(8)}.$$

Also since the resultant of (X, Y, Z), the normal at P and the direction of sliding must lie in one plane, we have the determinantal equation

$$X(qW - rV) + Y(rU - pW) + Z(pV - qU) = 0 \quad \text{..................(9)}.$$

Since the friction must act *opposite* to the direction of sliding, we must have $XU + YV + ZW$ negative. When this vanishes and changes sign, the point of contact ceases to slide.

If the body start from rest we must use the method explained in Art. 146 to determine whether the point of contact will begin to slide or not. Assume X, Y, Z to be the forces necessary to prevent sliding. Then since u, v, w, ω_1, ω_2, ω_3 are all initially zero, we have by differentiating (4) and eliminating the differential coefficients of u, v, w, ω_1, ω_2, ω_3 three linear equations to find X, Y, Z, in terms of the known initial values of (p, q, r) and (ξ, η, ζ). The point of contact will slide or not according as these values make the left-hand side of equation (8) less or greater than the right-hand side.

The equations to find X, Y, Z may be obtained by treating the forces as if they were indefinitely small impulses. In the time dt, we may regard the body as acted on by an impulse gdt at G and a blow whose components are Xdt, Ydt, Zdt at P. By Art. 296 we may consider these in succession. The effect of the first is to communicate to P a velocity gdt in a direction normal to the fixed plane and outwards. If P does not slide, the effect of the blow at P must be to destroy this velocity. Hence X, Y, Z may be found from the equations of Art. 304 if we write $u_1 = pg$,

$v_1 = qg$, $w_1 = rg$ and u_2, v_2, w_2 all equal to zero on the left-hand sides and (to suit the notation of this article) change p, q, r on the right-hand sides into ξ, η, ζ. Geometrically the point of contact will not slide if the diametral line of the fixed plane with regard to the ellipsoid called E in Art. 304 makes a less angle with the normal than $\tan^{-1}\mu$.

In any of these cases when p, q, r have been found, the inclinations of the principal axes to the vertical are known. Their motion round the vertical may then be deduced by the rule given in Art. 249. When u, v, w and the motions of the axes have been found, the velocity of the centre of gravity resolved along any straight line fixed in space may be found by resolution.

594. Some integrals of these equations are supplied by the principles of angular momentum and Vis Viva. If the plane is perfectly smooth we have

$$A\omega_1 p + B\omega_2 q + C\omega_3 r = \alpha,$$

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + \left(\frac{dz}{dt}\right)^2 = \beta - 2gz,$$

where α and β are two constants.

If the plane is perfectly rough we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + u^2 + v^2 + w^2 = \beta - 2gz.$$

595. Ex. 1. A body rests with a plane face on an imperfectly rough horizontal plane whose coefficient of friction is μ. The centre of gravity of the body is vertically over the centre of gravity of the face, and the form of the face is such that the radius of gyration of the face about any straight line in its plane through its centre of gravity is γ. The body is now projected along the plane so that the initial velocity of its centre of gravity is v_0 and the initial rotation about a vertical axis through its centre of gravity is ω_0. If ω_0 be very small, prove that the centre of gravity moves in a straight line and its velocity at the end of any time t is $v_0 - \mu gt$. Show also that the angular velocity at the same time is $\omega_0\left(1 - \frac{\mu gt}{v_0}\right)^{\frac{\gamma^2}{k^2}}$, where k is the radius of gyration of the body about a vertical through the centre of gravity. [*Poisson, Traité de Mecanique.*]

Ex. 2. A body of any form rests with a plane face in contact with a smooth fixed plane so that the perpendicular from the centre of gravity G on the plane falls within the face. If the body is then struck by a blow which passes through G or begins to move from rest under the action of any finite forces whose resultant passes through G, prove that it will not turn over, but will begin to slide along the plane, even if the line of action of the force cuts the plane outside the base. [*Cournot.*]

596. Whatever the shape of a body may be we may suppose it to be set in rotation about the normal at the point of contact with an angular velocity n. If this angular velocity be not zero, the normal must be a principal axis at the point of contact, and yet it must pass through the centre of gravity. This cannot be unless the normal be a principal axis at the centre of gravity. If however $n = 0$, this condition is not necessary. There are therefore two cases to be considered.

Case 1. *A body of any form is placed in equilibrium resting with the point* C *on a rough horizontal plane, with a principal axis at the centre of gravity vertical, and*

is then set in rotation with an angular velocity n *about* GC. *A small disturbance being given to the body, it is required to find the motion.*

Case 2. *A body of any form is placed in equilibrium on a rough horizontal plane with the centre of gravity over the point of contact. A small disturbance being given to the body, to find the motion.*

597. *Case* 1. Supposing the body not to depart far from its initial position, we have p, q, u, v, w, ω_1, ω_2 all small quantities and $r=1$ nearly. Hence by (2), when we neglect the squares of small quantities, we see that X, Y are also small, and $Z=-g$ nearly. It follows by (1) that ω_3 is constant and $\therefore =n$. Also ξ and η are small and $\zeta=h$ nearly, where h is the altitude of the centre of gravity above the horizontal plane before the motion was disturbed. The equation to the surface may, by Taylor's theorem, be written in the form

$$\zeta=h-\frac{1}{2}\left(\frac{\xi^2}{a}+\frac{2\xi\eta}{b}+\frac{\eta^2}{c}\right),$$

where (a, b, c) are some constants depending on the curvatures of the principal sections of the body at the point C.

The squares of all small quantities being neglected, the preceding equations become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(B-C)\,n\omega_2&=-g\eta-hY\\ B\frac{d\omega_2}{dt}-(C-A)\,n\omega_1&=hX+g\xi\end{aligned}\right\},$$

$$\frac{du}{dt}-nv=gp+X,\qquad \frac{dv}{dt}+nu=gq+Y,$$

$$\frac{dp}{dt}=nq-\omega_2,\qquad \frac{dq}{dt}=\omega_1-np,$$

$$u-n\eta+h\omega_2=0,\qquad v-h\omega_1+n\xi=0,$$

$$\frac{p}{\dfrac{\xi}{a}+\dfrac{\eta}{b}}=\frac{q}{\dfrac{\xi}{b}+\dfrac{\eta}{c}}=1.$$

Eliminating X, Y, u, v, ω_1, ω_2 from these equations, we get

$$(A+h^2)\frac{d^2q}{dt^2}+(A+B+2h^2-C)\,n\frac{dp}{dt}-\{(B-C)\,n^2+hg+h^2n^2\}\,q=-(g+hn^2)\,\eta+hn\frac{d\xi}{dt}$$

$$-(B+h^2)\frac{d^2p}{dt^2}+(A+B+2h^2-C)\,n\frac{dq}{dt}+\{(A-C)n^2+hg+h^2n^2\}\,p=(g+hn^2)\,\xi+hn\frac{d\eta}{dt}.$$

It will be found convenient to express ξ, η in terms of p, q. The right-hand sides of each of these equations will then take the form

$$Lp+Mq+L'\frac{dp}{dt}+M'\frac{dq}{dt}.$$

To solve these equations, we must then assume p, q to be of the form

$$\left.\begin{aligned}p&=P_0\cos\lambda t+P_1\sin\lambda t\\ q&=Q_0\cos\lambda t+Q_1\sin\lambda t\end{aligned}\right\}.$$

If the tangents to the lines of curvature of the moving body at C be parallel to the principal axes at the centre of gravity, these equations admit of considerable simplification. In that case the equation to the surface may be written in the form

$$\zeta = h - \frac{1}{2}\left(\frac{\xi^2}{a} + \frac{\eta^2}{c}\right),$$

where a and c are the radii of curvature of the lines of curvature. The right-hand sides of the equations then become respectively

$$-(g+hn^2)\,cq + hna\frac{dp}{dt} \text{ and } (g+hn^2)\,ap + hnc\frac{dq}{dt}.$$

To satisfy the equations, it will be sufficient to put

$$p = F\cos(\lambda t + f), \qquad q = G\sin(\lambda t + f).$$

This simplification is possible, because we can see beforehand that $\dfrac{P_0}{P_1} = \dfrac{Q_0}{Q_1}$.

Substituting and eliminating the ratio $\dfrac{F}{G}$, we get the following quadratic to determine λ^2.

$$[(A+h^2)\lambda^2 + \{B - C + h(h-c)\}n^2 + g(h-c)][(B+h^2)\lambda^2 + \{A - C + h(h-a)\}n^2 + g(h-a)]$$
$$= \lambda^2 n^2\{A + B + 2h^2 - C - ha\}\{A + B + 2h^2 - C - hc\}.$$

If λ_1, λ_2 be the roots of this equation, the motion is represented by the equations

$$\left.\begin{aligned} p &= F_1\cos(\lambda_1 t + f_1) + F_2\cos(\lambda_2 t + f_2) \\ q &= G_1\sin(\lambda_1 t + f_1) + G_2\sin(\lambda_2 t + f_2) \end{aligned}\right\},$$

where $\dfrac{G_1}{F_1}$, $\dfrac{G_2}{F_2}$ are known functions of λ_1, λ_2 respectively, and F_1, F_2, f_1, f_2 are constants to be determined by the initial values of p, q, $\dfrac{dp}{dt}$, $\dfrac{dq}{dt}$.

In order that the motion may be stable, it is necessary that the roots of this quadratic should be real and positive. These conditions may be easily expressed.

Ex. 1. A solid of revolution is placed with its axis vertical on a perfectly rough horizontal plane and is set in rotation about its axis with an angular velocity n.

If c be the radius of curvature at the vertex, h the altitude of the centre of gravity, k the radius of gyration about the axis, k' that about an axis through the vertex perpendicular to the axis of figure, show that the position of the body will be stable if $n > 2\dfrac{k'\sqrt{g(h-c)}}{k^2 + hc}$.

Ex. 2. An ellipsoid is placed with one of its vertices in contact with a smooth horizontal plane. What angular velocity of rotation must it have about the vertical axis in order that the equilibrium may be stable?

Result. Let a, b, c be the semi-axes, c the vertical axis, then the angular velocity must be greater than $\sqrt{\dfrac{5g}{c}}\cdot\dfrac{\sqrt{c^4-a^4}+\sqrt{c^4-b^4}}{a^2+b^2}$. [*Puiseux.*]

Ex. 3. A solid of any form is placed in equilibrium with the point C on a smooth horizontal plane, a principal axis GC at the centre of gravity being vertical, and an angular velocity n is then communicated to it about GC. A small disturb-

ance being given, show that the harmonic periods may be deduced from the quadratic

$$(A\lambda^2+E)(B\lambda^2+F)=(A+B-C)n^2\lambda^2+g^2(\rho'-\rho)^2\sin^2\delta\cos^2\delta,$$

where

$$E=(B-C)n^2+g\{(h-\rho)\sin^2\delta+(h-\rho')\cos^2\delta\},$$
$$F=(A-C)n^2+g\{(h-\rho)\cos^2\delta+(h-\rho')\sin^2\delta\}.$$

Also h is the altitude of the centre of gravity, ρ, ρ' are the principal radii of curvature at the vertex, and δ is the angle the principal axis GA makes with the plane of the section whose radius of curvature is ρ. [*Puiseux.*]

598. *Case* 2. Supposing the disturbance to be small, we have ω_1, ω_2, ω_3, u, v, w all small quantities. Hence when we neglect the squares of small quantities the equations (1) and (2) become respectively,

$$A\frac{d\omega_1}{dt}=\eta Z-\zeta Y,\quad B\frac{d\omega_2}{dt}=\zeta X-\xi Z,\quad C\frac{d\omega_3}{dt}=\xi Y-\eta X\ldots\ldots\ldots\ldots\text{(i)},$$

$$\frac{du}{dt}=gp+X,\quad \frac{dv}{dt}=gq+Y,\quad \frac{dw}{dt}=gr+Z\ldots\ldots\ldots\ldots\text{(ii)}.$$

Let ξ_0, η_0, ζ_0 be the co-ordinates of the point of contact in the position of equilibrium, and let $\xi=\xi_0+\xi'$, $\eta=\eta_0+\eta'$, $\zeta=\zeta_0+\zeta'$. Then in the small terms of equation (4) we may write ξ_0, η_0, ζ_0 for ξ, η, ζ. Hence differentiating these and eliminating X, Y, Z, u, v, w by help of equations (i) and (ii), we get

$$(A+\eta_0^2+\zeta_0^2)\frac{d\omega_1}{dt}-\xi_0\eta_0\frac{d\omega_2}{dt}-\xi_0\zeta_0\frac{d\omega_3}{dt}=-g(\eta r-\zeta q)\ldots\ldots\ldots\ldots\text{(iii)},$$

and two similar equations.

Let p_0, q_0, r_0 be the values of p, q, r in the position of equilibrium. Then $\frac{\xi_0}{p_0}=\frac{\eta_0}{q_0}=\frac{\zeta_0}{r_0}=\rho$, where ρ is the radius vector from G to the point of contact. Now in the small terms of equations (3) we may write p_0, q_0, r_0 for p, q, r. Hence equations (iii) become by substitution

$$A\frac{d\omega_1}{dt}=\eta_0\rho\frac{d^2r}{dt^2}-\zeta_0\rho\frac{d^2q}{dt^2}-g(\eta r-\zeta q)\ldots\ldots\ldots\ldots\text{(iv)},$$

and two similar equations. At the time t let $p=p_0+p'$, $q=q_0+q'$, and $r=r_0+r'$.

Then since $(p_0+p')^2+(q_0+q')^2+(r_0+r')^2=1$, we have $p_0p'+q_0q'+r_0r'=0$. The form of the surface being known we can find p', q', r' in terms of ξ', η', ζ', and thus express $\eta r-\zeta q$, $\zeta p-\xi r$, $\xi q-\eta p$ in the form $-g(\eta r-\zeta q)=Lp'+Mq'$.

The equations (iv) now become

$$A\frac{d\omega_1}{dt}=\eta_0\rho\frac{d^2r'}{dt^2}-\zeta_0\rho\frac{d^2q'}{dt^2}+Lp'+Mq'\ldots\ldots\ldots\ldots\text{(v)},$$

and two similar equations.

Differentiating equations (3), and substituting for $\frac{d\omega_1}{dt}$, $\frac{d\omega_2}{dt}$, $\frac{d\omega_3}{dt}$, r' and $\frac{d^2r'}{dt^2}$, we get equations of the form

$$\left.\begin{aligned}F\frac{d^2p'}{dt^2}+G\frac{d^2q'}{dt^2}&=Hp'+Kq'\\ F'\frac{d^2p'}{dt^2}+G'\frac{d^2q'}{dt^2}&=H'p'+K'q'\end{aligned}\right\}.$$

To solve these we put $p'=P\cos(\lambda t+f)$, $q'=Q\cos(\lambda t+f)$, substituting and eliminating the ratios $\frac{P}{Q}$, we have the quadratic

$$\begin{vmatrix} F\lambda^2+H, & G\lambda^2+K \\ F'\lambda^2+H', & G'\lambda^2+K' \end{vmatrix}=0 \quad\text{.......................... (vi)},$$

to determine λ^2.

Thus by virtue of the relation existing between p', q', r', each of these may be represented by an expression of the form

$$P_1\cos(\lambda_1 t+f_1)+P_2\cos(\lambda_2 t+f_2).$$

Substituting these values in equations (v) we see that ω_1, ω_2, ω_3 can each be represented by an expression

$$\Omega_1+E_1\cos(\lambda_1 t+f_1)+E_2\cos(\lambda_2 t+f_2),$$

where E_1, E_2 are known functions of P_1, P_2 ... and λ_1, λ_2, but Ω_1, Ω_2, Ω_3 are small arbitrary quantities. By substituting in equations (3) and equating the coefficients of $\cos(\lambda_1 t+f_1)$ and $\cos(\lambda_2 t+f_2)$, we may find the values of E_1 and E_2 without difficulty. And we also see that we must have

$$\frac{\Omega_1}{p_0}=\frac{\Omega_2}{q_0}=\frac{\Omega_3}{r_0},$$

so that, of the three Ω_1, Ω_2, Ω_3, only one is really arbitrary. We have therefore but five arbitrary constants, viz. P_1, P_2, f_1, f_2, and Ω_1. These are determined by the initial values of ω_1, ω_2, ω_3, p' and q'.

To find the motion of the principal axes round the vertical, let ϕ be the angle the plane containing GC and the vertical makes with the plane of AC. Then by drawing a figure for the standard case in which p, q, r are all positive, it will be seen that if μ be the rate at which GC goes round the vertical,

$$\mu\sqrt{1-r^2}=\omega_1\cos\phi+\omega_2\sin\phi=\frac{p_0\omega_1+q_0\omega_2}{\sqrt{1-r_0^2}}.$$

Substituting for ω_1, ω_2, this takes the form

$$\mu=n_3+N_1\cos(\lambda_1 t+f_1)+N_2\cos(\lambda_2 t+f_2),$$

where n_3, N_1, N_2 are all known constants.

In order that the equilibrium may be stable it is necessary that the roots of the quadratic (vi) should both be real and positive. These conditions may easily be expressed.

These conditions being supposed satisfied, the expressions for p', q', r' will only contain periodical terms, and thus the inclinations of the principal axes to the vertical will not be sensibly altered. But the expressions for ω_1, ω_2, ω_3 *may* each contain a non-periodical term, and if so the rate at which the principal axes will go round the vertical will also contain non-periodical terms. The body therefore may gradually turn with a slow motion round the normal at the point of contact. The expressions for u, v, w will contain only periodic terms, so that the body will have no motion of translation in space.

Motion of a Rod.

599. When the body whose motion is to be determined is a rod, it is often more convenient to recur to the original equations of motion supplied by D'Alembert's Principle. The equations of Lagrange may also be used with advantage. These methods will be illustrated by the following problem.

A uniform heavy rod, suspended from a fixed point O by a string, makes small oscillations about the vertical. Determine the motion.

Let O be taken as origin, and let the axis of z be measured vertically downwards; let $2a$ be the length of the rod, b the length of the string. Let (l, m, n) (p, q, r) be the direction-cosines of the string and rod. Then l, m, p, q are small quantities whose squares are to be neglected, and we may put n and r each equal to unity. Let u be the distance of any element du of the rod from that extremity A of the rod to which the string is attached. Let (x, y, z) be the co-ordinates of the element du, then we have

$$x = bl + up, \quad y = bm + uq, \quad z = b + u \quad \text{.........................(1)}.$$

Let M be the mass of the rod, MT the tension of the string, the equations of motion of the centre of gravity will be

$$\left.\begin{aligned} b\frac{d^2l}{dt^2} + a\frac{d^2p}{dt^2} &= -Tl \\ b\frac{d^2m}{dt^2} + a\frac{d^2q}{dt^2} &= -Tm \\ 0 &= g - T \end{aligned}\right\} \quad \text{............................(2)}.$$

By D'Alembert's Principle the equation of moments round x will be

$$\Sigma du\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma du\,(yZ - zY) = \Sigma du\,(yg).$$

By equations (1) this reduces to

$$\int_0^{2a} du\left\{-(b+u)\left(b\frac{d^2m}{dt^2} + u\frac{d^2q}{dt^2}\right)\right\} = 2ag\,(bm + aq).$$

Integrating, we get

$$-2ab\left(b\frac{d^2m}{dt^2} + a\frac{d^2q}{dt^2}\right) - 2ba^2\frac{d^2m}{dt^2} - \frac{8a^3}{3}\frac{d^2q}{dt^2} = 2ag\,(bm + aq),$$

which by equations (2) reduces to

$$b\frac{d^2m}{dt^2} + \frac{4}{3}a\frac{d^2q}{dt^2} = -gq.$$

Therefore the four equations of motion are

$$b\frac{d^2l}{dt^2} + a\frac{d^2p}{dt^2} = -gl, \quad b\frac{d^2l}{dt^2} + \frac{4}{3}a\frac{d^2p}{dt} = -gp \quad \text{....................(3)},$$

and two similar equations for m, q. These equations do not contain m or q, and on the other hand the equations to find m and q do not contain l or p. This shows that the oscillations in the plane xz are not affected by those in the perpendicular plane yz. See Art. 450.

To solve these equations, put $l = F\sin(\lambda t + \alpha)$, $p = G\sin(\lambda t + \alpha)$,

we get $$b\lambda^2 F + a\lambda^2 G = gF, \qquad b\lambda^2 F + \frac{4}{3}a\lambda^2 G = gG;$$

$$\therefore \lambda^4 - \frac{4a + 3b}{ab}g\lambda^2 + \frac{3g^2}{ab} = 0,$$

and the values of λ may be found from this equation.

In order to make a comparison of different methods, let us deduce the motion from Lagrange's equations. In this case we must determine the semi vis viva T true to the *squares* of the small quantities p, q, l, m, we cannot therefore put $r=1$, $n=1$. Since $p^2+q^2+r^2=1$, $l^2+m^2+n^2=1$, we have

$$r=1-\frac{p^2+q^2}{2}, \qquad n=1-\frac{l^2+m^2}{2},$$

we must therefore replace the third of equations (1) by

$$z=bn+ur=b+u-b\frac{l^2+m^2}{2}-u\frac{p^2+q^2}{2}.$$

If accents denote differential coefficients with regard to t, as in Lagrange's equations we have

$$\Sigma mx'^2=\Sigma m\,(b^2l'^2+2bl'p'u+p'^2u^2)=M\left(b^2l'^2+2bl'p'a+\frac{4a^2}{3}p'^2\right).$$

The value of $\Sigma my'^2$ may be found in a similar manner. The value of $\Sigma mz'^2$ is of the fourth order and may be neglected. Hence we have

$$2T=b^2\,(l'^2+m'^2)+2ab\,(l'p'+m'q')+\frac{4a^2}{3}\,(p'^2+q'^2).$$

Also we have $U=-g\left(b\dfrac{l^2+m^2}{2}+a\dfrac{p^2+q^2}{2}\right)+\text{constant}.$

The equation $\dfrac{d}{dt}\dfrac{dT}{dl'}-\dfrac{dT}{dl}=\dfrac{dU}{dl}$ becomes $bl''+ap''=-gl$;

similarly we get $$bl''+\frac{4a}{3}p''=-gp.$$

These are the same equations which we deduced from D'Alembert's Principle, and the solution may be continued as before.

EXAMPLES*.

1. A uniform rod, moveable about one extremity, moves in such a manner as to make always nearly the same angle a with the vertical; show that the time of a small oscillation is $2\pi\sqrt{\dfrac{2a}{3g}\cdot\dfrac{\cos a}{1+3\cos^2 a}}$, a being the length of the rod.

2. If a rough plane inclined at an angle a to the horizon be made to revolve with uniform angular velocity n about a normal Oz and a sphere be placed at rest upon it, show that the path in space of the centre will be a prolate, a common, or a curtate cycloid, according as the point at which the sphere is initially placed is without, upon, or within the circle whose equation is $x^2+y^2=\dfrac{35\,g\sin a}{2n^2}x$, the axis Oy being horizontal.

When the sphere is placed at rest on the moving plane, it should be noticed that a velocity is suddenly given to it by the impulsive frictions.

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* These Examples are taken from the Examination Papers which have been set in the University and in the Colleges.

3. A circular disc capable of motion about a vertical axis through its centre perpendicular to its plane is set in motion with angular velocity Ω. A rough uniform sphere is gently placed on any point of the disc, not the centre, prove that the sphere will describe a circle on the disc, and that the disc will revelve with angular velocity $\frac{7Mk^2}{7Mk^2+2mr^2}\Omega$, where Mk^2 is the moment of inertia of the disc about its centre, m is the mass of the sphere and r the radius of the circle traced out.

4. A sphere is pressed between two perfectly rough parallel boards which are made to revolve with the uniform angular velocities Ω and Ω' about fixed axes perpendicular to their planes. Prove that the centre of the sphere describes a circle about an axis which is in the same plane as the axes of revolution of the boards and whose distances from these axes are inversely proportional to the angular velocities about them.

Show that when the boards revolve about the same axis, their points of contact will trace on the sphere small circles, the tangents of whose angular radii will be $\frac{c}{a}\cdot\frac{\Omega\sim\Omega'}{\Omega+\Omega'}$, a being the radius of the sphere and c that of the circle described by its centre.

5. A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere being placed on it in a position of unstable equilibrium is so projected that the centre begins to move with a velocity V parallel to the axis of the cylinder. It is then slightly disturbed in a direction perpendicular to the axis. If θ be the angle the radius through the point of contact makes with the vertical, prove that the velocity of the centre parallel to the axis at any time t is $V\cos\sqrt{\frac{2}{7}}\,\theta$ and that the sphere will leave the cylinder when $\cos\theta=\frac{10}{17}$.

6. A uniform sphere is placed in contact with the exterior surface of a perfectly rough cone. Its centre is acted on by a force the direction of which always meets the axis of the cone at right angles and the intensity of which varies inversely as the cube of the distance from that axis. Prove that if the sphere be properly started the path described by its centre will meet every generating line of the cone on which it lies in the same angle.

See the *Solutions of Cambridge Problems* for 1860, page 92.

7. Every particle of a sphere of radius a, which is placed on a perfectly rough sphere of radius c, is attracted to a centre of force on the surface of the fixed sphere with a force varying inversely as the square of the distance; if it be placed at the extremity of the diameter through the centre of force and be set rotating about that diameter and then slightly displaced, determine its motion; and show that when it leaves the fixed sphere the distance of its centre from the centre of force is a root of the equation $20x^3-13(2c+a)x^2+7a(2c+a)^2=0$.

8. A perfectly rough plane revolves uniformly about a vertical axis in its own plane with an angular velocity n, a sphere being placed in contact with the plane rolls on it under the action of gravity, find the motion.

Take the axis of revolution as axis of z, and let the axis of x be fixed in the plane. Let a be the radius, m the mass of the sphere; F, F' the frictions resolved

parallel to the axes of x and z and R the normal reaction. The equations of motion are therefore by Art. 179 $\frac{d^2x}{dt^2} - n^2x = \frac{F}{m}$, $-an^2 + 2n\frac{dx}{dt} = \frac{R}{m}$ and $\frac{d^2z}{dt^2} = -g + \frac{F'}{m}$. The equations of rotation by Art. 255 are $\frac{d\omega_x}{dt} - n\omega_y = -\frac{F'a}{k^2}$, $\frac{d\omega_y}{dt} + n\omega_x = 0$, $\frac{d\omega_z}{dt} = \frac{Fa}{k^2}$. Since the point of contact has the same motion as the plane the geometrical equations by Art. 244 are $\frac{dx}{dt} - an + a\omega_z = 0$, $\frac{dz}{dt} - a\omega_x = 0$. Solving these equations we find that the sphere will not fall down. If the sphere start from *relative* rest at a point in the axis of x, we have $z = -\frac{g}{n^2}\tan^2 i\{1 - \cos(nt\cos i)\}$ where $\sin i = \sqrt{\frac{5}{7}}$. The sphere will therefore never descend more than $\frac{5g}{n^2}$ below its original position.

9. A perfectly rough vertical plane revolves with a uniform angular velocity μ about an axis perpendicular to itself, and also with a uniform angular velocity Ω about a vertical axis in its own plane which meets the former axis. A heavy uniform sphere of radius c is placed in contact with the plane; prove that the position of its centre at any time t, will be determined by the equations

$$7\frac{d^2\xi}{dt^2} - 5\Omega^2\xi - 2\mu\frac{dz}{dt} = 0,$$

$$7\frac{d^2z}{dt^2} + 2\Omega^2\frac{dz}{dt} + 2\mu\left(\frac{d^2\xi}{dt^2} + \Omega^2\xi\right) = 0,$$

z denoting the distance of the centre from the horizontal plane through the horizontal axis of revolution, and ξ that from the plane through the two axes.

Prove also that $7u = 7c\Omega + 2\mu b$, $7v + 2\mu a = 0$, if a and b be the initial values of ξ and z, u and v those of $\frac{d\xi}{dt}$ and $\frac{dz}{dt}$.

10. A hoop $AGBF$ revolves about AB its diameter as a fixed vertical axis. GF is a horizontal diameter of the same circle which is without mass and which is rigidly connected to the circle; DC is a smaller concentric hoop which can turn freely about GF as diameter. If Ω, Ω', ω, ω', be the greatest and least angular velocities about AB, GF respectively, prove that $\Omega \,.\, \Omega' = \omega^2 - \omega'^2$.

11. OA, OB, OC are the principal axes of a rigid body which is in motion about a fixed point O. The axis OC has a constant inclination α to a line OZ fixed in space, and revolves with uniform angular velocity Ω round it, and the axis OA always lies in the plane ZOC. Prove that the constraining couple has its axis coincident with OB, and that its moment is $-(A - C)\,\Omega^2 \sin\alpha\cos\alpha$.

CHAPTER XI.

PRECESSION AND NUTATION,

&c. &c.

On the Potential.

600. *To find the potential of a body of any form at any external distant point.*

Let the centre of gravity G of the body be taken as the origin of co-ordinates and let the axis of x pass through S the external point. Let the distance $GS=\rho$. Let (x, y, z) be the co-ordinates of any element dm of the body situated at any point P and let $GP=r$, then $PS^2=\rho^2+r^2-2\rho x$. The potential of the body is

$$V=\Sigma\frac{dm}{PS};\quad \therefore\ V=\Sigma\frac{dm}{\rho}\left\{1-\frac{2\rho x-r^2}{\rho^2}\right\}^{-\frac{1}{2}}$$

$$=\Sigma\frac{dm}{\rho}\left\{1+\frac{1}{2}\frac{2\rho x-r^2}{\rho^2}+\frac{3}{8}\left(\frac{2\rho x-r^2}{\rho^2}\right)^2+\frac{5}{16}\left(\frac{2\rho x-r^2}{\rho^2}\right)^3+\frac{35}{128}\left(\frac{2\rho x-r^2}{\rho^2}\right)^4+\dots\right\};$$

arranging these terms in descending powers of ρ, we get

$$V=\Sigma\frac{dm}{\rho}\left\{1+\frac{x}{\rho}+\frac{3x^2-r^2}{2\rho^2}+\frac{5x^3-3xr^2}{2\rho^3}+\frac{35x^4-30x^2r^2+3r^4}{8\rho^4}+\dots\right\}.$$

Let M be the mass of the body, then $\Sigma dm=M$. Also since the origin is at the centre of gravity, we have $\Sigma x dm=0$.

Let A, B, C be the principal moments of inertia at the centre of gravity, I the moment of inertia about the axis of x, which in our case is the line joining the centre of gravity of the body to the attracted point. Then

$$\Sigma dm\, r^2=\frac{1}{2}(A+B+C),$$

$$\Sigma dm\, x^2=\Sigma dm\,(r^2-y^2-z^2)=\frac{1}{2}(A+B+C)-I.$$

Let l be any linear dimension of the body, then if ρ be so great compared with l that we may neglect the fraction $\left(\frac{l}{\rho}\right)^3$ of the potential, we have

$$V = \frac{M}{\rho} + \frac{A+B+C-3I}{2\rho^3}.$$

If we wish to make a nearer approximation to the value of V, we must take account of the next terms, viz.

$$\frac{5\Sigma mx^3 - 3\Sigma mxr^2}{2\rho^4}.$$

Let (ξ, η, ζ) be the co-ordinates of m referred to any fixed rectangular axes having the origin at G, and let (α, β, γ) be the angles GS makes with these axes. Then

$$x = \xi \cos\alpha + \eta \cos\beta + \zeta \cos\gamma;$$

$$\therefore\ \Sigma mx^3 = \cos^3\alpha\, \Sigma m\xi^3 + 3\cos^2\alpha \cos\beta\, \Sigma m\xi^2\eta + \ldots\ldots$$

If the body be symmetrical about any set of rectangular axes meeting at G, we have $\Sigma m\xi^3 = 0$, $\Sigma m\xi^2\eta = 0$, &c. $= 0$, so that this next term in the expression for the potential vanishes altogether. Thus the error of the preceding expression for V is comparable to only the fraction $\left(\frac{l}{\rho}\right)^4$ of the potential. This is the case with the earth, the form and structure of which are very nearly symmetrical about the principal axes at its centre of gravity.

This theorem is due to Poisson, but it was put into the convenient form just given by Prof. MacCullagh. See *Royal Irish Transactions* for 1855, page 387.

601. In the investigation of this value for the potential, S has been supposed to be at a very great distance. But the expression is also very nearly correct wherever the point S be situated, provided the body be an ellipsoid whose strata of equal density are concentric ellipsoids of small ellipticity.

To prove this, we may use a theorem in attractions due to Maclaurin, viz. The potentials of confocal ellipsoids at any external point are proportional to their masses. Let us first consider the case of a solid homogeneous ellipsoid. Describe an internal confocal ellipsoid of very small dimensions and let a', b', c' be its semi-axes. Then because the ellipticity is very small, we can take a', b', c' so small that S may be regarded as a distant point with regard to the internal ellipsoid. Hence the potential due to the internal ellipsoid is

$$V' = \frac{M'}{\rho} + \frac{A'+B'+C'-3I'}{2\rho^3},$$

where accented letters have the same meaning relatively to the internal ellipsoid that unaccented letters have with regard to the given ellipsoid. The error made in this expression is of the order $\left(\frac{a'}{\rho}\right)^4 V'$. Hence, by Maclaurin's theorem, the potential V of the given ellipsoid is

$$V = \frac{M}{\rho} + \frac{M}{M'}\,\frac{A' + B' + C' - 3I'}{2\rho^3},$$

and the error is of the order $\left(\frac{a'}{\rho}\right)^4 V$.

If a, b, c be the semi-axes of the given ellipsoid, we have

$$a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2 = \lambda^2;$$

$$\therefore\ A = M\,\frac{b^2 + c^2}{5} = M\left(\frac{b'^2 + c'^2}{5} + \frac{2}{5}\lambda^2\right) = \frac{M}{M'}A' + \frac{2}{5}M\lambda^2.$$

Similarly, $B = \frac{M}{M'}B' + \frac{2}{5}M\lambda^2, \quad C = \frac{M}{M'}C' + \frac{2}{5}M\lambda^2.$

Also if (α, β, γ) be the direction-angles of the line GS with reference to the principal axes at G, we have

$$I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma = \frac{M}{M'}I' + \frac{2}{5}M\lambda^2.$$

Hence, substituting, we have

$$V = \frac{M}{\rho} + \frac{A + B + C - 3I}{2\rho^3}.$$

If a, b, c be arranged in descending order of magnitude, we can by diminishing the size of the internal ellipsoid make c' as small as we please. In this case we have ultimately $a' = \sqrt{a^2 - c^2}$. Let ϵ be the ellipticity of the section containing a and c the greatest and least semi-axis. Then $a' = a\sqrt{2\epsilon}$, and the error of the above expression for V is of the order $4\left(\frac{a}{\rho}\right)^4 \epsilon^2 V$.

The theorem being true for any solid homogeneous ellipsoid is also true for any homogeneous shell bounded by concentric ellipsoids of small ellipticity. For the potential of such a shell may be found by subtracting the potentials of the bounding ellipsoids, $A + B + C$ by Art. 5 being independent of the directions of the axes.

Lastly, suppose the body to be an ellipsoid whose strata of equal density are concentric ellipsoids of small ellipticity, the external boundary being homogeneous. Then the proposition being true for each stratum, is also true for the whole body.

This theorem was first given by Prof. MacCullagh as a problem, and was published in the *Dublin University Calendar* for 1834, page 268. Some years after, about 1846, he gave his proof of the theorem in his lectures, which is substantially the same as that given in this Article. See the *Transactions of the Royal Irish Academy*, Vol. XXII., Parts I. and II., *Science*.

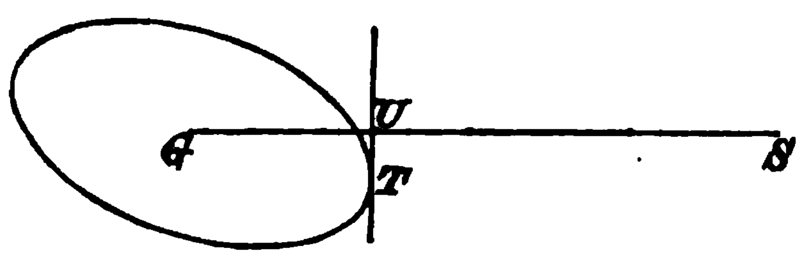
602. The following geometrical interpretation of the formula of Art. 600 is also due to Prof. MacCullagh. His demonstration and another by the Rev. R. Townsend may be found in the *Irish Transactions* for 1855.

A system of material points attracts a point S *whose distance from the centre of gravity* G *of the attracting mass is very great compared with the mutual distances of the particles. If a tangent plane be drawn to the ellipsoid of gyration perpendicular to* GS, *touching the ellipsoid in* T *and cutting* GS *in* U, *then the resultant attraction on* S *lies in the plane* SGT. *The component of the attraction on* S *in the direction* $\mathrm{TU} = -\frac{3\mathrm{M}}{\rho^4}\mathrm{GU}\,.\,\mathrm{UT}$. *The component of the attraction on* S *in the direction* $\mathrm{UG} = \frac{M}{\rho^2} + \frac{3}{2}\frac{\mathrm{A}+\mathrm{B}+\mathrm{C}-3\mathrm{I}}{\rho^4}$.

These theorems are also true if we replace the ellipsoid of gyration by any confocal ellipsoid. Let a, b, c be the semi-axes of this confocal, and let p be the perpendicular GU on the tangent plane. Since by Art. 26, $A = Ma^2 + \lambda$, $B = Mb^2 + \lambda$, &c. where λ is some constant, we have $V = \frac{M}{\rho} + \frac{M(a^2+b^2+c^2-3p^2)}{2\rho^3}$.

To prove that the resultant force on S lies in the plane SGT, let us displace S to S' where SS' is perpendicular to this plane and is equal to $\rho\, d\psi$. By Art. 326 the force on S in the direction SS' is $\frac{1}{\rho}\frac{dV}{d\psi}$. But after this displacement the tangent plane perpendicular to GS intersects along TU the former tangent plane, hence $\frac{dp}{d\psi} = 0$, and $\therefore \frac{dV}{d\psi} = 0$.

To find the force P acting at S in the direction TU, let us displace S to S'' where



SS'' is parallel to TU and is equal to $\rho\, d\psi$. Since GU is perpendicular to UT we have, exactly as in the Differential Calculus, $TU = \frac{dp}{d\psi}$. Hence

$$P = \frac{1}{\rho}\frac{dV}{d\psi} = -\frac{3M}{\rho^4}p\,.\,TU.$$

Lastly, to find the force R in the direction SG we have by Art. 326

$$R = -\frac{dV}{d\rho} = \frac{M}{\rho^2} + \frac{3}{2}\frac{A+B+C-3I}{\rho^4}.$$

Ex. Show that the product $GU\,.\,TU$ is the same for all confocals.

603. Ex. If GP be a straight line through the centre of gravity such that the moment of inertia about it is equal to the mean of the three principal moments of inertia at G, then the resolved attraction of the body on any point S in the direction SG is more nearly the same as if the body were collected into its centre of gravity when S lies in GP, than when S lies in any other straight line through G.

Show also that the moment of inertia about GP is equal to the mean of the moments of inertia about all straight lines passing through G.

If two of the principal moments of inertia are equal, prove that GP makes with the axis of unequal moment an angle equal to $\cos^{-1}\frac{1}{\sqrt{3}}$.

604. Ex. 1. If two bodies exert equal attractions on all external points, prove that their centres of gravity must coincide and their masses must be equal. The principal axes at their common centre of gravity must be coincident in direction, and the difference of their moments of inertia about any straight line constant.

Ex. 2. Thence show that two Chaslesian shells of the same body have the same principal axes at their common centre of gravity and the difference of their moments of inertia about any straight line constant.

Ex. 3. If the attraction of a body on every external point be the same as that of a single particle placed at some point, then the mass of the particle is equal to the mass of the body, the point is the centre of gravity, and unless the law of attraction be as the direct distance, every axis through the centre of gravity is a principal axis at the centre of gravity. See the *Quarterly Mathematical Journal*, 1857, Vol. II. page 136.

These results follow readily from Ex. 1.

Ex. 4. Let an ellipsoid be described having its semi-axes a, b, c such that $M\frac{2}{3}a^2=B+C-A+\lambda$, $M\frac{2}{3}b^2=C+A-B+\lambda$, $M\frac{2}{3}c^2=A+B-C+\lambda$, where λ is at our disposal, and may be any quantity positive or negative which does not make a, b, c imaginary. Let an indefinitely thin shell of mass M be constructed bounded by similar ellipsoids and having this ellipsoid for one bounding surface. Then the attractions of the given body and this shell on any distant external point are the same in direction and magnitude.

The attraction of such a shell on any external point is normal to the confocal through that point and is equal to $\frac{M}{a'b'c'}p'$, where a', b', c' are the semi-axes of the confocal and p' the perpendicular on the tangent plane at the attracted point. See the *Quarterly Journal of Pure and Applied Mathematics*, 1867, Vol. VIII. page 322.

Ex. 5. The attraction of a body two of whose principal moments at the centre of gravity A and B are equal and greater than the third attracts a distant point as if its mass were equally distributed over a straight line $2\sqrt{3\frac{A-C}{M}}$ placed perpendicular to the plane of A, B with its middle point at the centre of gravity. This proposition is accurately true if the body be an indefinitely thin shell bounded by similar prolate spheroids. In any case it is necessary that the equal moments A, B should be greater than the third moment of inertia C.

Ex. 6. Whatever be the relative magnitudes of the three principal moments of inertia, the attraction on a distant point is the same as if the mass was distributed

over the focal conic of the ellipsoid described in (4) so that the density at any point P is proportional to $\dfrac{AB}{\sqrt{AP \cdot PB}}$, where AB is the diameter through P.

Ex. 7. The attraction of any body of mass M on a distant particle may be found in the following manner. Let an indefinitely thin shell of mass $3M$ be constructed bounded by similar ellipsoids and having the ellipsoid of gyration at the centre of gravity for one bounding surface. Also let a particle of mass $4M$ be collected at the centre of gravity. Then the attraction of M on any distant particle is the same in direction and magnitude as if $4M$ attracted it and $3M$ repelled it.

605. Ex. If the law of attraction had been $-\phi$ (dist.) instead of the inverse square, the potential of a body on any external point S would have been represented by $\Sigma m\phi_1(PS)$, where $\phi(\rho)$ is the differential coefficient of $\phi_1(\rho)$. In this case, by reasoning in the same way as in Art. 600, we get

$$V = M\phi_1(\rho) + \phi'(\rho)\frac{A+B+C}{4} - \frac{\rho}{2}\frac{d}{d\rho}\left(\frac{\phi(\rho)}{\rho}\right)I,$$

where A, B, C and I have the same meanings as before.

If (x', y', z') be the co-ordinates of S referred to the principal axes at G, the moment of the attraction of S about the axis of y is $= \dfrac{1}{\rho}\dfrac{d}{d\rho}\dfrac{\phi(\rho)}{\rho}\cdot(C-A)x'z'$. See Art. 326.

606. Ex. An indefinitely thin stratum is placed on a sphere and the density at any point P is equal to $al^2+bm^2+cn^2$, where l, m, n are the direction-cosines of the radius through P referred to any rectangular axes. Show that the potential of the stratum at any external point is equal to

$$\frac{E(a+b+c)}{\rho f} + \frac{Ef}{5}\,\frac{3(al^2+bm^2+cn^2)-a-b-c}{\rho^3},$$

where f is the radius of the sphere and E its volume.

607. *To find the Force-function due to the attraction of any body on any other distant body.*

Let G, G' be the centres of gravity of the two bodies, and let $GG' = R$. Let A, B, C; A', B', C' be the principal moments of inertia of the two bodies at G and G' respectively; I, I' the moments of inertia about GG', and let M, M' be the masses of the two bodies.

Let m' be any element of the body M' situated at the point S, and let $GS = \rho$. Then the potential of the body M at m' is $m'\left\{\dfrac{M}{\rho} + \dfrac{A+B+C-3I_1}{2\rho^3}\right\}$, where I_1 is the moment of inertia of the body M about GS. We have now to sum this expression for all values of m'. This gives

$$M\Sigma\frac{m'}{\rho} + \Sigma m'\frac{A+B+C-3I_1}{2\rho^3}.$$

The first term by the same reasoning as before gives

$$\frac{MM'}{R}+M\frac{A'+B'+C'-3I'}{2R^3}.$$

In the second term, let x', y', z' be the co-ordinates of m' referred to G' as origin. Then

$$\rho = R\left(1+\frac{x'}{R}+\text{squares of } x', y', z'\right),$$

$$I_1 = I(1+\alpha x'+\beta y'+\gamma z'+\text{squares}),$$

where α, β, γ are some constants. Substituting these, and remembering that $\Sigma m'x'=0$, $\Sigma m'y'=0$, $\Sigma m'z'=0$, we get

$$M'.\frac{A+B+C-3I}{2R^3}\left\{1+\begin{pmatrix}\text{terms depending on the}\\ \text{squares of } x', y', z'\end{pmatrix}\right\}.$$

Hence the required force-function is

$$V=\frac{MM'}{R}+M\frac{A'+B'+C'-3I'}{2R^3}+M'\frac{A+B+C-3I}{2R^3}.$$

The error of this expression is of the order $\left(\frac{ll'}{R^2}\right)^2 V$, where l, l' are any linear dimensions of the two bodies respectively.

608. *To find the moment of the attraction of the sun and moon about one of the principal axes of the earth at its centre of gravity.*

Let the principal axes of the earth at its centre of gravity be taken as the axes of reference, and let α, β, γ be the direction-angles of the centre of gravity G' of the sun. Then if V be the potential of the sun or moon on the earth, we have

$$V=\frac{MM'}{R}+M\frac{A'+B'+C'-3I'}{2R^3}+M'\frac{A+B+C-3I}{2R^3},$$

where unaccented letters refer to the earth, and accented letters to the sun or moon. Let θ be the angle the plane through the sun and the axis of y makes with the plane of xy, then $\frac{dV}{d\theta}$ is the required moment in the direction in which we must turn the body to increase θ. From the above expression, since θ enters only through I, we have

$$\frac{dV}{d\theta}=-\frac{3}{2}\frac{M'}{R^3}\frac{dI}{d\theta}.$$

Now $I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma$, and by Spherical Trigonometry, we have

$$\left.\begin{aligned}\cos\gamma &= \sin\beta\sin\theta\\ \cos\alpha &= \sin\beta\cos\theta\end{aligned}\right\};$$

$$\therefore \frac{dI}{d\theta} = -2(A-C)\sin^2\beta\sin\theta\cos\theta;$$

$$\therefore \left.\begin{array}{l}\text{the moment required}\\ \text{about the axis of } y\end{array}\right\} = -3\frac{M'}{R^3}(C-A)\cos\alpha\cos\gamma.$$

In this expression the mass of the attracting body is measured in astronomical units. We may eliminate this unit in the following manner. Let n' be the mean angular velocity of the sun about the earth, R_0 its mean distance, so that if M be the mass of the earth, we have $\frac{M'+M}{R_0^3} = n'^2$. Now M is very small compared with M', so small that $\frac{M}{M'}$ is of the order of terms already neglected. Hence we may in the same terms put $\frac{M'}{R_0^3} = n'^2$, and therefore

$$\left.\begin{array}{l}\text{the moment of the sun's at-}\\ \text{traction about the axis of } y\end{array}\right\} = -3n'^2(C-A)\cos\alpha\cos\gamma\left(\frac{R_0}{R}\right)^3.$$

Let n'' be the mean angular velocity of the moon about the earth, so that, if M'' be the mass of the moon, R'_0 the mean distance, we have $\frac{M''+M}{R'^3_0} = n''^2$. Let ν be the ratio of the mass of the earth to that of the moon, then we have $\frac{M''(1+\nu)}{R'^3_0} = n''^2$, and therefore if R' be the distance of the moon

$$\left.\begin{array}{l}\text{the moment of the moon's}\\ \text{attraction about the axis of } y\end{array}\right\} = -\frac{3n''^2}{1+\nu}(C-A)\cos\alpha\cos\gamma\left(\frac{R'_0}{R'}\right)^3.$$

In the same way the moments about the other axes may be found. Putting κ for the coefficient, we have

$$\text{moment about axis of } x = -3\kappa(B-C)\cos\beta\cos\gamma,$$

$$\text{moment about axis of } z = -3\kappa(A-B)\cos\alpha\cos\beta.$$

609. Ex. 1. A body free to move about its centre of gravity is acted on by any number of attracting particles arranged in any way at a constant distance ρ from the centre of gravity. If $A_1, B_1, C_1, D_1, E_1, F_1$ be the moments and products of inertia of the body referred to any rectangular axes meeting in the centre of gravity,

and if accented letters represent corresponding quantities for the particles referred to the same axes, prove that the mutual potential of the body and the particles is

$$V=\frac{MM'}{\rho}+\frac{3(A_1A_1'+B_1B_1'+C_1C_1'+2F_1F_1'+2D_1D_1'+2E_1E_1')-(A_1+B_1+C_1)(A_1'+B_1'+C_1')}{2\rho^5},$$

where M' is the mass of all the particles. If the axes of reference be principal axes for either body, this result admits of considerable simplification.

Show that the numerator of the second term may be expressed in terms of the invariants of the momental ellipsoids of the body and of the system of particles.

Ex. 2. The force function between a body of any form and a uniform circular ring whose centre is at the centre of gravity of the body and whose mass is M' is

$$V=\frac{MM'}{\rho}-M'\frac{A+B+C-3J}{4\rho^3},$$

where J is the moment of inertia of the body about an axis through its centre of gravity perpendicular to the plane of the ring, and A, B, C are the principal moments of inertia at the centre of gravity.

This follows from Ex. 1.

Ex. 3. Thence show that Saturn's ring supposed uniform will have the same moments to turn Saturn about its centre of gravity as if half the whole mass were collected into a particle and placed in the axis of the ring at the same distance from Saturn, provided the particle repelled instead of attracted Saturn.

Ex. 4. If the earth be formed of concentric spheroidal strata of small but different ellipticities and of different densities, show that

$$\frac{C-A}{C}=\frac{\int\rho\frac{d(a^5\epsilon)}{da}da}{\int\rho\frac{da^5}{da}da},$$

where ϵ is the ellipticity and ρ the density of a stratum, the major-axis of which is a; the square of ϵ being neglected. It follows that if ϵ be constant, $\frac{C-A}{C}$ is independent of the law of density.

If we assume the law of density and the law of ellipticity given in the Figure of the Earth, this formula gives $\frac{C-A}{C}=\cdot 00313593$. See Pratt's *Figure of the Earth.*

Ex. 5. A body free to turn about a fixed straight line passing through the centre of gravity is in equilibrium under the attraction of a distant fixed particle. Show that the time of a small oscillation is $2\pi\left\{\frac{B\rho^5}{3M'\xi\{(C-A)\xi+F\eta\}}\right\}^{\frac{1}{2}}$, where the fixed straight line is the axis of y, the plane of xy in equilibrium passes through the attracting particle, and ξ, η are the co-ordinates of the particle. Also A, B, C, D, E, F are the moments and products of inertia of the body about the axes.

If the straight line did not pass through the centre of gravity show that the time would be proportional to ρ.

Motion of the Earth about its Centre of Gravity.

610. *To find the motion of the pole of the earth about its centre of gravity when disturbed by the attraction of the sun and moon, the figure of the earth being taken to be one of revolution.*

Let us consider the effect of these two bodies separately. Then, provided we neglect terms depending on the square of the disturbing force, we can by addition determine their joint effect.

The sun attracts the parts of the earth nearer to it with a force slightly greater than that with which it attracts the parts more remote, and thus produces a small couple which tends to turn the earth about an axis lying in the plane of the equator and perpendicular to the line joining the centre of the earth to the centre of the sun. It is the effect of this couple which we have now to determine. It clearly produces small angular velocities about axes perpendicular to the axis of figure. We shall also suppose that the initial axis of rotation so nearly coincides with the axis of figure, that we may regard the angular velocities about axes lying in the plane of the equator to be small compared with the angular velocity about the axis of figure.

Let us take as axes of reference in the earth, GC the axis of figure, GA and GB moving in the earth with an angular velocity θ_3 round GC. Then following the notation of Art. 252, we have

$$h_1' = A\omega_1, \quad h_2' = A\omega_2, \quad h_3' = C\omega_3,$$

$$\theta_1 = \omega_1, \quad \theta_2 = \omega_2.$$

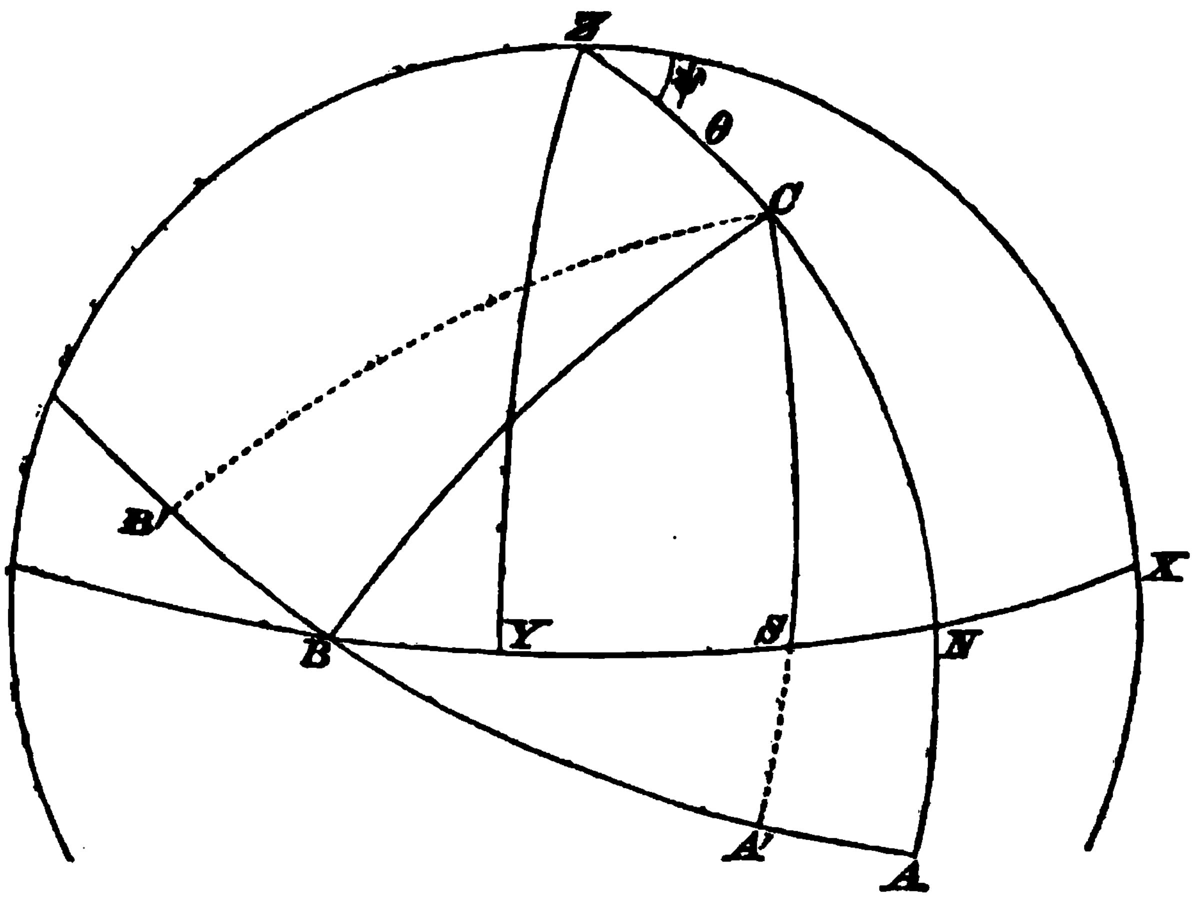
The equations of motion are therefore

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - A\omega_2\theta_3 + C\omega_3\omega_2 &= L \\ A\frac{d\omega_2}{dt} - C\omega_3\omega_1 + A\omega_1\theta_3 &= M \\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad &= 0 \end{aligned}\right\} \text{..............} (1).$$

The last of these equations shows that ω_3 is constant. Let this constant be denoted by n.

The other two angular velocities are to be found by solving the other two equations. This solution must be conducted by the method of continued approximation, ω_1 and ω_2 being regarded as small compared with n.

In the first instance let us suppose the orbit of the disturbing body to be fixed in space. This is very nearly true in the case of the sun, less nearly so for the moon. This limitation of the problem proposed will be found greatly to simplify the solution. We can now choose as our axes of reference in space two straight lines GX, GY at right angles to each other in the plane of the orbit and a third axis GZ normal to the plane.



611. In these equations of motion the quantity θ_3 is at our choice, let it be so chosen* that the plane containing the

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* We might also very conveniently have chosen as axes of reference, GC the axis of figure and axes GA', GB' moving on the earth so that GB' is the axis of the resultant couple produced by the action of the disturbing body on the earth. In this case the plane CA' moves so as always to contain the disturbing body S, so that θ_3 is the angular velocity of CS round C and is therefore a small quantity of the order n'. We shall therefore reject the small terms $\omega_2\theta_3$ and $\omega_1\theta_3$ in equations (1). The equations now become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + Cn\omega_2 &= 0 \\ A\frac{d\omega_2}{dt} - Cn\omega_1 &= M = -3\kappa(C-A)\cos\alpha\cos\gamma \end{aligned}\right\},$$

where the value of M is at once obtained from Art. 608, and in our case $\alpha = \frac{\pi}{2} - \gamma$. Eliminating ω_2 we have

$$\frac{d^2\omega_1}{dt^2} + \left(\frac{Cn}{A}\right)^2 \omega_1 = -\frac{Cn}{A^2} M.$$

Since the angular distance γ of the disturbing body from the pole of the earth

axes GC, GA also contains GZ. Then θ_3 is the angular velocity of the plane ZGC round GC. If ω_1 and ω_2 were zero, and the earth merely turned round its axis GC, it is clear that GC and therefore also the plane ZGC would be fixed in space. Hence θ_3 is a small quantity of the same order at least as ω_1 or ω_2. For a first approximation we neglect the squares of the small quantities to be found. We therefore reject the small terms $\omega_2\theta_3$, $\omega_1\theta_3$ in the equations (1). The equations now become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + Cn\omega_2 = L \\ A\frac{d\omega_2}{dt} - Cn\omega_1 = M \end{aligned}\right\} \text{..................... (2).}$$

Following the usual notation let θ be the angle ZC and ψ the angle the plane ZC makes with the fixed plane ZX. We have then the two geometrical equations

$$\omega_1 = -\sin\theta\frac{d\psi}{dt}, \quad \omega_2 = \frac{d\theta}{dt} \text{..................(3).}$$

These follow at once from a mere inspection of the figure, or we may deduce them from Art. 235, by putting $\phi = 0$.

We have now to find the magnitudes of L and M. Let S be the disturbing body and let it move in the direction X to Y. According to the usual rule in Astronomy, we shall suppose the longitude l of S to be measured in the direction of motion

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varies very slowly, the term on the right-hand side is very nearly constant. If this be regarded as a sufficient approximation we have

$$\omega_1 = -\frac{3\kappa}{2n}\frac{C-A}{C}\sin 2\gamma, \text{ and } \omega_2 = 0.$$

But in fact these are nearly true when we take account of the periodical term provided only S moves slowly. For suppose

$$M = M_0 + \Sigma P \sin(pt + Q),$$

where p is small; we have in that case

$$\omega_1 = -\frac{M_0}{Cn} - \Sigma\frac{CnP}{C^2n^2 - A^2p^2}\sin(pt+Q),$$

neglecting the small term p^2 in the denominator we have as before $\omega_1 = -\dfrac{M}{Cn}$.

The motion of the axis C in space is therefore simply that due to an angular velocity ω_1 about the axis A'. Since the plane $A'C$ moves so as always to contain the disturbing body S, the axis of figure GC is at any instant moving perpendicular to the plane containing it and the disturbing body (*i.e.* in the figure C is always moving perpendicular to SC) with an angular velocity equal to $\dfrac{3\kappa}{2n}\dfrac{C-A}{C}\sin 2\gamma$. If we resolve this in the direction along and perpendicular to ZC we easily deduce the equations (7) in the text and the solution may be continued as above.

from the point on the sphere *opposite* to B. This point is usually called the first point of Aries. Then

$$BS = \pi - l \text{ and } SN = l - \frac{\pi}{2}.$$

By Art. 608 we have

$$L = -3\kappa (B - C) \cos\beta \cos\gamma = -3\kappa (A - C) \sin SN \cos SN \sin\theta$$

$$= \frac{3}{2}\kappa (A - C) \sin\theta \sin 2l \quad\text{.........(4)},$$

$$M = -3\kappa (C - A) \cos\alpha \cos\gamma = -3\kappa (C - A) \cos^2 SN \sin\theta \cos\theta$$

$$= -\frac{3}{2}\kappa (C - A) \sin\theta \cos\theta (1 - \cos 2l) \quad\text{..........(5)}.$$

Since the motion of the disturbing body is very slow compared with the angular velocity of the earth about its axis, l and therefore L and M are very nearly constant. If this be regarded as a sufficiently near approximation we have at once by (2)

$$\omega_1 = -\frac{M}{Cn}, \quad \omega_2 = \frac{L}{Cn} \quad\text{....................(6)}.$$

That these are the integrals of equations (2) when we take some account of the variability of L and M may be shown by substitution in those equations. We see that they are satisfied if we may neglect such a term as

$$\frac{dL}{dt} = -\frac{3}{2}\kappa (B - C) \left\{\cos\theta \sin 2l \frac{d\theta}{dt} + 2 \sin\theta \cos 2l \frac{dl}{dt}\right\}.$$

Since $\kappa (B - C)$ and $\frac{d\theta}{dt}$ are both small quantities of the order ω_1 or ω_2, the first of these terms is of the order ω_2^2 and such terms we have already agreed to neglect. The last term is of the order $\frac{n'}{n}\omega_2$, where n' is the mean angular velocity of the disturbing body about the earth. Rejecting these terms also, we have by (3), (4) and (5),

$$\left.\begin{aligned} \frac{d\theta}{dt} &= -\frac{3\kappa}{2n}\frac{C - A}{C} \sin\theta \sin 2l \\ \frac{d\psi}{dt} &= -\frac{3\kappa}{2n}\frac{C - A}{C} \cos\theta (1 - \cos 2l) \end{aligned}\right\} \quad\text{..........(7)}.$$

612. To find the motion of the pole of the earth in space referred to the pole of the orbit of the disturbing body as origin, we have merely to integrate the equations (7). For a

first approximation, in which we reject the squares of the small quantities to be found, we may regard θ on the right-hand side as constant and equal to its mean value. If we write for l its approximate value

$$l = n't + \epsilon',$$

we find by integration

$$\left.\begin{aligned} \theta &= \text{const.} + \frac{3\kappa}{4nn'}\frac{C-A}{C}\sin\theta\cos 2l \\ \psi &= \text{const.} - \frac{3\kappa}{2nn'}\frac{C-A}{C}\cos\theta\left(l - \frac{1}{2}\sin 2l\right) \end{aligned}\right\} \ldots\ldots\ldots(8).$$

613. We may also solve equations (2) in the following manner. Since we reject the squares of the small quantities to be found, we may in calculating the values of L and M to a first approximation suppose θ to be constant and l to be measured from a fixed point in space. We then have by the theory of elliptic motion

$$l = n't + \epsilon' + P_1 \sin(p_1 t + q_1) + P_2 \sin(p_2 t + q_2) + \&c.,$$

where the coefficients of the trigonometrical terms are all known small quantities, and all the coefficients of t are very small compared with n. In the case of the sun the coefficient of t in the greatest of the trigonometrical terms is $\frac{1}{365}n$ and in the case of the moon $\frac{1}{27}n$.

We may also include in this formula the secular inequalities in the value of l. For, we shall presently find that θ has no secular inequalities, and that the first point of Aries from which l is measured has a very slow motion which is very nearly uniform on the plane of the orbit of the disturbing body. This slow motion may obviously be included in the n'.

If we eliminate ω_2 between equations (2) we have

$$\frac{d^2\omega_1}{dt^2} + \frac{C^2n^2}{A^2}\omega_1 = \frac{1}{A}\frac{dL}{dt} - \frac{Cn}{A^2}M.$$

The first term on the right-hand side we have already agreed to neglect. Substituting in the expression for M given in (5) the value of l, suppose we have

$$M = \Sigma F \cos(\lambda t + f),$$

where the constant part of M is given by $\lambda = 0$ and all the other values of λ are very small. Then solving, we find

$$\omega_1 = -\Sigma \frac{FCn}{C^2n^2 - A^2\lambda^2}\cos(\lambda t + f).$$

Since F and λ^2 are both very small we may reject the small term λ^2 in the denominator, we then have

$$\omega_1 = -\frac{1}{Cn}\Sigma F \cos(\lambda t + f) = -\frac{M}{Cn}.$$

This result is strictly true for the constant term and very nearly true for the periodical terms. In the same way we may prove that $\omega_2 = \dfrac{L}{Cn}$.

When we proceed to find θ and ψ from the values of ω_1 and ω_2 by the help of equations (3), it will be seen that no term will rise on integration in which λ is not small. These rejected terms will not therefore afterwards become important.

614. The integration of equation (7) may be effected without neglecting the terms containing the powers of e' in the expression for l. By the theory of elliptic motion we have

$$R^2\frac{dl}{dt}=\text{constant}=R_0^2n'\sqrt{1-e'^2},$$

where a very small term has been rejected on the left-hand side depending on the motion of Aries. Substituting for κ its value given in Art. 608 we find

$$\left.\begin{aligned}\frac{d\theta}{dl}&=-\frac{3n'}{2n}\frac{1}{1+\nu}\frac{C-A}{C}\frac{R_0}{R\sqrt{1-e'^2}}\sin\theta\sin 2l\\ \frac{d\psi}{dl}&=-\frac{3n'}{2n}\frac{1}{1+\nu}\frac{C-A}{C}\frac{R_0}{R\sqrt{1-e'^2}}\cos\theta(1-\cos 2l)\end{aligned}\right\},$$

where ν is to be put equal to zero when the disturbing body is the sun. From the equation to the ellipse, we have

$$\frac{R_0(1-e'^2)}{R}=1+e'\cos(l-L).$$

If this value of R be substituted in the equations, the integrations can be effected without difficulty. But it is clear that all the terms which contain e' are periodic and do not rise on integration so as to become equally important with the others. Since then e' is small, being equal in the case of the sun to about $\frac{1}{60}$, it will be needless to calculate these terms.

615. Let us now examine the geometrical meaning of the equations (8). For the sake of brevity, let us put $S=\frac{3\kappa}{2nn'}\frac{C-A}{C}$, so that by Art. 608 $S=\frac{3}{2}\frac{C-A}{C}\frac{n'}{n}$ or $S=\frac{3}{2}\frac{C-A}{C}\frac{n''}{n}\frac{1}{1+\nu}$ according as the sun or moon is the disturbing body, the orbit of the disturbing body being in both cases regarded as circular.

Let us consider first the term $-S\cos\theta\, l$ in the value of ψ. Let a point C_0 describe a small circle round Z the pole of the orbit of the disturbing planet, the distance CZ being constant and equal to the mean value of θ. Let the velocity be uniform and equal to $Sn'\cos\theta\sin\theta$, and let the direction of motion be *opposite* to that of the disturbing body. Then C_0 represents the motion of the pole of the earth so far as this term is concerned. This uniform motion is called Precession.

Next let us consider the two terms

$$\delta\theta=\frac{1}{2}S\sin\theta\cos 2l,\ \delta\psi=\frac{1}{2}S\cos\theta\sin 2l.$$

If we put $x = \sin\theta\,\delta\psi$, $y = \delta\theta$, we have

$$\frac{x^2}{\left(\frac{1}{2} S\cos\theta\sin\theta\right)^2} + \frac{y^2}{\left(\frac{1}{2} S\sin\theta\right)^2} = 1,$$

which is the equation to an ellipse.

Let us then describe round C_0 as centre an ellipse whose semi-axes are $\frac{1}{2} S\cos\theta\sin\theta$ and $\frac{1}{2} S\sin\theta$ respectively perpendicular to and along ZC; and let a point C_1 describe this ellipse in a period equal to half the periodic time of the disturbing body. Also let the velocity of C_1 be the same as if it were a material point attracted by a centre of force in the centre varying as the distance. Then C_1 represents the motion of the pole of the earth as affected both by Precession and the principal parts of Nutation.

If we had chosen to include in our approximate values of θ and ψ any small term of higher order, we might have represented its effect by the motion of a point C_2 describing another small ellipse having C_1 for centre. And in a similar manner by drawing successive ellipses we could represent geometrically all the terms of θ and ψ.

616. In this solution we have not yet considered the Complementary Functions. To find these we must solve

$$A\frac{d\omega_1}{dt} + Cn\omega_2 = 0, \quad A\frac{d\omega_2}{dt} - Cn\omega_1 = 0.$$

We easily find $\omega_1 = H\sin\left(\frac{Cn}{A}t + K\right)$, $\omega_2 = -H\cos\left(\frac{Cn}{A}t + K\right)$.

The quantities H and K depend on the initial values of $\omega_1\,\omega_2$. As these initial values are unknown H and K must be determined by observation. If H had any sensible value it would be discovered by the variations produced by it in the position in space of the pole of the earth. The period of these would be $\frac{2\pi}{n}\frac{A}{C}$, as A and C are nearly equal in the case of the earth, this period is nearly equal to a day. No such inequalities have been found. If however any such inequality existed we might consider these two terms together as a separate inequality to be afterwards added to that produced by the other terms of $\omega_1\,\omega_2$ whose period is half a year.

The effect of the complementary function on the motion of the pole of the earth has been already considered. The motion is the same as if the earth were at any instant set in

rotation about an axis whose direction-cosines are proportional to $H\sin\left(\frac{Cn}{A}t+K\right)$, $-H\cos\left(\frac{Cn}{A}t+K\right)$ and n and then left to itself. The instantaneous axis will describe a right cone of small angle round the axis of figure and also a right cone of small angle in space. Hence from this cause there can be no permanent change in the position in space of the axis of the earth. See Art. 522.

617. The preceding investigations are of course approximations. In the first instance we neglected in the differential equations the squares of the ratios of ω_1 and ω_2 to n, and afterwards some periodical terms which are an $\frac{n'}{n}$ th of those retained. We see by equations (3) and (8) that the second set of terms rejected is much greater than the first, and yet when the sun is the disturbing body these terms are only about $\frac{1}{365}$ th part of those retained, and when the moon is the disturbing body these are only $\frac{1}{27}$ th part of terms which themselves are imperceptible.

We have also regarded the earth as a solid of revolution so that $A-B$ may be taken zero, a supposition which cannot be strictly correct.

618. In the case of the sun we have $S=\frac{3}{2}\frac{C-A}{C}\frac{n'}{n}$, so that the precession in one year is $\frac{3}{2}\frac{C-A}{C}\frac{n'}{n}\cos\theta\, 2\pi$. It is shown in treatises on the Figure of the Earth that there is reason to put $\frac{C-A}{C}=\cdot 0031$. Also we have $\frac{n'}{n}=\frac{1}{365}$, and $\theta=23^\circ.8'$. This gives a precession of about $15''\cdot 42$ per annum. Similarly the coefficients of Solar Nutation in ψ and θ are respectively found to be $1''\cdot 23$ and $0''\cdot 53$. If we supposed the moon's orbit to be fixed, we could find in a similar manner the motion of the pole produced by the moon referred to the pole of the moon's orbit. In this case $S=\frac{3}{2}\frac{C-A}{C}\frac{n''}{n}\frac{1}{1+\nu}$. The value of θ varies between the limits $23^\circ \pm 5^\circ$. Putting $\frac{n'}{n}=\frac{1}{27}$, $\nu=80$, $\theta=23^\circ$, we find a precession in one year a little more than double that produced by the sun. But the coefficients of what would be the nutations are about one-sixth of those produced by the sun.

619. We have hitherto considered the orbit of the disturbing body to be fixed in space. If it be not fixed, we must take the

plane CA perpendicular to its instantaneous position at the moment under consideration. The quantity θ_3 will not be the same as before*, but if the motion of the orbit in space be very slow, θ_3 will still be very small. We may therefore neglect the small terms $\theta_3\omega_1$ and $\theta_3\omega_2$ as before. The dynamical equations will not therefore be materially altered. With regard to the geometrical equations (3) it is clear that ω_2, ω_1 will continue to express the resolved parts of the velocity of C in space along and perpendicular to the instantaneous position of ZC. To this degree of approximation therefore, all the change that will be necessary is to refer the velocities as given by equations (7) to axes fixed in space and then by integration we shall find the motion of C. This is the course we shall pursue in the case of the moon.

The attractions of the planets on the earth and sun slightly alter the plane of the earth's motion round the sun, so that the position of the ecliptic in space varies slowly. It can oscillate nearly five degrees on each side of its mean position. If the earth were spherical there would be no precession caused by the attractions of the sun and moon. The direction of the plane of the equator would then be fixed in space, and the changes of its obliquity to the ecliptic would be wholly caused by the motion of the latter, and would be very considerable. But, as Laplace remarks, the attractions of the sun and moon on the terrestrial spheroid cause the plane of the equator to vary along with the ecliptic so that the possible change of the obliquity is reduced to about one and a third degrees which is about one-quarter of what it would have been without those actions.

At present the obliquity is decreasing at the rate of about 48″ per century. After an immense number of years, it will begin to increase and will oscillate about its mean value. These inequalities we do not propose to discuss in this treatise. We must refer the reader to the second volume of the *Mécanique Céleste*, livre cinquième. He may also consult the *Connaissance des Temps* for 1827, page 234.

620. Ex. 1. If the earth were a homogeneous shell bounded by similar ellipsoids, the interior being empty, the precession would be the same as if the earth were solid throughout.

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* The value of θ_3 may be found in the following manner. The orbit at any instant is turning about the radius vector of the planet as an instantaneous axis. Let u be this angular velocity which we shall suppose known. Let Z, Z'; B, B' be two successive positions of the pole of the orbit and the extremity of the axis of B respectively. Then ZB = a right angle = $Z'B'$. Hence the projections of ZZ', BB', on ZB are equal. This gives, since ZB is at right angles to both CZ and SB, $B\hat{S}B' \sin BS = Z\hat{C}Z' \sin ZC$. Now the angle $ZCZ' = -\delta\theta_3$ and the angle $BSB' = u$, hence $\delta\theta_3 \,.\, \sin\theta = -u \sin l$. The value of $\delta\theta_3$ must be added to the former value of θ_3.

Ex. 2. If the earth were a homogeneous shell bounded externally by a spheroid and internally by a concentric sphere, the interior being filled with a perfect fluid of the same density as the earth, show that the precession would be greater than if the earth were solid throughout.

Let (a, a, c) be the semi-axes of the spheroid, r the radius of the sphere. Then since the precession varies as $\dfrac{C-A}{C}$ by Art. 615, the precession is increased in the ratio $a^4c : a^4c - r^5$.

Ex. 3. If the sun were removed to twice its present distance show that the solar precession per unit of time would be reduced to one-eighth of its present value; and the precession per year to about one-third of its present value.

Ex. 4. A body turning about a fixed point is acted on by forces which tend to *produce* rotation about an axis at right angles to the instantaneous axis, show that the angular velocity cannot be uniform unless the momental ellipsoid at the fixed point is a spheroid.

The axis about which the forces tend to produce rotation is that axis about which it would begin to turn if the body were placed at rest.

Ex. 5. A body free to turn about its centre of gravity is in stable equilibrium under the attraction of a distant fixed particle. Show that the axis of least moment is turned toward the particle. Show also that the times of the principal oscillations are respectively $2\pi\left\{\dfrac{B\rho^3}{3M'(C-A)}\right\}^{\frac{1}{2}}$ and $2\pi\left\{\dfrac{C\rho^3}{3M'(B-A)}\right\}^{\frac{1}{2}}$.

If the body be the earth and M' be the sun, show that the smaller of these two periods is about ten years.

621. *To give a general explanation of the manner in which the attraction of the Sun causes Precession and Nutation.*

If a body be set in rotation about a fixed point O under the action of no forces, we know that the momenta of all the particles are together equivalent to a couple which we shall represent by G about an axis called the invariable line. Let T be the Vis Viva of the body. If a plane be drawn perpendicular to the axis of G at a distance $\dfrac{\sqrt{MT}}{G}\epsilon^2$ from the fixed point, then the whole motion is represented by making the momental ellipsoid whose parameter is ϵ roll on this plane. In the case of the earth, the axis OI of instantaneous rotation so nearly coincides with OC the axis of figure that the fixed plane on which the ellipsoid rolls is very nearly a tangent plane at the extremity of the axis of figure. This is so very nearly the case that we shall neglect the *squares* of all small terms depending on the resolved part of the angular velocity about any axis of the earth perpendicular to the axis of figure.

Let us now consider how this motion is disturbed by the action of the sun. The sun attracts the parts of the earth nearer to it with a slightly greater force than it attracts those more remote.

Hence when the sun is either north or south of the equator its attraction will produce a couple tending to turn the earth about that axis in the plane of the equator which is perpendicular to the line joining the centre of the earth to the centre of the sun. Let the magnitude of this couple be represented by α, and let us suppose that it acts impulsively at intervals of time dt.

At any one instant this couple will generate a new momentum αdt about the axis of the couple α. This has to be compounded with the existing momentum G, to form a resultant couple G'. If the axis of α were exactly perpendicular to that of G we should have $G' = \sqrt{G^2 + (\alpha dt)^2} = G$ ultimately.

Let θ be the angle that the axis of G makes with OC, then θ is a quantity of that order of small quantities whose square is to be neglected. Taking the case when OG, OC and the axis of α are in one plane, for this is the case in which G' will most differ from G, we have

$$G'^2 = (G\cos\theta)^2 + (G\sin\theta + \alpha dt)^2$$
$$= G^2 + 2G\alpha \sin\theta dt \quad\text{..........................(1)}.$$

Then α and θ being of the same order of small quantities, the term $\alpha \sin\theta$ is to be neglected. Hence we have $G' = G$. But the axis of G is altered in space by an angle $\frac{\alpha dt}{G}$ in a plane passing through OG and the axis of α.

Next let us consider how the Vis Viva T is altered. If T' be the new Vis Viva we have

$$T' - T = \text{twice the work done by the couple } \alpha$$
$$= 2\alpha\,(\omega\cos\beta)\,dt \quad\text{..........................(2)},$$

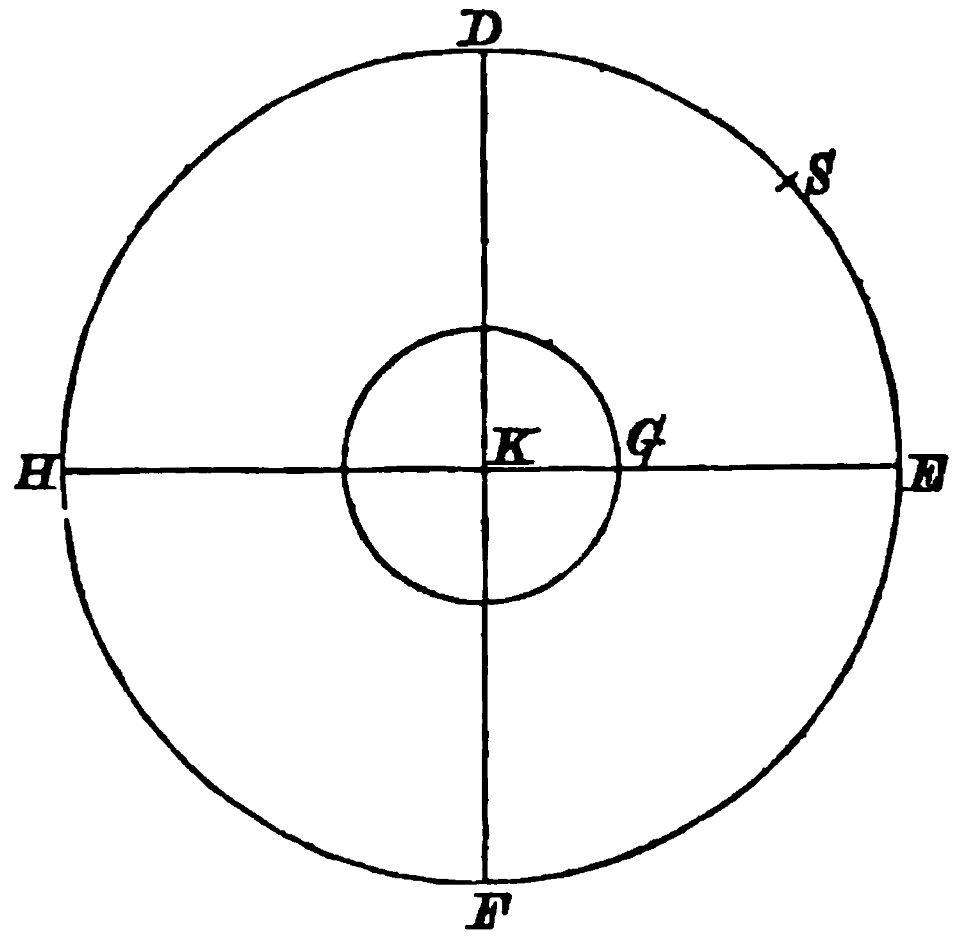
where $\omega\cos\beta$ is the resolved part of the angular velocity about the axis of α. For the same reason as before the product of this angular velocity and α is to be neglected. Hence we have $T' = T$. It follows from these results that the distance $\frac{\sqrt{MT}}{G}\epsilon^2$ of the fixed plane from the fixed point is unaltered by the action of α.

Thus the fixed plane on which the ellipsoid rolls keeps at the same distance from the fixed point, so that the three lines OC, OI, OG being initially very near each other will always remain very close to each other. But the normal OG to this plane has a motion in space, hence the others must accompany it. This motion is what we call Precession and Nutation.

Lastly these small terms which have been neglected will not continually accumulate so as to produce any sensible effect. As the earth turns round in one day, the axis OC will describe

a cone of small angle θ round OG. The axis about which the sun generates the angular velocity α is always at right angles to the plane containing the sun and OC. Hence, regarding the sun as fixed for a day, the angle θ in equation (1) changes its sign every half day. Thus G' is alternately greater and less than G. Similarly since the instantaneous axis describes a cone about OG it may be shown that T' is alternately greater and less than T.

622. Let us trace the motion of the axis OG through a whole year. Describe a sphere whose centre is at O and let us refer the



motion to the surface of this sphere. Let K be the pole of the ecliptic and let the sun S describe the circle $DEFH$ of which K is the pole. Let DF be a great circle perpendicular to KG, then since OG and the axis of figure of the earth are so close that we may treat them as coincident, D and F will be the intersections of the equator and ecliptic. When the sun is north or south of the equator, its attraction generates the couple α, which will be positive or negative according as the sun is on one side or the other. This couple vanishes when the sun is passing through the equator at D or F. If the sun be anywhere in DEF, i.e. north of the equator, G is moved in a direction perpendicular to the arc GS towards D. If the sun be anywhere in FHD, α has the opposite sign and hence G is again moved perpendicular to the instantaneous position of GS but still towards D. Considering the whole effect produced in one year while the sun describes the circle $DEFH$, we see that G will be moved a very small space towards D, i.e. in the direction opposite to the sun's motion. Resolving this along the tangent to the circle centre K and radius KG, we see that the motion of G is made up of (1) a uniform

motion of G along this circle backwards, which is called Precession and (2) an inequality in this uniform motion which is one part of Solar Nutation. Again as the sun moves from D to E, G is moved inwards so that the distance KG is diminished, but as the sun moves from E to F, KG is as much increased. So that on the whole the distance KG is unaltered, but it has an inequality which is the other part of Solar Nutation.

It is evident that each of these inequalities goes through its period in half a year.

623. *To explain the cause of Lunar Nutation.*

The attraction of the sun on the protuberant parts at the earth's equator causes the pole C of the earth to describe a small circle with uniform velocity round K the pole of the ecliptic with two inequalities, one in latitude and one in longitude, whose period is half a year. These two inequalities are called Solar Nutations. In the same way the attraction of the moon causes the pole of the earth to describe a small circle round M the pole of the lunar orbit with two inequalities. These inequalities are very small and of short period, viz. a fortnight, and are therefore generally neglected. All that is taken account of is the uniform motion of C round M. Now K is the origin of reference, hence if M were fixed the motion of C round M would be represented by a slow uniform motion of C round K together with two inequalities whose magnitude would be equal to the arc MK, or 5 degrees, and whose period would be very long, viz. equal to that of C round K produced by the uniform motion. But we know by Lunar Theory that M describes a circle round K as centre with a velocity much more rapid than that of C. Hence the motion of C will be represented by a slow uniform motion round K, together with two inequalities which will be the smaller the greater the velocity of M round K, and whose period will be nearly equal to that of M round K. This period we know to be about 19 years. These two inequalities are called the Lunar Nutations. It will be perceived that their origin is different from that of Solar Nutation.

624. *To calculate the Lunar Precession and Nutation.*

Let K be the pole of the ecliptic, M that of the lunar orbit, C the pole of the earth. Let KX be any fixed arc, $KC=\theta$, $XKC=\psi$, then we have to find θ and ψ in terms of t. By Art. 615 the velocity of C in space is at any instant in a direction perpendicular to MC, and equal to

$$-\frac{3n''^2}{2n}\frac{C-A}{C}\frac{1}{1+\nu}\cos MC\sin MC.$$

For the sake of brevity let the coefficient of $\cos MC \sin MC$ be represented by P. Then resolving this velocity along and perpendicular to KC, we have

$$\left.\begin{aligned} \frac{d\theta}{dt} &= -P \sin MC \cos MC \sin KCM \\ \sin\theta \frac{d\psi}{dt} &= -P \sin MC \cos MC \cos KCM \end{aligned}\right\}.$$

By Lunar theory we know that M regredes round K uniformly, the distance KM remaining unaltered. Let then $KM = i$, and the angle $XKM = -mt + \alpha$.

Now by spherical trigonometry,

$$\cos MC = \cos i \cos\theta + \sin i \sin\theta \cos MKC,$$

$$\sin MC \cos KCM = \frac{\cos i - \cos MC \cos\theta}{\sin\theta}$$

$$= \cos i \sin\theta - \sin i \cos\theta \cos MKC,$$

$$\sin MC \,.\, \sin KCM = \sin i \sin MKC.$$

Substituting these we have

$$\frac{d\theta}{dt} = -P\left\{\sin i \cos i \cos\theta \sin MKC + \frac{1}{2}\sin^2 i \sin\theta \sin 2MKC\right\},$$

$$\sin\theta\frac{d\psi}{dt} = -P\left\{\sin\theta\cos\theta\left(\cos^2 i - \frac{1}{2}\sin^2 i\right)\right.$$

$$\left. - \sin i \cos i \cos 2\theta \cos MKC - \frac{1}{2}\sin^2 i \sin\theta\cos\theta\cos 2MKC\right\}.$$

For a first approximation we may neglect the variations of θ and ψ when multiplied by the small quantity P. Hence $\frac{d\theta}{dt}$ contains only periodic terms, and the inclination θ has no permanent alteration. But $\frac{d\psi}{dt}$ contains a term independent of MKC; considering only this term, we have

$$\psi = \text{constant} - P\cos\theta\left(\cos^2 i - \frac{1}{2}\sin^2 i\right)t.$$

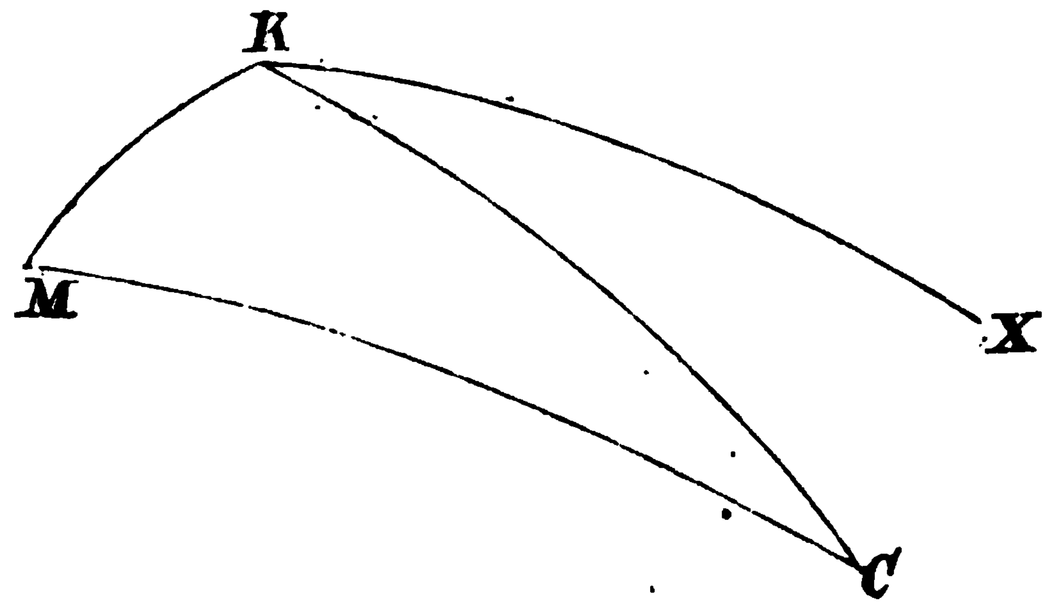
This equation expresses the precessional motion of the pole due to the attraction of the moon. We may write this equation in the form $\psi = \psi_0 - pt$.

To find the nutations, we must substitute for MKC its approximate value

$$MKC = (-m + p)\,t + \alpha - \psi_0.$$

We then have after integration

$$\theta = \text{const.} - \frac{P \sin i \cos i \cos \theta}{m - p} \cos MKC - \frac{P \sin^2 i \sin \theta}{4(m-p)} \cos 2MKC.$$



The second of these two periodic terms being about one-fiftieth part of the first, which is itself very small, is usually neglected. Also p is very small compared with m, hence we have

$$\theta = \theta_0 - \frac{P \sin i \cos i \cos \theta}{m} \cos MKC.$$

This term expresses the Lunar Nutation in the obliquity.

In the same way by integrating the expression for ψ, and neglecting the very small terms, we have

$$\psi = \psi_0 - P \cos \theta \left(\cos^2 i - \frac{1}{2} \sin^2 i\right) t - P \frac{\sin 2i}{2m} \cdot \frac{\cos 2\theta}{\sin \theta} \sin MKC.$$

The angle MKC is the longitude of the moon's descending node, and the line of nodes is known to complete a revolution in about 18 years and 7 months. If we represent this period by T we have $MKC = -\frac{2\pi}{T} t + \text{constant}.$

The pole M of the lunar orbit moves round the point of reference K with an angular velocity which is rapid compared with p, but yet is sufficiently small to make the Lunar Nutations greater than the Solar. We may also notice that if M had moved round K with an angular velocity more nearly equal to p the Nutations would have been still larger. This may explain why a slow motion of the ecliptic in space may produce some corresponding nutations of very long period and of considerable magnitude.

Motion of the Moon about its centre of gravity.

625. In discussing the precession and nutation of the equinoxes, the earth has been regarded as a rigid body two of whose principal moments at the centre of gravity are equal to each other. One consequence of this supposition was that the rotation about the axis of unequal moment is not directly altered by the attraction of the disturbing bodies. As an example of the effect of these forces on the rotation when all the three principal moments are unequal, we shall now consider the case of the moon as disturbed by the attraction of the earth. As our object is to examine the mode in which the forces alter the several motions of the moon about its centre of gravity rather than to obtain arithmetical results of the greatest possible accuracy, we shall separate the problem into two. In the first place we shall suppose the moon to describe an orbit which is very nearly circular in a plane which is one of the principal planes at its centre of gravity. In the second case we shall remove the latter restriction and examine the effects of the obliquity of the moon's orbit to the moon's equator.

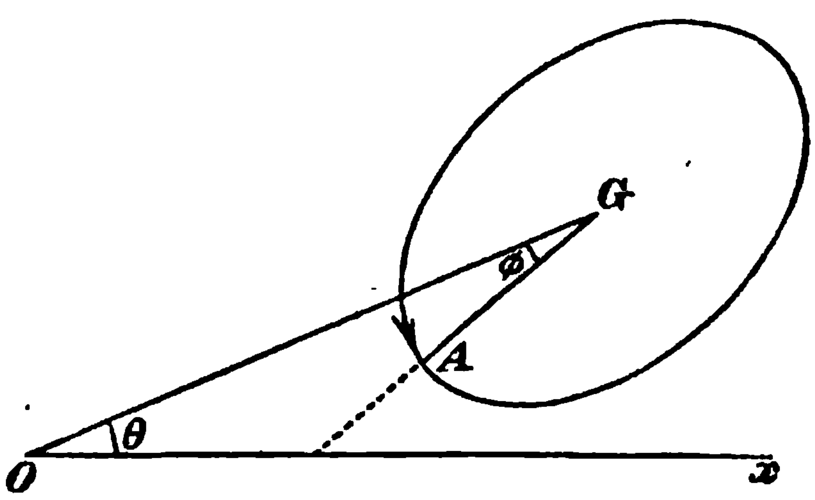
626. *The moon describes an orbit about the centre of the earth which is very nearly circular. Supposing the plane of the orbit to be one of the principal planes of the moon at its centre of gravity, find the motion of the moon about its centre of gravity.*

Let GA, GB, GC be the principal axes at G the centre of gravity of the moon, and let GC be the axis perpendicular to the plane in which G moves. Let A, B, C be the moments of inertia about GA, GB, GC respectively, and let M be the mass of the moon, and let accented letters denote corresponding quantities for the earth.

Let O be the centre of the earth, and let Ox be the initial line. Let $OG=r$, $GOx=\theta$. Let us suppose the moon turns round its axis GC in the same direction that the centre of gravity describes its orbit about O, and let the angle $OGA=\phi$.

The mutual potential of the earth and moon is by Art. 607

$$V=\frac{MM'}{r}+M\frac{A'+B'+C'-3I'}{2r^3}+M'\frac{A+B+C-3I}{2r^3}.$$



Here $I=A\cos^2\phi+B\sin^2\phi$ and therefore the moment of the forces tending to turn the moon round GC is

$$\frac{dV}{d\phi}=-\frac{3}{2}\frac{M'}{r^3}(B-A)\sin 2\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since $\theta+\phi$ is the angle which GA, a line fixed in the body, makes with Ox, a line fixed in space, the equation of the motion of the moon round GC is

$$\frac{d^2\theta}{dt^2}+\frac{d^2\phi}{dt^2}=-\frac{3}{2}\frac{M'}{r^3}\frac{B-A}{C}\sin 2\phi \quad\text{.........................(2).}$$

The motion of the centre of gravity of the moon referred to the centre of the earth as a fixed point is found in the Lunar Theory. It is there shown that r and θ may be expressed in the form

$$r=c\{1+L\cos(pt+\alpha)+\&c.\},$$

$$\frac{d\theta}{dt}=n+\beta t+Mn\cos(pt+\alpha)+\&c.,$$

where βt is a very small term which represents a secular change in the moon's angular velocity about the earth, and is really the first term of the expansion of a trigonometrical expression.

If we substitute the value of $\dfrac{d\theta}{dt}$ in equation (2) we have the following equation to determine ϕ,

$$\frac{d^2\phi}{dt^2}=-\frac{1}{2}q^2\sin 2\phi-\beta+npM\sin(pt+\alpha)+\&c. \quad\text{....................(3),}$$

where for the sake of brevity we have put $n^2\dfrac{3}{2}\dfrac{B-A}{C}=\dfrac{q^2}{2}$.

Now we know by observation that the moon always turns the same face towards the earth, so that amongst the various motions which may result from different initial conditions, the one which we wish to examine is characterized by ϕ being nearly constant. Let us then introduce into this equation the assumption that ϕ is nearly constant; we may then deduce from the integral how far this assumption is compatible with any given initial conditions which we may suppose to have been imposed on the moon. Putting $\phi=\phi_0+\phi'$, where ϕ_0 is supposed to contain all the constant part of ϕ, we easily find

$$\left.\begin{aligned}&\frac{1}{2}q^2\sin 2\phi_0=-\beta\\ &\frac{d^2\phi'}{dt^2}+q^2\cos 2\phi_0\phi'=npM\sin(pt+\alpha)+\&c.\end{aligned}\right\}\quad\text{......................(4).}$$

The numerical value of q depends on the structure of the moon and can therefore only be found by comparing the results of this investigation or some other results with observation. The first of equations (4) shows that 2β must be less than q^2. But for various reasons, though q is very small, we must yet suppose that $\dfrac{2\beta}{q^2}$ is also extremely small. Assuming this, we see that ϕ_0 must also be very small. It follows also that we may write $2\phi_0$ for $\sin 2\phi_0$ and unity for $\cos 2\phi_0$ in these equations. Solving the second equation, we find,

$$\phi=H\sin(qt+K)-\frac{\beta}{q^2}+M\frac{np}{q^2-p^2}\sin(pt+\alpha)+\&c. \quad\text{..............(5),}$$

where H and K are two arbitrary constants whose values depend on the initial conditions. The angular velocity of the moon about its axis is therefore given by the formula

$$\frac{d\theta}{dt}+\frac{d\phi}{dt}=n+\beta t+Hq\sin(qt+K)+M\frac{nq^2}{q^2-p^2}\sin(pt+\alpha)+\&c. \quad\text{........(6).}$$

If q^2 were negative or zero, the character of the solution of (3) would be altered. In the former case the expression for ϕ would contain real exponentials. If the initial conditions were so nicely adjusted that the coefficient of the term containing the positive exponent were zero, the value of ϕ would still be always small. But this motion would be unstable, the smallest disturbance would alter the values of the arbitrary constants and then ϕ would become large. If we also examine the solution when $q^2=0$, we easily see that ϕ could not remain small. We therefore infer that of the axes GA, GB of the moon, the axis of least moment is turned towards the earth and that these two principal moments are not equal.

In order that the expression (5) for ϕ may represent the actual motion it is necessary and sufficient that H when found from the initial conditions should be small. We see, by differentiation, that Hq is of the same order of small quantities as $\frac{d\phi}{dt}$. Hence H will be small if at any instant the angular velocity, viz. $\frac{d\theta}{dt}+\frac{d\phi}{dt}$, of the moon about GC were so nearly equal to the angular velocity, viz. $\frac{d\theta}{dt}$, of its centre of gravity round the earth, that the ratio of the difference to q is very small.

If therefore we suppose the moon at any instant to be moving with its axis of least moment pointed towards the earth and its angular velocity about its axis of rotation to be nearly equal to that of the moon round the earth, then the axis of least moment will continue always to point very nearly to the earth. The mean angular velocity of the moon about its axis will immediately become equal to that of the moon about the earth and will partake of all its secular changes. This is Laplace's theorem. It shows that the present state of motion of the moon is stable, rather than explains how the angular velocity about the axis came to be so nearly equal to the angular velocity about the earth.

627. By comparing the value of the angular velocity of the moon about its axis obtained by theory with the results of observation, we may hope to obtain some indications of the value of q^2 and thence of $\frac{B-A}{C}$. If the term $Hq\sin(qt+K)$ could be detected by observation, we should deduce the value of $\frac{B-A}{C}$ from its period.

Among the other terms of the expression for the angular velocity of the moon about its axis, those will be best suited to discover the value of q which have the largest coefficients, that is those in which either the numerator M is the greatest or the denominator q^2-p^2 the least possible. By examining the numerical value of their coefficients Laplace has shown that if $\frac{B-A}{C}$ were as great as ·03 the elliptic inequality could be recognized by observation, and if it were between ·0014 and ·003 the annual equation could be observed.

628. We may also calculate by the help of Art. 326 the radial and transverse forces which act on the centre of gravity of the moon due to the mutual attractions of the earth and moon. Since the principal moments of the moon are nearly equal and its linear size small compared with its distance from the earth, these forces are very nearly the same as if the moon were collected

into its centre of gravity. The effect of the small forces neglected by this assumption will be insignificant compared with the other forces which act on the centre of gravity of the moon. The motion of the centre of gravity of the moon is therefore very nearly the same as if the whole mass were collected into its centre of gravity.

Since however there are no other forces which have a moment round GC besides those found above, the effect of these may be perceptible. The effects of tidal friction on the rotation of the moon may be omitted, at least at the present time.

Ex. The centre of gravity G of a rigid body describes an orbit which is nearly circular about a very distant fixed centre of force O attracting according to the Newtonian law and situated in one of the principal planes through G. If $r=c(1+\rho)$, $\theta=nt+n\psi$ be the polar co-ordinates of G referred to O, show that the equations of motion are

$$\left.\begin{aligned} \frac{d^2\rho}{dt^2}-3n^2\rho-2n^2\frac{d\psi}{dt} &= -\frac{9}{4}n^2\gamma'-\frac{9}{4}n^2\gamma\cos 2\phi \\ 2\frac{d\rho}{dt}+\frac{d^2\psi}{dt^2} &= \frac{3}{2}n\gamma\sin 2\phi \\ \frac{d^2\phi}{dt^2}+n\frac{d^2\psi}{dt^2} &= -\frac{q^2}{2}\sin 2\phi \end{aligned}\right\},$$

where $\gamma=\dfrac{B-A}{Mc^2}$, $\gamma'=\dfrac{2C-A-B}{3Mc^2}$.

We may notice that the values of γ and γ' are much smaller than q^2 and might therefore be rejected in a first approximation.

If the body always turns the same face to the centre of force so that ϕ is nearly constant and is small, show that there will be two small inequalities in the value of ϕ of the form $L\sin(pt+a)$, where p is given by

$$(p^2-n^2)(p^2-q^2)-3n^2\gamma(p^2+3n^2)=0,$$

one of these periods being nearly the same as that of the body round the centre of force and the other being very long.

If the body turns very nearly uniformly round its axis GC, so that $\phi=n't+\epsilon'$ nearly, show that there will be two small inequalities in the value of ϕ, one in which $p=n$ and another in which $p=2n'$.

629. Ex. 1. Show that the moon always turns the same face very nearly to that focus of her orbit in which the earth is not situated. [Smith's Prize.]

Ex. 2. If the centre of gravity G of the moon were constrained to describe a circle with a uniform angular velocity n about a fixed centre of force O attracting according to the Newtonian law; show that the axis GA of the moon will oscillate on each side of GO or will make complete revolutions relatively to GO according as the angular velocity of the moon about its axis at the moment when GA and GO coincide in direction is less or greater than $n+q$. Find also the extent of the oscillations.

Ex. 3. A particle m moves without pressure along a smooth circular wire of mass M with uniform velocity under the action of a central force situated in the centre of the wire attracting according to the law of nature. Show that this system of motion is stable if $\dfrac{m}{M}>\dfrac{8+12\sqrt{6}}{25}$. The disturbance is supposed to be given to the particle or the wire, the centre of force remaining fixed in space.

Ex. 4. A uniform ring of mass M and of very small section is loaded with a heavy particle of mass m at a point on its circumference, and the whole is in uniform motion about a centre of force attracting according to the law of nature. Show that the motion cannot be stable unless $\frac{m}{M+m}$ lies between ·815865 and ·8279.

This example shows (1) that if a ring, such as Saturn's ring, be in motion about a centre of force, its position cannot be stable, if the ring be uniform; and (2) that if, to render the motion stable, the ring be weighted, a most delicate adjustment of weights is necessary. A very small change in the distribution of the weights would change a stable combination to one that is unstable. This example is taken from *Prof. Maxwell's Essay on Saturn's Rings.*

Ex. 5. The centre of gravity of a body of mass M, symmetrical about the plane of xy, is G; and O is a point such that the resultant attraction of the body on O is along the line GO. Then if the body be placed with O coinciding with a fixed centre of force S, and be set in rotation about an axis through O perpendicular to the plane of xy with an angular velocity ω, G will, if undisturbed, revolve uniformly in a circle, always turning the same face towards O, provided $Ma\omega^2$ is equal to the resultant attraction along GO, where a is the distance GO. It is required to determine the conditions that this motion should be stable.

The motion being disturbed, O will no longer coincide with the centre of force S. Let two straight lines at right angles revolving uniformly round S as origin with an angular velocity ω be chosen as co-ordinate axes, and let x be initially parallel to OG. Let (x, y) be the co-ordinates of O, ϕ the angle OG makes with the axis of x, then x, y, ϕ are all small. Let V be the potential of the body at O, and let $\frac{d^2V}{dx^2}=\alpha$, $\frac{d^2V}{dxdy}=\gamma$, $\frac{d^2V}{dy^2}=\beta$. Let S be the amount of matter in the centre of force. Then the equations of motion of G, Art. 179, will reduce to

$$\left(\frac{d^2}{dt^2}-\omega^2-\frac{S}{M}\alpha\right)x-\left(2\omega\frac{d}{dt}+\frac{S}{M}\gamma\right)y-2\omega a\frac{d}{dt}\phi=0,$$

$$\left(2\omega\frac{d}{dt}-\frac{S}{M}\gamma\right)x+\left(\frac{d^2}{dt^2}-\omega^2-\frac{S}{M}\beta\right)y+a\frac{d^2}{dt^2}\phi=0,$$

and the equation of angular momentum about S will lead to

$$2\omega ax+a\frac{d}{dt}y+(a^2+k^2)\frac{d}{dt}\phi=0,$$

where k is the radius of gyration of the body about O. Combining these equations as a determinant and reducing we find that the differential equation in ξ, η, or ϕ is of the form

$$A\frac{d^4}{dt^4}+B\frac{d^2}{dt^2}+C=0.$$

The condition of stability is that the roots of this equation should be real and negative. Hence A, B, C must be of the same sign and $B^2>4AC$. This proposition is due to Sir W. Thomson and is given in *Prof. Maxwell's Essay on Saturn's Rings.*

630. The motion of a rigid body about a distant centre of force has been investigated on the supposition that the motion takes place entirely in one plane. We see by equation (2) of Art. 626 that the case in which the centre

of gravity describes a circular orbit, and the rigid body always turns the axis of least moment towards the centre of force, is one of *steady motion*. The preceding investigation also shows that this motion is *stable* for all disturbances which do not alter the plane of motion. It remains now to determine the effect of these disturbances in the more general case when the motion takes place in three dimensions.

The whole attraction of the centre of force on the body is equivalent to a single force acting at the centre of gravity, and a couple. If the size of the body be small compared with its distance from the centre of force, we may neglect the effect of the motion of the body about its centre of gravity in modifying the resultant force. The motion of the centre of gravity will then be the same as if the whole were collected into a single particle. The problem is therefore reduced to the following. A rigid body turns about its centre of gravity G, and is acted on by a centre of force E which moves in a given manner. In the case in which the rigid body is the moon, this centre of force, i.e. the earth, moves in a nearly circular orbit in a plane which itself also has a slow motion in space. This motion is such that a normal GM to the instantaneous orbit describes a cone of small angle about a normal GK to the ecliptic. The two normals maintain a nearly constant inclination of about $5^0.8'$; and the motion of the normal to the instantaneous orbit is nearly uniform.

631. It will clearly be convenient to refer the motion to axes GX, GY, GZ fixed in space such that GZ is normal to the ecliptic. Let GA, GB, GC be the principal axes of the moon at the centre of gravity G. Let (p, q, r) be the direction-cosines of GZ referred to the co-ordinate axes GA, GB, GC. Then we have, since GZ is fixed in space,

$$\left.\begin{aligned} \frac{dp}{dt} - \omega_3 q + \omega_2 r = 0 \\ \frac{dq}{dt} - \omega_1 r + \omega_3 p = 0 \\ \frac{dr}{dt} - \omega_2 p + \omega_1 q = 0 \end{aligned}\right\} \quad \text{.................................(I)}.$$

Now our object is to find the small oscillations about the state of steady motion in which GZ, GC, GM all coincide. We shall therefore have p, q, ω_1, ω_2 all small, and r very nearly equal to unity. The equations (I) will therefore become

$$\left.\begin{aligned} \frac{dp}{dt} - nq + \omega_2 = 0 \\ \frac{dq}{dt} - \omega_1 + np = 0 \end{aligned}\right\},$$

where n is the mean value of ω_3.

Let λ, μ, ν be the direction-cosines of the centre of force E as seen from G. Then we have by Euler's equations and Art. 608,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 = -3n'^2(B-C)\mu\nu \\ B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 = -3n'^2(C-A)\nu\lambda \\ C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 = -3n'^2(A-B)\lambda\mu \end{aligned}\right\} \quad \text{..................(II)}.$$

In the case of steady motion, the rigid body always turns the axis (GA) of least moment towards the centre of force, and $\omega_3 = n'$. We have then both μ and ν small quantities, so that in the first equation we may neglect their product $\mu\nu$, and in the second equation we may put $\nu\lambda = \nu$. Also, we may put $\omega_3 = n = n'$ in the small terms.

If l be the latitude of the earth as seen from the moon, we have

$$\sin l = \cos ZE = p\lambda + q\mu + r\nu = p + \nu \text{ nearly.}$$

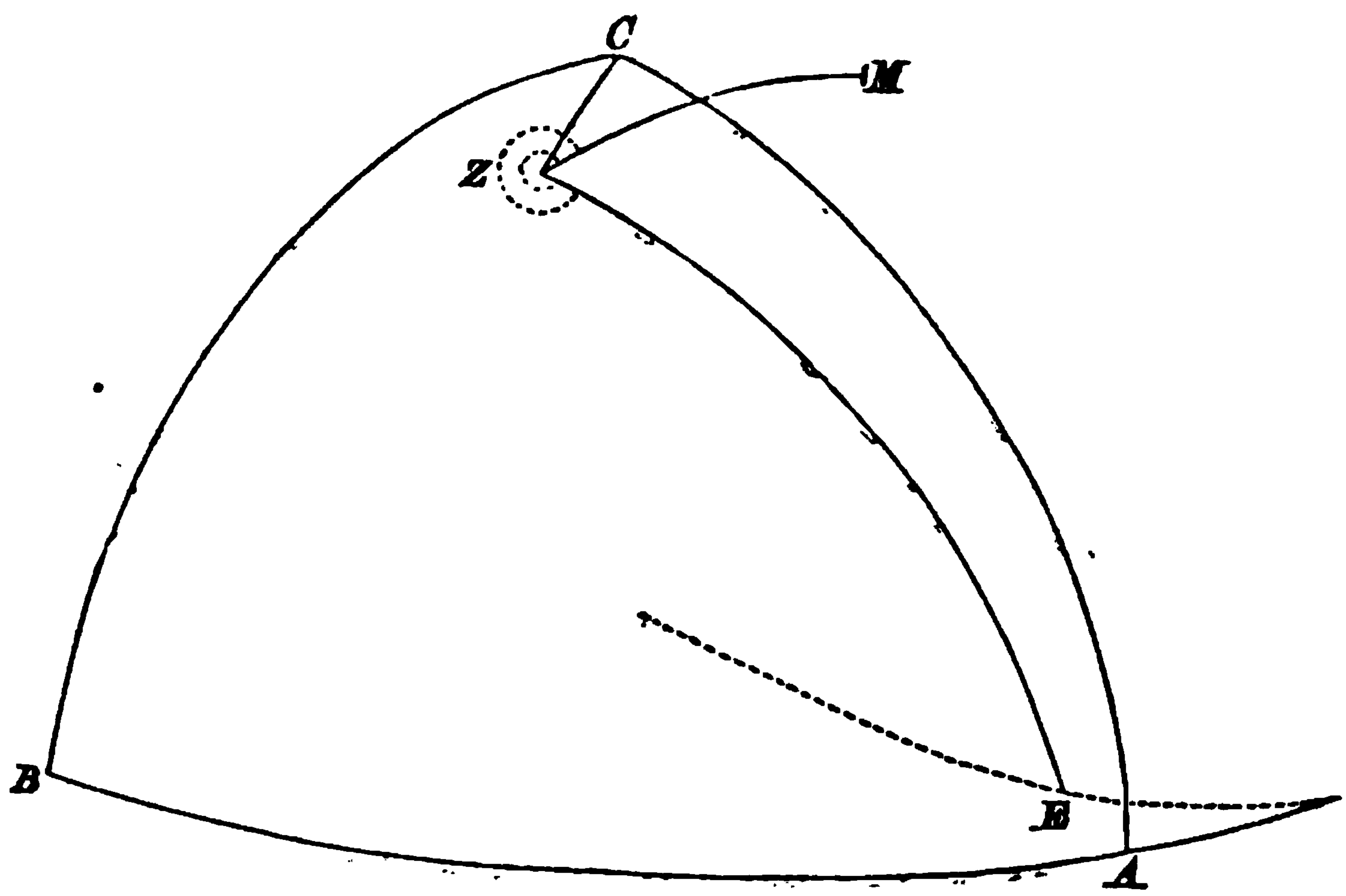
Hence the two first of Euler's equations take the form

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\,n\omega_2 &= 0 \\ B\frac{d\omega_2}{dt} - (C-A)\,n\omega_1 &= -3n^2(C-A)(-p+\sin l) \end{aligned}\right\} \text{...........(III).}$$

If the earth, as seen from the moon, be supposed to move in a circular orbit in a plane making a constant inclination $\tan^{-1} k$ with the ecliptic, and the longitude of whose node is $-gt+\beta$, we shall have

$$\sin l = k \sin(n't + gt - \beta),$$

In this expression g measures the rate at which the node regredes, and is about the two hundred and fiftieth part of n. We shall therefore regard $\frac{g}{n}$ as a small quantity.



To solve these equations, it will be found convenient to substitute for ω_1, ω_2 their values in terms of p, q. We then have

$$\left.\begin{aligned} A\frac{d^2q}{dt^2} + (A+B-C)\,n\frac{dp}{dt} - n^2(B-C)\,q &= 0 \\ B\frac{d^2p}{dt^2} - (A+B-C)\,n\frac{dq}{dt} + 4n^2(C-A)\,p &= 3n^2(C-A)\sin l \end{aligned}\right\}.$$

To find p, q, let us put

$$p = P\sin\{(n'+g)t-\beta\},\quad q = Q\cos\{(n'+g)t-\beta\},$$

where P, Q are some constants to be determined by substitution in the equation. We have

$$\left.\begin{array}{l} Q\{A(n+g)^2+(B-C)n^2\} = P(A+B-C)n(n+g) \\ P\{B(n+g)^2-4(C-A)n^2\} - Q(A+B-C)n(n+g) = -3n^2k(C-A) \end{array}\right\}.$$

We may solve these equations to find P and Q accurately. In the case of the moon the ratios $\frac{A-B}{C}$, $\frac{B-C}{A}$, $\frac{C-A}{B}$ and $\frac{g}{n}$ are all small. If then we neglect the products of these small quantities, the first equation gives us $\frac{Q}{P}=1-\frac{g}{n}$. The second equation will then give

$$P = \frac{3nk(C-A)}{3n(C-A)-2Bg}.$$

As g is very small compared with n, we may regard P and Q as equal.

632. The complementary functions may be found in the usual manner by assuming

$$p = F\sin(st+H),\quad q = G\cos(st+H),$$

on substituting we have the quadratic

$$ABs^4 - \{(A+B-C)^2 - B(B-C) - 4A(A-C)\}n^2s^2 + 4(A-C)(B-C)n^4 = 0,$$

to find s^2, and
$$\frac{G}{F} = \frac{(A+B-C)ns}{As^2+(B-C)n^2},$$

to find the ratio of the coefficients of corresponding terms in p and q. If the roots of this equation were negative p and q would be represented by exponential values of t, and thus they would in time cease to be small. It is therefore necessary for stability that the coefficient of s^2 should be negative and the product $(A-C)(B-C)$ positive. Both these conditions are probably satisfied in the case of the moon. For since $B-C$ and $A-C$ are both small, the term $(A+B-C)^2$ is much greater than the two other terms in the coefficient of s^2. Also, since the moon is flattened at its poles, we shall probably have A and B both less than C.

633. Let M be the pole of the moon's orbit, which is the same as that of the earth's orbit as seen from the centre of the moon. Then M is the pole of the dotted line in the figure of Art. 631. Therefore the angle EZM measured by turning ZE in the positive direction round Z until it comes into coincidence with ZM, is $=\frac{3\pi}{2}-\{(n+g)t-\beta\}$. Again, if the angle EZC be measured in the same direction, we have

$$\cos EZC = \frac{\cos EC - \cos CZ\cos ZE}{\sin CZ\sin ZE} = \frac{\nu - r(p\lambda+q\mu+r\nu)}{\sqrt{p^2+q^2}\sin ZE} = \frac{-p}{\sqrt{p^2+q^2}},\text{ nearly.}$$

Hence we easily find
$$\sin EZC = \frac{-q}{\sqrt{p^2+q^2}}.$$

But
$$\sin CZM = \sin EZM\cos EZC - \cos EZM\sin EZC$$
$$= \frac{\cos\{(n+g)t-\beta\}p - \sin\{(n+g)t-\beta\}q}{\sqrt{p^2+q^2}}.$$

If now we substitute for p and q their values, it is clear that the terms in p and q, whose argument is $n+g$, disappear. So that if F and G were zero, the sine of the angle CZM would be absolutely zero. In this case the three poles C, Z, M must lie in an arc of a great circle, or, which is the same thing, *the moon's equator, the moon's orbit, and the ecliptic must cut each other in the same line of nodes.*

If however F and G be not zero, but only very small, we have

$$\sin CZM = \frac{\Sigma F' \sin(s't+H')}{\sqrt{P^2 + \Sigma G'^2 \sin(s't+H')}},$$

where F', G' contain either F or G as a factor, and are therefore small. If then F and G be both small compared with P, the angle CZM will remain either always small or always nearly equal to π.

The intersection of the moon's equator with the ecliptic will then oscillate about the intersection of the moon's orbit with the ecliptic as its mean position. Since these oscillations are insensible, it follows that in the case of nature, the complementary functions must be extremely small compared with the terms depending directly on the disturbing force.

634. If we disregard the complementary functions we have $p = P\sin\phi$, $q = P\cos\phi$, where $\phi = (n'+g)t - \beta$. Now $\sin^2 CZ = p^2 + q^2$; therefore $CZ = -P$ very nearly. The value of CZ, the inclination of the lunar equator to the ecliptic, is known to be about $1^\circ.28'$. Hence, since $\frac{g}{n} = \cdot004$, we may deduce from the expression for P at the end of Art. 631 an approximation to the value of $\frac{C-A}{B}$. In this manner Laplace finds $\frac{C-A}{B} = \cdot000599$.

CHAPTER XII.

MOTION OF A STRING OR CHAIN.

The Equations of Motion.

635. PROP. *To determine the general equations of motion of a string under the action of any forces.*

First. *Let the string be inextensible.*

Let Ox, Oy, Oz be any axes fixed in space. Let $Xmds$, $Ymds$, $Zmds$ be the impressed forces that act on any element ds of the string whose mass is mds. Let u, v, w be the resolved parts of the velocities of this element parallel to the axes. Then, by D'Alembert's principle, the element ds of the string is in equilibrium under the action of the forces

$$mds\left(X-\frac{du}{dt}\right),\quad mds\left(Y-\frac{dv}{dt}\right),\quad mds\left(Z-\frac{dw}{dt}\right),$$

and the tensions at its two ends.

Let T be the tension at the point (x, y, z), then $T\frac{dx}{ds}$, $T\frac{dy}{ds}$, $T\frac{dz}{ds}$ are its resolved parts parallel to the axes. The resolved parts of the tensions at the other end of the element will be

$$T\frac{dx}{ds}+\frac{d}{ds}\left(T\frac{dx}{ds}\right)ds,$$

and two similar quantities with y and z written for x.

Hence the equations of motion are

$$\left.\begin{aligned} m\frac{du}{dt}&=\frac{d}{ds}\left(T\frac{dx}{ds}\right)+mX\\ m\frac{dv}{dt}&=\frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY\\ m\frac{dw}{dt}&=\frac{d}{ds}\left(T\frac{dz}{ds}\right)+mZ\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In these equations the variables s and t are independent. For any the same element of the string, s is always constant, and its path is traced out by variation of t. On the other hand, the curve in which the string hangs at any proposed time is given by variations of s, t being constant. In this investigation s is measured from any arbitrary point, fixed in the string, to the element under consideration.

To find the geometrical equations. We have

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

Differentiating this with respect to t, we get

$$\frac{dx}{ds}\frac{du}{ds} + \frac{dy}{ds}\frac{dv}{ds} + \frac{dz}{ds}\frac{dw}{ds} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations (1) and (2) are sufficient to determine x, y, z, and T, in terms of s and t.

Ex. If V be the Vis Viva of any arc AB of the chain; T_1, T_2 the tensions at the extremities of this arc; u_1', u_2' the velocities of the extremities resolved along the tangents at those extremities, prove that

$$\frac{1}{2}\frac{dV}{dt} = T_2u_2' - T_1u_1' + \int (Xu + Yv + Zw)\, mds,$$

the integration extending over the whole arc.

636. The equations of motion may be put under another form. Let ϕ, ψ, χ be the angles made by the tangent at x, y, z, with the axes of co-ordinates. Then the equations (1) become

$$m\frac{du}{dt} = \frac{d}{ds}(T\cos\phi) + mX \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

with similar equations for v and w.

To find the geometrical equations, differentiate $\cos\phi = \frac{dx}{ds}$ with respect to t;

$$\therefore -\sin\phi\frac{d\phi}{dt} = \frac{du}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Similarly, by differentiating $\cos\psi = \frac{dy}{ds}$ and $\cos\chi = \frac{dz}{ds}$, we get two other similar equations for ψ and χ. Taking these six equations in conjunction with the following

$$\cos^2\phi + \cos^2\psi + \cos^2\chi = 1 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

we have seven equations to determine u, v, w, ϕ, ψ, χ and T.

If the motion takes place in one plane, these reduce to the four following equations:

$$\left.\begin{aligned} m\frac{du}{dt} &= \frac{d}{ds}(T\cos\phi) + mX \\ m\frac{dv}{dt} &= \frac{d}{ds}(T\sin\phi) + mY \end{aligned}\right\}\text{..................} (6),$$

$$\left.\begin{aligned} -\sin\phi\,\frac{d\phi}{dt} &= \frac{du}{ds} \\ \cos\phi\,\frac{d\phi}{dt} &= \frac{dv}{ds} \end{aligned}\right\}\text{......................} (7).$$

The arbitrary constants and functions which enter into the solutions of these equations must be determined from the peculiar circumstances of each problem.

637. Secondly. *Let the string be elastic.*

Let σ be the unstretched length of the arc s, and let $m d\sigma$ be the mass of an element $d\sigma$ of unstretched length or ds of stretched length. Then by the same reasoning as before, the equations of motion become

$$m\frac{du}{dt} = \frac{d}{d\sigma}\left(T\frac{dx}{ds}\right) + mX \text{..............................} \text{(i)},$$

and two similar equations for v and w. To find the geometrical equations we must differentiate

$$\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2 = \left(\frac{ds}{d\sigma}\right)^2,$$

the independent variables being now σ and t. Differentiating with regard to t we have

$$\frac{dx}{d\sigma}\frac{du}{d\sigma} + \frac{dy}{d\sigma}\frac{dv}{d\sigma} + \frac{dz}{d\sigma}\frac{dw}{d\sigma} = \frac{ds}{d\sigma}\frac{d}{dt}\left(\frac{ds}{d\sigma}\right).$$

But if λ be the modulus of elasticity of the string, we have

$$\frac{ds}{d\sigma} = 1 + \frac{T}{\lambda} \text{.......................................} \text{(ii)}.$$

Substituting we have

$$\frac{dx}{d\sigma}\frac{du}{d\sigma} + \frac{dy}{d\sigma}\frac{dv}{d\sigma} + \frac{dz}{d\sigma}\frac{dw}{d\sigma} = \left(1 + \frac{T}{\lambda}\right)\frac{1}{\lambda}\frac{dT}{dt} \text{.....................} \text{(iii)}.$$

The two equations (ii) and (iii) together with the three equations (i) will suffice for the determination of u, v, w, s and T in terms of σ and t.

If we wish to use the equations of motion in the forms corresponding to (3) or (6), the dynamical equations become

$$m\frac{du}{dt} = \frac{d}{d\sigma}(T\cos\phi) + mX,$$

with similar equations for v and w.

The geometrical equations corresponding to (4) or (7) may be found thus. We have

$$\frac{dx}{d\sigma}=\cos\phi\frac{ds}{d\sigma}=\cos\phi\left(1+\frac{T}{\lambda}\right).$$

Differentiating, we have

$$\frac{du}{d\sigma}=-\sin\phi\frac{d\phi}{dt}+\frac{1}{\lambda}\frac{d}{dt}(T\cos\phi),$$

with similar expressions for v and w.

638. When the motion of the string takes place in one plane, it is often convenient to resolve the velocities along the tangent and normal to the curve.

Let u', v' be the resolved parts of the velocity of the element ds along the tangent and normal to the curve at that element. Let ϕ be the angle the tangent to the element ds makes with the axis of x. Then by Art. 179 or 252, the equations of motion are

$$\left.\begin{aligned}\frac{du'}{dt}-v'\frac{d\phi}{dt}&=X'+\frac{dT}{mds}\\ \frac{dv'}{dt}+u'\frac{d\phi}{dt}&=Y'+\frac{T}{m\rho}\end{aligned}\right\}\quad\text{.................(1).}$$

The geometrical equations may be obtained as follows. We have

$$u=u'\cos\phi-v'\sin\phi.$$

Differentiating with respect to s, we have by Art. 636,

$$-\frac{d\phi}{dt}\sin\phi=\left(\frac{du'}{ds}-\frac{v'}{\rho}\right)\cos\phi-\left(\frac{dv'}{ds}+\frac{u'}{\rho}\right)\sin\phi,$$

where ρ is the radius of curvature, and is equal to $\dfrac{ds}{d\phi}$. Since the axis of x is arbitrary in position, take it so that the tangent during its motion is parallel to it at the instant under consideration; then $\phi=0$ and we have

$$0=\frac{du'}{ds}-\frac{v'}{\rho}\quad\text{.........................(2).}$$

Similarly, by taking the axis of x parallel to the normal,

$$\frac{d\phi}{dt}=\frac{dv'}{ds}+\frac{u'}{\rho}\quad\text{.........................(3).}$$

These four equations are sufficient to determine u', v', ϕ and T in terms of s and t.

If the string be extensible, the dynamical equations become

$$\left.\begin{aligned}\frac{du'}{dt}-v'\frac{d\phi}{dt}&=X'+\frac{dT}{md\sigma}\\ \frac{dv'}{dt}+u'\frac{d\phi}{dt}&=Y'+\frac{T}{m\rho}\frac{ds}{d\sigma}\end{aligned}\right\}.$$

To find the geometrical equations, we may differentiate

$$u=u'\cos\phi-v'\sin\phi$$

with regard to σ. This gives by Art. 637

$$-\sin\phi\frac{d\phi}{dt}+\frac{1}{\lambda}\frac{d}{dt}(T\cos\phi)=\left(\frac{du'}{d\sigma}-\frac{v'}{\rho}\frac{ds}{d\sigma}\right)\cos\phi-\left(\frac{dv'}{d\sigma}+\frac{u'}{\rho}\frac{ds}{d\sigma}\right)\sin\phi.$$

By the same reasoning as in Art. 638, this reduces to

$$\frac{1}{\lambda}\frac{dT}{dt}=\frac{du'}{d\sigma}-\frac{v'}{\rho}\left(1+\frac{T}{\lambda}\right),$$

$$\frac{d\phi}{dt}\left(1+\frac{T}{\lambda}\right)=\frac{dv'}{d\sigma}+\frac{u'}{\rho}\left(1+\frac{T}{\lambda}\right).$$

639. The equations (2) and (3) may also be obtained in the following manner. The motion of the point P of the string being represented by velocities u' and v' along the tangent PA and normal PO at P, the motion of a consecutive point Q will be represented by velocities $u'+du'$ and $v'+dv'$ along the tangent QB, and normal QO at Q. Let the arc $PQ=ds$, and let QN be a perpendicular on PA. Since the string is inextensible, the resultant velocity of Q resolved along the tangent at P must be ultimately the same as the resolved part of the velocity of P in the same direction. Hence

$$(u'+du')\cos d\phi-(v'+dv')\sin d\phi=u',$$

or, proceeding to the limit,

$$du'-v'd\phi=0;\ \therefore\ \frac{du'}{ds}-\frac{v'}{\rho}=0.$$

Again, $\dfrac{d\phi}{dt}$ is the angular velocity of PQ round P. Hence the difference of the velocities of P and Q resolved in any direction which is ultimately perpendicular to PQ must be equal to $PQ\,\dfrac{d\phi}{dt}$;

$$\therefore\ (u'+du')\sin d\phi+(v'+dv')\cos d\phi-v'=ds\frac{d\phi}{dt},$$

or in the limit

$$\frac{d\phi}{dt}=\frac{dv'}{ds}+\frac{u'}{\rho}.$$

640. Ex. 1. An elastic ring without weight, whose length when unstretched is given, is stretched round a circular cylinder. The cylinder is suddenly annihilated,

show that the time which the ring will take to collapse to its natural length is $\sqrt{\frac{Ma\pi}{8\lambda}}$, where M is the mass of the string, λ its modulus of elasticity, and a is the natural radius.

Ex. 2. A homogeneous light inextensible string is attached at its extremities to two fixed points, and turns about the straight line joining those points with uniform angular velocity. Find the form of the string, supposing its figure permanent.

Result. Let the straight line joining the fixed points be the axis of x, then the form of the string is a plane curve whose equation is $1+\left(\frac{dy}{dx}\right)^2=\left(\frac{a-y^2}{b}\right)^2$, where a and b are two constants.

On Steady Motion.

641. Def. When the motion of a string is such that the curve which it forms in space is always equal, similar, and similarly situated to that which it formed in its initial position, that motion may be called steady.

642. Prop. *To investigate the steady motion of an inextensible string.*

It is obvious that every element of the string is animated with two velocities, one due to the motion of the curve in space, and the other to the motion of the string along the curve which it forms in space. Let a and b be the resolved parts along the axes of the velocity of the curve at the time t, and let c be the velocity of the string along its curve.

Then, following the usual notation, we have

$$\left.\begin{aligned} u &= a + c\cos\phi \\ v &= b + c\sin\phi \end{aligned}\right\} \quad\text{.........................}\quad (1).$$

Now a, b, c are functions of t only, hence $\frac{du}{ds}=-c\sin\phi\,\frac{d\phi}{ds}$. Therefore by equation (7) of Art. 636 we have

$$\frac{d\phi}{dt}=c\,\frac{d\phi}{ds}\quad\text{.........................}\quad (2).$$

Substituting the values of u and v in the equations of motion, Art. 635, we get

$$\left.\begin{aligned} \frac{da}{dt}+\frac{dc}{dt}\cos\phi-c\sin\phi\,\frac{d\phi}{dt} &= X+\frac{d}{ds}\left(\frac{T}{m}\cos\phi\right) \\ \frac{db}{dt}+\frac{dc}{dt}\sin\phi+c\cos\phi\,\frac{d\phi}{dt} &= Y+\frac{d}{ds}\left(\frac{T}{m}\sin\phi\right) \end{aligned}\right\}.$$

Substituting for $\frac{d\phi}{dt}$, these equations reduce to

$$\left.\begin{aligned}\frac{da}{dt} &= \left(X - \frac{dc}{dt}\cos\phi\right) + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\cos\phi\right\}\\ \frac{db}{dt} &= \left(Y - \frac{dc}{dt}\sin\phi\right) + \frac{d}{ds}\left\{\left(\frac{T}{m} - c^2\right)\sin\phi\right\}\end{aligned}\right\}\text{.........}(3).$$

The form of the curve is to be independent of t; hence, on eliminating T, the resulting equation must not contain t. This will not generally be the case unless $\frac{da}{dt}$, $\frac{db}{dt}$, $\frac{dc}{dt}$ are all constants. In any case their values will be determined by the known circumstances of the Problem. The above equations must then be solved, s being supposed to be the only independent variable, and t being constant.

643. If the string move uniformly in space, and the elements of the string glide uniformly along the string, $\frac{da}{dt} = 0$, $\frac{db}{dt} = 0$, $\frac{dc}{dt} = 0$. It will then follow from the above equations, that the form of the string will be the same as if it was at rest, but the tension will exceed the stationary tension by mc^2.

644. Ex. 1. *Let an electric cable be deposited at the bottom of a sea of uniform depth from a ship moving with uniform velocity in a straight line, and let the cable be delivered with a velocity* c *equal to that of the ship. Find the equation to the curve in which the string hangs.*

The motion may be considered *steady*, and the form of the curve of the string will be always the same.

If the friction of the water on the string be neglected, gravity diminished by the buoyancy of the water will be the only force acting on the string, let this be represented by g'. Hence the form of the travelling curve will be the common catenary, and the tension at any point will exceed the tension in the catenary by the weight of a length of string equal to $\frac{c^2}{g'}$.

Next let the friction of the water on any element of the cable be supposed to vary as the velocity of the element, and to act in a direction opposite to the direction of motion of the element*. Let μ be the coefficient of friction.

Let the axis of x be horizontal, and let x' be the abscissa of any point of the cable measured from the place where the cable touches the ground, in the direction

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* Each element of the string has a motion both along the cable and transversely to it. The coefficients of these frictions are probably not the same, but they have been taken equal in the above investigation.

of the ship's motion. Also let s' be the length of the curve measured from the same point. Then $x=x'+ct$, and $s=s'+ct$.

Following the same notation as before, we have

$$X=-\mu u, \qquad Y=-g'-\mu v.$$

But $$u=c-c\cos\phi, \qquad v=-c\sin\phi.$$

Hence the equations (3) become

$$\left.\begin{aligned} 0&=-\mu c+\mu c\cos\phi+\frac{d}{ds}\left\{\left(\frac{T}{m}-c^2\right)\cos\phi\right\}\\ 0&=-g'+\mu c\sin\phi+\frac{d}{ds}\left\{\left(\frac{T}{m}-c^2\right)\sin\phi\right\}\end{aligned}\right\}.$$

To integrate these put $\sin\phi=\frac{dy}{ds}$, $\cos\phi=\frac{dx}{ds}$. Hence,

$$\left.\begin{aligned} g'A&=-\mu cs+\mu cx+\left(\frac{T}{m}-c^2\right)\cos\phi\\ g'B&=-g's+\mu cy+\left(\frac{T}{m}-c^2\right)\sin\phi\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(1),$$

where A and B are two arbitrary constants.

At the point where the cable meets the ground, we must have either $T=0$ or $\phi=0$. For if ϕ be not zero, the tangents at the extremities of an infinitely small portion of the string make a finite angle with each other. Then, if T be not zero, resolving the tensions at the two ends in any direction, we have an infinitely small mass acted on by a finite force. Hence the element will in that case alter its position with an infinite velocity. First, let us suppose that $\phi=0$. Also at the same point, $y=0$ and $s'=0$. Hence $B=-ct$.

Putting $\frac{\mu c}{g'}=e$, we get by division

$$\frac{dy}{dx'}=\frac{s'-ey}{A-ex'+es'}\quad\ldots\ldots\ldots\ldots(2).$$

This is the differential equation to the curve in which the cable hangs.

To solve this equation*, let us find s' in terms of the other quantities,

$$s'=\frac{A\dfrac{dy}{dx'}-ex'\dfrac{dy}{dx'}+ey}{1-e\dfrac{dy}{dx'}}.$$

Differentiating, we have

$$\sqrt{1+\left(\frac{dy}{dx'}\right)^2}=\frac{\dfrac{d^2y}{dx'^2}\cdot(A-ex'+e^2y)}{\left(1-e\dfrac{dy}{dx'}\right)^2}.$$

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* The problem of the mechanical conditions of the deposit of a submarine cable has been considered by the Astronomer Royal in the *Phil. Mag.* July 1858. His solution is different from that given above, but his method of integrating the differential equation (2) has been followed.

Put p for $\frac{dy}{dx}$ where convenient, and put v for $A - ex' + e^2y$; the equation then becomes

$$\frac{1}{v}\frac{dv}{dx'} = \frac{-e\frac{dp}{dx'}}{(1-ep)\sqrt{1+p^2}},$$

in which the variables are separated, and the integrations can be effected. The equation can be integrated a second time, but the result is very long. The arbitrary constant A may have any value, depending on the length of the cable hanging from the ship at the time $t=0$.

The curve in its lower part resembles a circular arc or the lower part of a common catenary. But in its upper part the curve does not tend to become vertical, but tends to approach an asymptote making an angle $\cot^{-1}e$ with the horizon. The asymptote does not pass through the point where the cable touches the ground but below it, its smallest distance being $\frac{A}{e\sqrt{e^2+1}}$; the asymptote also passes below the ship.

If the conditions of the question be such that the tension at the lowest point of the cable is equal to nothing, the tangent to the curve at that point will not necessarily be horizontal. Let λ be the angle this tangent makes with the horizon, Referring to equations (1) we have simultaneously

$$x'=0,\ y=0,\ s'=0,\ T=0, \text{ and } \phi=\lambda.$$

Hence $$A = -\frac{c^2}{g'}\cos\lambda,\ B = -\frac{c^2}{g'}\sin\lambda - ct.$$

The differential equation to the curve will now become

$$\frac{dy}{dx'} = \frac{-\frac{c^2}{g'}\sin\lambda + s' - ey}{-\frac{c^2}{g'}\cos\lambda + es' - ex'} \qquad (3),$$

which can be integrated in the same manner as before. One case deserves notice; viz. when $e = \cot\lambda$. The equation is then evidently satisfied by $y = \frac{1}{e}x'$. The two constants in the integral of (3) are to be determined by the condition that when $x'=0$, $y=0$, then $\frac{dy}{dx'} = \tan\lambda$. Both these conditions are satisfied by the relation $y = \frac{1}{e}x'$. Hence this is the required integral. The form of the cable is therefore a straight line, inclined to the horizon at an angle $\lambda = \cot^{-1}e$; and the tension may be found from the formula $T = \frac{mg'y}{1+\cos\lambda}$.

Ex. 2. Let a cable be delivered with velocity c' from a ship moving with uniform velocity c in a straight line on the surface of a sea of uniform depth. If the resistance of the water to the cable be proportional to the square of the velocity, the coefficient B, of resistance for longitudinal motion being different from the coefficient A, for lateral motion, prove that the cable may take the form of a straight line making an angle λ with the horizon, such that $\cot^2\lambda = \sqrt{e^4+\frac{1}{4}} - \frac{1}{2}$, where e is the ratio of the speed of the ship to the terminal velocity of a length of

cable falling laterally in water. Prove also that the tension will be found from the equation

$$T = \left\{ y - \frac{B}{A} c^2 \left(\frac{c'}{c} - \cos\lambda \right)^2 \frac{y}{\sin\lambda} \right\} mg'. \quad \text{[Phil. Mag.]}$$

On Initial Motions.

645. *A string, under the action of any forces in one plane, begins to move from a state of rest in the form of any given curve. To find the initial tension at any given point.*

Let $mPds$, $mQds$ be the resolved parts of the forces respectively along the tangent and normal to any element ds. The force P is taken positively when it acts in the direction in which s is measured, and Q is positive when it acts in the direction in which ρ is measured along the normal, viz. inwards. Let m be the mass of a unit of length.

Let u, v be the velocities of the element along the tangent and normal. Then the equations of motion are by Art. 638

$$\frac{du}{dt} - v\frac{d\phi}{dt} = P + \frac{1}{m}\frac{dT}{ds} \quad\text{.......................}\quad (1),$$

$$\frac{dv}{dt} + u\frac{d\phi}{dt} = Q + \frac{1}{m}\frac{T}{\rho} \quad\text{.......................}\quad (2),$$

where T is the tension, ρ the radius of curvature, and ϕ the angle the tangent makes with any fixed straight line. The geometrical equations are

$$\frac{du}{ds} - \frac{v}{\rho} = 0 \quad\text{.........}\quad (3), \qquad \frac{dv}{ds} + \frac{u}{\rho} = \frac{d\phi}{dt} \quad\text{.............}\quad (4).$$

Differentiating (1) and multiplying (2) by $\frac{1}{\rho}$, we get

$$\left.\begin{aligned} \frac{d^2u}{ds\,dt} - v\frac{d^2\phi}{ds\,dt} - \frac{dv}{ds}\frac{d\phi}{dt} &= \frac{dP}{ds} + \frac{1}{m}\frac{d^2T}{ds^2} \\ \frac{1}{\rho}\frac{dv}{dt} + \frac{u}{\rho}\frac{d\phi}{dt} &= \frac{Q}{\rho} + \frac{1}{m}\frac{T}{\rho^2} \end{aligned}\right\} \quad\text{............}\quad (5).$$

But by differentiating (3) we have, since $\frac{1}{\rho} = \frac{d\phi}{ds}$,

$$\frac{d^2u}{ds\,dt} - v\frac{d^2\phi}{ds\,dt} - \frac{1}{\rho}\frac{dv}{dt} = 0 \quad\text{..................}\quad (6).$$

Hence, subtracting the second of equations (5) from the first, we have by (4) and (6)

$$\frac{1}{m}\left(\frac{d^2T}{ds^2} - \frac{T}{\rho^2}\right) + \frac{dP}{ds} - \frac{Q}{\rho} = -\left(\frac{d\phi}{dt}\right)^2.$$

In the beginning of the motion just after the string has been cut we may reject the squares of small quantities, hence $\left(\frac{d\phi}{dt}\right)^2$ may be rejected. Hence we have

$$\frac{d^2T}{ds^2} - \frac{T}{\rho^2} = -m\frac{dP}{ds} + m\frac{Q}{\rho} \quad \text{.................(7)}.$$

This is the general equation to determine the tension of a string just after it has been cut.

The two arbitrary constants introduced in the solution of this equation are to be determined by the circumstances of the case. If both ends of the string are free, we must have $T=0$ at both ends.

Since the string begins to move from a state of rest we have initially $u=0$, $v=0$. At the end of a time dt, $\frac{du}{dt}dt$ and $\frac{dv}{dt}dt$ will be the velocities of any element of the string. Hence if ψ be the angle the initial direction of motion of any element of the string makes with the tangent to the element, we have by equations (1) and (2)

$$\tan\psi = \frac{Q + \frac{1}{m}\frac{T}{\rho}}{P + \frac{1}{m}\frac{dT}{ds}} \quad \text{........................(8)}.$$

It must be remembered that the constants of integration are necessarily constant only throughout the length of the string at the time $t=0$. They may be functions of t and may be either continuous or discontinuous. For example, if a point of the string be absolutely fixed in space, the transverse action of the fixed point on the string may cause the constants to become discontinuous at that point. In this case equation (8) is not necessarily true in the immediate neighbourhood of the fixed point.

646. If the string be heterogeneous we may easily show in the same way, that the initial tension is given by

$$\frac{d}{ds}\left(\frac{1}{m}\frac{dT}{ds}\right) - \frac{1}{m}\frac{T}{\rho^2} = -\frac{dP}{ds} + \frac{Q}{\rho}.$$

647. *A string is in equilibrium under the action of forces in one plane. Supposing the string to be cut at any given point, find the instantaneous change of tension.*

Let T_0 be the tension at any point (x, y) just before the string was cut. Then the forces P, Q satisfy the equations of equilibrium

$$0 = P + \frac{1}{m}\frac{dT_0}{ds}, \quad 0 = Q + \frac{1}{m}\frac{T_0}{\rho}.$$

Hence
$$-\frac{dP}{ds}+\frac{Q}{\rho}=\frac{1}{m}\frac{d^2T_0}{ds^2}-\frac{1}{m}\frac{T_0}{\rho^2}.$$

If T' be the instantaneous change of tension, we have $T'=T-T_0$. The equation of the last article therefore becomes
$$\frac{d^2T'}{ds^2}-\frac{T'}{\rho^2}=0.$$

648. Ex. 1. *A string is in equilibrium in the form of a circle about a centre of repulsive force in the centre. If the string be now cut at any point* A, *prove that the tension at any point* P *is instantaneously changed in the ratio of*
$$1-\frac{e^{\pi-\theta}+e^{-(\pi-\theta)}}{e^{\pi}+e^{-\pi}}:1,$$
where θ is the angle subtended at the centre by the arc AP.

Let F be the central force, then $P=0$, and $mQ=-F$. Let a be the radius of the circle. Then the equation of Art. 645 to determine T becomes
$$\frac{d^2T}{ds^2}-\frac{T}{a^2}=-\frac{F}{a}.$$

Let s be measured from the point A towards P, then $s=a\theta$; also F is independent of s. Hence we have
$$T=Fa+A\epsilon^{\theta}+B\epsilon^{-\theta}.$$

To determine the arbitrary constants A and B we have the condition $T=0$ when $\theta=0$ and $\theta=2\pi$;
$$\therefore\ T=Fa.\left\{1-\frac{\epsilon^{(\pi-\theta)}+\epsilon^{-(\pi-\theta)}}{\epsilon^{\pi}+\epsilon^{-\pi}}\right\}.$$

But just before the string was cut $T=Fa$. Hence the result given in the enunciation follows.

Ex. 2. A string is wound round the under part of a vertical circle and is just supported in equilibrium at the ends of a horizontal diameter by two forces. The circle being suddenly removed, prove that the tension at the lowest point is instantly decreased in the ratio $4 : e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}$.

Ex. 3. The extreme links of a uniform chain can slide freely on two given curves in a vertical plane, and the whole is in equilibrium under the action of gravity. Supposing the chain to break at any point, prove that the initial tension at any point is $T=y\,(A\phi+B)$, where y is the altitude of the point above the directrix of the catenary, ϕ the angle the tangent makes with the horizon, and A, B two arbitrary constants. Explain how the constants are to be determined.

Ex. 4. A string rests on a smooth table in the form of an arc of an equiangular spiral and begins to move from rest under the action of a central force F which tends from the pole and varies as the n^{th} power of the distance, show that the initial tension is given by
$$T=-rF\frac{n\cos^2\alpha+\sin^2\alpha}{n\,(n+1)\cos^2\alpha-\sin^2\alpha}+Ar^p+Br^q,$$

where a is the angle of the spiral, p and q are the roots of the quadratic

$$x(x-1)=\tan^2 a.$$

Show that the solution changes its form when a is such that the first term is infinite, and find the new form.

649. *A string rests on a smooth horizontal table and is acted on at one extremity by an impulsive tension, find the impulsive tension at any point and the initial motion.*

Let T be the impulsive tension at any point P, $T+dT$ the tension at a consecutive point Q, then the element PQ is acted on by the tensions T and $T+dT$ at the extremities. Let ϕ be the angle the tangent at P to the string makes with any fixed line; u, v the initial velocities of the element resolved respectively along the tangent and normal at P to the string. Then, resolving along the tangent and normal, we have

$$\left.\begin{aligned} muds &= (T+dT)\cos d\phi - T \\ mvds &= (T+dT)\sin d\phi \end{aligned}\right\};$$

therefore proceeding to the limit

$$u=\frac{1}{m}\frac{dT}{ds},\quad v=\frac{1}{m}\frac{T}{\rho}.$$

But by Art. 639, we have $\dfrac{du}{ds}=\dfrac{v}{\rho}$. Hence the equation to find T becomes

$$\frac{d^2T}{ds^2}-\frac{T}{\rho^2}=0.$$

This, as might have been expected from mechanical considerations, is the same as the equation in Art. 647.

If the chain be heterogeneous we easily find in the same way

$$\frac{d}{ds}\left(\frac{1}{m}\frac{dT}{ds}\right)=\frac{1}{m}\frac{T}{\rho^2}.$$

The two results in this article appear to have been first given in College Examination Papers.

650. Ex. If T_1, T_2 be the impulsive tensions at the extremities of any arc of the chain, u_1, u_2 the initial velocities at the extremities resolved along the tangents at the extremities, prove that the initial kinetic energy of the whole arc is

$$\frac{1}{2}(T_2u_2-T_1u_1).$$

This readily follows by integrating $m(u^2+v^2)\,ds$ along the whole length of the arc. But it also follows at once from Art. 331, for the work done at either extremity is the product of the tension into half the initial tangential velocity.

Small Oscillations of a loose chain.

651. *A heavy heterogeneous chain is suspended by one extremity and hangs in a straight line under the action of gravity. A small disturbance being given to the chain in a vertical plane, it is required to find the equations of motion*.*

Let O be the point of support, let the axis Ox be measured vertically downwards and Oy horizontally in the plane of disturbance. Let mds be the mass of any elementary arc whose length PQ is ds, and let T be the tension at P. Let l be the length of the string, and let us suppose that a weight Mg is attached to the lower extremity.

The equations of motion as in Art. 635 will be

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{1}{m}\frac{d}{ds}\left(T\frac{dx}{ds}\right) + g \\ \frac{d^2y}{dt^2} &= \frac{1}{m}\frac{d}{ds}\left(T\frac{dy}{ds}\right) \end{aligned}\right\}\quad\text{.................(1).}$$

Since the motion is very small, the point P will oscillate in a very small arc, the tangent at the middle point being horizontal. Hence we may put $\dfrac{dx}{dt} = 0$. For a similar reason we may put $dx = ds$. We therefore have by integrating the first of equations (1)

$$T = \text{constant} - g\int m dx.$$

But $T = Mg$ when $x = l$, hence we find

$$T = Mg + g\int_x^l m dx \quad\text{......................(2).}$$

* In the *Seventh Volume of the Journal Polytechnique*, Poisson discusses the oscillations of a heavy homogeneous chain suspended by one extremity. Putting $(l-x)^{\frac{1}{2}} \pm \frac{1}{2} g^{\frac{1}{2}} t$ equal to s or s' according as the upper or lower sign is taken, and $y' = y(l-x)^{\frac{1}{4}}$, he reduces the equation to the form $\dfrac{d^2y'}{dsds'} = -\dfrac{1}{4}\dfrac{y'}{(s+s')^2}$. He obtains the integral by means of two definite integrals and two infinite series. After a rather long discussion of the forms of the arbitrary functions which occur in the integral, he finds that a solitary wave will travel up the chain with a uniform acceleration and down with a uniform retardation each equal to half that of gravity.

When the chain is homogeneous, this equation takes the simple form

$$T = Mg + mg(l - x) \text{.....................} (3).$$

It may be noticed that this expression is independent of the time; the tension at any point of the chain is equal to the total weight of matter below that point.

The second equation may be written in either of the forms

$$\left.\begin{aligned} \frac{d^2y}{dt^2} &= \frac{1}{m}\frac{d}{dx}\left(T\frac{dy}{dx}\right) \\ &= \frac{1}{m}T\frac{d^2y}{dx^2} + \frac{1}{m}\frac{dT}{dx}\frac{dy}{dx} \end{aligned}\right\} \text{.....................} (4),$$

where T is a function of x given by the equations (2) or (3).

652. Let us suppose that the displacements of the particles forming any finite portion of the chain during a finite time, are represented by $y = \phi(x, t)$, where ϕ is a continuous function of x and t. Let P be a geometrical point within this portion of the chain which moves so that the particle-velocity at P, *i.e.* $\frac{dy}{dt}$ is always equal to some constant quantity A. Let v be the velocity with which P moves, then following in our mind the motion of P, we have

$$\frac{d^2y}{dt^2} + \frac{d^2y}{dx\,dt}v = 0 \text{.......................} (5).$$

Let Q be a point also within the portion, such that the tangent to the chain at Q makes with the vertical an angle whose tangent, *i.e.* $\frac{dy}{dx}$, is $\frac{B}{T}$, where B is some constant quantity.

Let v' be the velocity with which Q moves, then

$$T\frac{d^2y}{dx\,dt} + \frac{d}{dx}\left(T\frac{dy}{dx}\right)v' = 0 \text{..................} (6).$$

Eliminating the second differential coefficients of y from equations (4), (5) and (6), we easily deduce that if P and Q coincide at any instant,

$$vv' = \frac{T}{m} \text{..............................} (7).$$

This reasoning requires that all the second differential coefficients should be finite, and that y should be a continuous function of x and t. It would not apply to any point P, if the discontinuous extremities of two waves were passing over P in opposite directions. But the consideration of these exceptions is unnecessary for our present purpose.

Let AB be a disturbed portion of the chain travelling in the direction AB on a chain otherwise in equilibrium. At the confines of the disturbance the two portions of the string must not make a finite angle with each other. If they did, an element of the string would be acted on by a finite moving force, which is the resultant of the two finite tensions at its extremities. In such a case the disturbance would instantly extend itself further along the chain and take up some new form. Supposing we exclude any such case as this, we must have, as long as the motion is finite, both $\dfrac{dy}{dt}=0$, and $\dfrac{dy}{dx}=0$, at both the upper and lower extremity of the disturbance. If then P be a point at which $\dfrac{dy}{dt}=0$, and Q a point at which $\dfrac{dy}{dx}=0$, P and Q may be considered as taken just within the boundary of the wave; P and Q will therefore each travel with the velocity of that boundary. Hence putting $v=v'$, we find for the velocity of either point

$$v^2=\frac{T}{m}\quad\text{.........................}(8).$$

It appears therefore that if a solitary wave travel up the chain, the velocity increases as the wave approaches the upper extremity. The upper end of the wave will travel a little quicker than the lower end, because the tension at the upper end exceeds that at the lower; thus the length of the wave will gradually increase. When the wave travels down the chain, the velocity for the same reason decreases.

653. Ex. 1. If the chain be homogeneous, show that the boundaries of a solitary wave will travel up the chain with an acceleration equal to half that of gravity, and down the chain with a retardation of the same numerical amount.

Ex. 2. Let the law of density be $m=A(l+l'-x)^{-\frac{1}{2}}$ where l is the length of the chain and A, l' two constants. Also let a weight equal to $2Ag\sqrt{l'}$ be fastened to the lower extremity, prove that

$$y=f\left\{(l+l'-x)^{\frac{1}{2}}-\left(\frac{g}{2}\right)^{\frac{1}{2}}t\right\}+F\left\{(l+l'-x)^{\frac{1}{2}}+\left(\frac{g}{2}\right)^{\frac{1}{2}}t\right\}.$$

This integration may be effected by writing $\theta=(l+l')^{\frac{1}{2}}-(l+l'-x)^{\frac{1}{2}}$. The equation of motion then takes the form $\dfrac{d^2y}{dt^2}=\dfrac{g}{2}\dfrac{d^2y}{d\theta^2}$, which can be solved in the usual manner.

Ex. 3. The chain is said to sound an harmonic note when its motion can be represented by an expression of the form $y=\phi(x)\sin(\kappa t+a)$; so that the motion of every element repeats itself at the same constant interval. Show that the harmonic periods of the chain and weight are given by

$$\kappa l'^{\frac{1}{2}}\tan\kappa\{(l+l')^{\frac{1}{2}}-l'^{\frac{1}{2}}\}=1\quad\text{...... ....}(1).$$

To prove this, we substitute $y=f(\theta)\sin(\kappa t+\alpha)$ in the differential equation obtained in the last Example; we thus find $f(\theta)$ to be trigonometrical. Since $y=0$ when $x=0$ for all values of t, the expression for y reduces to

$$y=\sin\kappa\theta\left\{A_\kappa\sin\kappa t\left(\frac{g}{2}\right)^{\frac{1}{2}}+B_\kappa\cos\kappa t\left(\frac{g}{2}\right)^{\frac{1}{2}}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where A_κ and B_κ are two arbitrary constants. But when $x=l$, y must satisfy the equation of motion of the weight, viz. $\frac{d^2y}{dt^2}=-g\frac{dy}{dx}$. Whence the result follows by substitution.

Ex. 4. If the initial motion of the chain and weight be given by the equations $y=f(x)$, $\frac{dy}{dt}=F(x)$ when $t=0$, then y can be expanded in a series, the general term of which is expressed by equation (2) of the last example. Find the values of A_κ and B_κ.

We notice that equation (1) of the last example may be written in the form

$$\cos\kappa\theta_1=\kappa\sqrt{l'}\sin\kappa\theta_1,$$

where θ_1 is the value of θ when $x=l$. We then easily find that

$$\int_0^{\theta_1}\sin\kappa\theta\sin\kappa'\theta d\theta=-\sqrt{l'}\sin\kappa\theta_1\sin\kappa'\theta_1,$$

$$\int_0^{\theta_1}\sin^2\kappa\theta d\theta=\frac{1}{2}\theta_1-\frac{1}{2}\sqrt{l'}\sin^2\kappa\theta_1.$$

These results may be obtained by integrating the left-hand sides and substituting for $\cos\kappa\theta_1$ and $\cos\kappa'\theta_1$ their values in terms of $\sin\kappa\theta_1$ and $\sin\kappa'\theta_1$.

If we now multiply both sides of equation (2) by $\sin\kappa\theta$ and integrate from $\theta=0$ to $\theta=\theta_1$, we find by the use of these two results

$$\frac{1}{2}B_\kappa(\theta_1+\sqrt{l'}\sin^2\kappa\theta_1)=\int_0^{\theta_1}y\sin\kappa\theta d\theta+f(l)\sqrt{l'}\sin\kappa\theta_1.$$

Differentiating (2) and performing the same process, we have

$$\frac{\kappa}{2}A_\kappa\sqrt{\frac{g}{2}}(\theta_1+\sqrt{l'}\sin^2\kappa\theta_1)=\int_0^{\theta_1}\frac{dy}{dt}\sin\kappa\theta d\theta+F(l)\sqrt{l'}\sin\kappa\theta_1.$$

654. *An inelastic heterogeneous chain is suspended from two fixed points under the action of gravity. Any small disturbance being given in its own plane, it is required to find the small oscillations.*

Let the axis of x be horizontal and that of y vertical. Let C be any point on the chain when hanging in equilibrium, and let the arc s be measured from C. Let (x, y) be the co-ordinates of any point P determined by $CP=s$. Let T be the tension at P, $mgds$ the weight of an element ds situated at P. The equations of equilibrium are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)=0,\quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)-mg=0.$$

Let α be the angle the tangent at P makes with the axis of x, then we easily find

$$T = \frac{wg}{\cos\alpha}, \quad m = w\frac{d\tan\alpha}{ds} \quad\text{.....................} \quad (1),$$

where w is an undetermined constant.

When the chain is in motion, let $(x+\xi, y+\eta)$ be the coordinates of the position of the particle P at the time t, and let the tension at that point be $T' = T + U$. The equations of motion will be

$$\frac{d^2\xi}{dt^2} = \frac{1}{m}\frac{d}{ds}\left\{T'\left(\frac{dx}{ds} + \frac{d\xi}{ds}\right)\right\},$$

$$\frac{d^2\eta}{dt^2} = \frac{1}{m}\frac{d}{ds}\left\{T'\left(\frac{dy}{ds} + \frac{d\eta}{ds}\right)\right\} - g,$$

which, by subtracting the equations of equilibrium, reduce to

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= \frac{1}{m}\frac{d}{ds}\left(T\frac{d\xi}{ds} + U\frac{dx}{ds}\right) \\ \frac{d^2\eta}{dt^2} &= \frac{1}{m}\frac{d}{ds}\left(T\frac{d\eta}{ds} + U\frac{dy}{ds}\right) \end{aligned}\right\} \quad\text{.................} \quad (2),$$

when the squares of small quantities are neglected.

Since the string is inelastic, we have

$$(dx + d\xi)^2 + (dy + d\eta)^2 = (ds)^2.$$

Expanding and rejecting the squares of small quantities, this becomes

$$\frac{dx}{ds}\frac{d\xi}{ds} + \frac{dy}{ds}\frac{d\eta}{ds} = 0 \quad\text{.....................} \quad (3).$$

We have thus three equations to find ξ, η and U as functions of s and t.

655. *To find the velocity with which a solitary wave will travel along the chain.*

If we suppose a small disturbance to travel along this chain, so that there is no abrupt change of direction of the chain at the boundaries of the wave, we must have at those points $\frac{d\xi}{ds} = 0$, $\frac{d\eta}{ds} = 0$, $\frac{d\xi}{dt} = 0$, $\frac{d\eta}{dt} = 0$, and $U = 0$. Let v be the velocity with which one boundary of this wave travels along the chain, then, following that boundary in our mind, we have

$$\frac{d^2\xi}{dt^2} + v\frac{d^2\xi}{ds\,dt} = 0, \quad \frac{d^2\xi}{dt\,ds} + v\frac{d^2\xi}{ds^2} = 0,$$

and therefore $$\frac{d^2\xi}{dt^2}=v^2\frac{d^2\xi}{ds^2},$$

with a similar equation for η. Thus the dynamical equations become at the boundary

$$\left.\begin{aligned}\left(v^2-\frac{T}{m}\right)\frac{d^2\xi}{ds^2}&=\frac{1}{m}\frac{dU}{ds}\frac{dx}{ds}\\ \left(v^2-\frac{T}{m}\right)\frac{d^2\eta}{ds^2}&=\frac{1}{m}\frac{dU}{ds}\frac{dy}{ds}\end{aligned}\right\},$$

and the geometrical equation becomes

$$\frac{d^2\xi}{ds^2}\frac{dx}{ds}=-\frac{d^2\eta}{ds^2}\frac{dy}{ds}.$$

From these we easily get $v^2=\frac{T}{m}$. Substituting for T and m their values, we have if ρ be the radius of curvature at P,

$$v=\sqrt{(g\rho\cos\alpha)}\ \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

so that the velocity of either boundary of the wave is that due to one quarter of the vertical chord of curvature at that point.

Ex. 1. A chain is in equilibrium under the action of any forces which are functions only of the position in space of the element acted on. Show that the velocity of either boundary of a solitary wave is that due to one quarter of the chord of curvature in the direction of the resultant force at that boundary.

656. *To solve as far as possible the equations of motion of a heavy slack heterogeneous chain.*

It will be convenient to express the unknown quantities ξ, η, U in terms of some *one* function ϕ.

Let $\alpha+\phi$ be the angle the tangent at P makes with the horizon at the time t. Then

$$\cos(\alpha+\phi)=\frac{dx+d\xi}{ds},\qquad \sin(\alpha+\phi)=\frac{dy+d\eta}{ds};$$

$$\therefore\ -\phi\sin\alpha=\frac{d\xi}{ds},\qquad \phi\cos\alpha=\frac{d\eta}{ds}\ \ldots\ldots\ldots\ldots(5);$$

$$\therefore\ \frac{d\xi}{d\alpha}=-\rho\phi\sin\alpha,\qquad \frac{d\eta}{d\alpha}=\rho\phi\cos\alpha\ \ldots\ldots\ldots\ldots(6),$$

$$\xi=-\int\rho\phi\sin\alpha d\alpha+A,\qquad \eta=\int\rho\phi\cos\alpha d\alpha+B\ \ldots\ldots\ldots\ldots(7),$$

where A and B are two undetermined functions of t.

The equations (2) now become

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}\frac{1}{\cos^2\alpha}&=\frac{d}{d\alpha}\left(-g\phi\tan\alpha+\frac{U}{w}\cos\alpha\right)\\ \frac{d^2\eta}{dt^2}\frac{1}{\cos^2\alpha}&=\frac{d}{d\alpha}\left(g\phi+\frac{U}{w}\sin\alpha\right)\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(8).$$

For the sake of brevity let accents denote differentiations with regard to t. Expanding the differentiations on the right-hand side, these equations may be written in the form

$$\left.\begin{aligned} -\xi''\sin\alpha+\eta''\cos\alpha-g\left(\phi\sin\alpha+\frac{d\phi}{d\alpha}\cos\alpha\right)&=U\frac{\cos^2\alpha}{w}\\ \xi''\cos\alpha+\eta''\sin\alpha+g\phi\cos\alpha\qquad\qquad&=\frac{dU}{d\alpha}\frac{\cos^2\alpha}{w}\end{aligned}\right\}.$$

Differentiating the first with regard to α and adding the result to the second, we obtain

$$\frac{\rho\phi''}{\cos\alpha}-g\frac{d^2\phi}{d\alpha^2}=2\frac{d}{d\alpha}\left(\frac{U\cos\alpha}{w}\right).$$

Differentiating the second and subtracting the first from the result, we obtain

$$2g\frac{d\phi}{d\alpha}=\frac{d^2}{d\alpha^2}\left(\frac{U\cos\alpha}{w}\right).$$

These equations evidently give

$$U\cos\alpha=wg\left(2\int\phi d\alpha+C\alpha+D\right)\text{..........................(9)},$$

$$\frac{d^2\phi}{dt^2}=g\frac{\cos\alpha}{\rho}\left(\frac{d^2\phi}{d\alpha^2}+4\phi+2C\right)\text{..........................(10)},$$

where C and D are two undetermined functions of t. These are the general equations to determine the small oscillations of a slack chain.

The undisturbed form of the curve being given, ρ is known as a function of α. We may then use the equation (10) to find ϕ as a function of α and t. The tension is then found from the equation (9), and the displacements ξ, η of any point of the chain by equations (7).

657. The determination of the whole motion depends therefore on the solution of a single equation. Supposing the integration to have been effected, the expression for ϕ will contain two new arbitrary functions of α and t. These we may represent by $\psi(P)$ and $\chi(Q)$ where ψ and χ are arbitrary functions of two determinate combinations P and Q of the variables. The arbitrary functions A and B are not independent of C and D, and the relations between them may be found by substituting in equations (8).

We have thus four arbitrary functions whose values have to be determined from the conditions of the question. Let α_0, α_1, be the values of α which correspond to the two extremities of the string. Then the values of ϕ and $\frac{d\phi}{dt}$ are given by the question when $t=0$ for all values of α from $\alpha=\alpha_0$ to $\alpha=\alpha_1$; also the initial values of A and B are given. Thus the values of $\psi(P)$ and $\chi(Q)$ are determined for all values of P and Q between the two limits which correspond to $\alpha=\alpha_0$, $t=0$ and $\alpha=\alpha_1$, $t=0$. The forms of ψ and χ for values of P and Q exterior to these limits, and the values of A and B when t is not zero, are to be found from the conditions at the extremities of the chain. If the extremities be fixed, we have both ξ and η equal to zero for all values of t when $\alpha=\alpha_0$ and $\alpha=\alpha_1$. It may thus happen that the arbitrary functions A, B, ψ and χ are discontinuous.

In many cases the circumstances of the problem will enable us to determine at once the form of C. Thus, suppose the string when in equilibrium to be symmetrical about a vertical line, say the axis of y, and let the points of support be

fixed in the same horizontal line. Then if the initial motion be also symmetrical about the axis of y, the whole subsequent motion will be symmetrical. Thus ϕ must be a function of a, containing when expanded only odd powers of a. Substituting such a series in equation (10) we see that C must be zero.

658. There are several cases in which the equation to find the small motions of a chain may be more or less completely integrated. One of the most interesting of these is that in which the chain hangs in equilibrium in the *form of a cycloid.* In this case we have, if b be the radius of the generating circle, $\rho = 4b \cos a$. The density of the chain at any point is given by $m = \dfrac{w}{4b \cos^2 a}$, so that all the lower part of the chain is of nearly uniform density, but the density increases rapidly higher up the chain and is infinite at the cusp.

The equation to find the oscillations now takes the simple form

$$\frac{d^2\phi}{dt^2} = \frac{g}{4b}\left\{\frac{d^2\phi}{da^2} + 4\phi + 2C\right\} \quad\text{.............................(11)},$$

in which all the coefficients are constants.

There are two cases of motion to be discussed, (1) when the chain swings up and down, and (2) when it swings from side to side. The results are indicated in the two following examples.

Ex. 1. *A heavy chain suspended from two points in the same horizontal line hangs under gravity in the form of a cycloid. Find the symmetrical oscillations of the chain, when the lowest point moves only up and down.*

In this case we have $C = 0$. To find the nature and time of a small oscillation, we put

$$\phi = \Sigma R \sin \kappa t + \Sigma R' \cos \kappa t,$$

where Σ implies summation for all values of κ, and R, R' are functions of a only. Substituting, we have

$$\frac{d^2R}{da^2} + 4\left(1 + \frac{b\kappa^2}{g}\right)R = 0;$$

with a similar equation to find R'. Therefore

$$R = L \sin 2\sqrt{\left(1 + \frac{b\kappa^2}{g}\right)}\, a,$$

where L is an arbitrary constant, the other constant being determined by the consideration that the motion is symmetrical about the axis of y. For the sake of brevity, put $\lambda = 2\sqrt{\left(1 + \dfrac{b\kappa^2}{g}\right)}$. Substituting in (7), we find that the terms derived from R become

$$\xi = \Sigma L \frac{2b}{\lambda^2 - 4}\{\lambda \cos \lambda a \sin 2a - 2 \sin \lambda a \cos 2a\} \sin \kappa t,$$

$$\eta = \Sigma\left[-L\frac{2b}{\lambda^2 - 4}\{\lambda \cos \lambda a \cos 2a + 2 \sin \lambda a \sin 2a\} - L\frac{2b}{\lambda}\cos \lambda a + H\right]\sin \kappa t,$$

where H is a constant depending on the position of the points of support. The terms derived from R' must be added to these, but have been omitted for the sake of brevity. They may be derived from those just written down by writing $\cos \kappa t$ for $\sin \kappa t$ and changing the constants L, H into two other constants L', H'.

Let the length of the chain be $2l$, then at either end $\sin \alpha_0 = \frac{l}{4b}$. At both extremities we must have $\xi = 0$, $\eta = 0$. All these four conditions can be satisfied if

$$\frac{\tan \lambda \alpha_0}{\lambda} = \frac{\tan 2\alpha_0}{2}.$$

This equation therefore determines the possible times of symmetrical vibration of a heterogeneous chain hanging in the form of a cycloid.

659. *If a be not very large, the oscillations are nearly the same as those of a uniform chain**. In this case since α_0 is small but $\lambda\alpha_0$ is not necessarily small, the equation to determine λ is approximately

$$\tan \lambda \alpha_0 = \lambda \alpha_0.$$

The least value of $\lambda\alpha$ which can be taken is a little less than $\frac{3\pi}{2}$. Hence λ is great, and therefore $\kappa = \sqrt{\left(\frac{g}{4b}\right)}\lambda$ nearly. The expressions for ξ and η now take the simple forms

$$\left.\begin{aligned} \xi &= \Sigma L \frac{4b}{\lambda^2} \{\lambda\alpha \cos \lambda\alpha - \sin \lambda\alpha\} \sin \left(\sqrt{\frac{g}{4b}}\,\lambda t + \epsilon\right) \\ \eta &= \Sigma L \frac{4b}{\lambda} \{\cos \lambda\alpha_0 - \cos \lambda\alpha\} \sin \left(\sqrt{\frac{g}{4b}}\,\lambda t + \epsilon\right) \end{aligned}\right\}.$$

The terms depending on $\cos \kappa t$ have been included in these expressions for ξ and η by introducing ϵ into the trigonometrical factor.

The roots of the equation $\tan \lambda\alpha_0 = \lambda\alpha_0$ may be found by continued approximation. The first is zero, but since λ occurs in the denominator of some of the small terms, this value is inadmissible. The others may be expressed by the formula $\lambda\alpha_0 = (2i+1)\frac{\pi}{2} - \theta$, where θ is not very large. This makes the time of vibration nearly equal to $\frac{4}{2i+1} \cdot \frac{l}{\sqrt{4gb}}$. Thus the times of vibration of the chain are all short.

This result will explain why the marching of troops in time along a suspension bridge may cause oscillations which are so great as to be dangerous to the bridge. It is clearly possible that the "marching time" may be equal to, or very nearly equal to some one of the times of vibrations of the bridge. If this should occur it follows from Arts. 498 and 503 that the stability of the bridge may be severely strained.

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* The reader who may wish to see another method of discussing the small oscillations of a suspension chain may consult a memoir by Mr Röhrs in the ninth volume of the *Cambridge Transactions*. Mr Röhrs considers the chain to be homogeneous, symmetrical about the vertical, and nearly horizontal from the beginning of the process. In the second edition of this treatise the small oscillations were also treated on the same hypothesis, but in a different manner. That method, however, is not nearly so simple as the one here given in which the approximate oscillations for a catenary are deduced from the accurate ones for a cycloid.

It should be noticed that the terms in the expression for ξ have the square of λ in the denominator, while those in the expression for η have the first power of λ. Since λ is great we might as a first approximation reject the values of ξ altogether, and regard each element of the chain as simply moving up and down.

660. Ex. 2. *A heavy chain suspended from two points hangs under gravity in the form of a cycloid. If it swings from side to side in its own plane so that the middle point has only a lateral motion without any perceptible vertical motion, find the times of oscillation.*

As in the last example, we put

$$\phi = \Sigma R \sin \kappa t + \Sigma R' \cos \kappa t,$$

where R and R' are functions of a only. Substituting in equation (11) we see that $2C = \Sigma h \sin \kappa t + \Sigma k \sin \kappa t$ where h and k are arbitrary constants. The equation to find R becomes

$$\frac{d^2R}{da^2} + 4\left(1 + \frac{b\kappa^2}{g}\right) R = -h.$$

If we put $\lambda^2 = 4\left(1 + \dfrac{b\kappa^2}{g}\right)$ as before, we find $R = -\dfrac{h}{\lambda^2} + L \sin(\lambda a + M)$.

Thence taking the term of ϕ which contains $\sin \kappa t$,

$$\frac{\xi}{\sin \kappa t} = \frac{h' - hb \cos 2a}{\lambda^2} + L \frac{2b}{\lambda^2 - 4} \{\lambda \cos(\lambda a + M) \sin 2a - 2 \sin(\lambda a + M) \cos 2a\},$$

where h' is an arbitrary constant introduced on integration. Substituting in equation (8), we find $h' = -h\left(b + \dfrac{g}{\kappa^2}\right)$. Also, we have in the same way

$$\frac{\eta}{\sin \kappa t} = -\frac{hb}{\lambda^2}(2a + \sin 2a)$$

$$-L \frac{2b}{\lambda^2 - 4}\{\lambda \cos(\lambda a + M) \cos 2a + 2 \sin(\lambda a + M) \sin 2a\} - L \frac{2b}{\lambda} \cos(\lambda a + M) + H.$$

If we suppose the two supports to be on the same horizontal line, we must have $\xi = 0$ and $\eta = 0$, when $a = \pm a_0$. These conditions may be satisfied if we take $M = \dfrac{\pi}{2}$, $H = 0$, for then ξ becomes an even and η an odd function of a. In this case $\eta = 0$ at the lowest point of the chain. We have then two equations to find $\dfrac{L}{h}$, equating these values, we have

$$\frac{2 \tan 2a_0 - \lambda \tan \lambda a_0 - \dfrac{\tan \lambda a_0}{\cos 2a} \dfrac{\lambda^2 - 4}{\lambda}}{2a_0 + \sin 2a_0} = \frac{\lambda \tan \lambda a_0 \tan 2a_0 + 2}{2 \cos^2 a_0 + \dfrac{4}{\lambda^2 - 4}}.$$

661. If a_0 be small, this equation is very nearly satisfied by $\lambda a_0 = i\pi$ where i is any integer. In this case the complete expressions for ξ and η take the simple forms

$$\left.\begin{aligned} \xi &= \Sigma L \frac{4b}{\lambda^2}(\cos \lambda a_0 - \cos \lambda a - \lambda a \sin \lambda a) \sin\left(\sqrt{\frac{g}{4b}}\, \lambda t + \epsilon\right) \\ \eta &= \Sigma L \frac{4b}{\lambda} \sin \lambda a \sin\left(\sqrt{\frac{g}{4b}}\, \lambda t + \epsilon\right) \end{aligned}\right\}.$$

662. Ex. 1. If we change the variables from a, t to p, q where

$$p=t+\int\sqrt{\frac{\rho}{g\cos a}}\,da,\qquad q=-t+\int\sqrt{\frac{\rho}{g\cos a}}\,da,$$

show that the general equation (10) of small oscillations takes the form

$$\frac{d^2\phi'}{dpdq}+\frac{\mu^3}{4}\left(\frac{d^2\mu}{da^2}+4\mu\right)\phi'=-\frac{\mu^3}{2}C,$$

where $\mu^4=\dfrac{g\cos a}{\rho}$ and $\phi=\mu\phi'$.

Show also that the coefficient of ϕ' is a function of $p+q$, the form of the function depending on the law of density of the chain.

This transformation may be useful, because it follows from Art. 655 that p is constant for the boundaries of a solitary wave travelling in one direction, and q for a wave travelling in the other direction.

Ex. 2. A heavy string hangs in equilibrium under gravity in such a form that its intrinsic equation is $\dfrac{\cos a}{\rho}=\dfrac{b^4}{g}\sin^4(2a+c)$ where b and c are any constants. Show that its law of density is given by $m=w\,\dfrac{b^4}{g}\,\dfrac{\sin^4(2a+c)}{\cos^3 a}$. If such a chain be set in motion in any symmetrical manner, prove that its motion is given by

$$\phi=b\sin(2a+c)\left\{F\left(t-\frac{\cot(2a+c)}{2b^2}\right)+f\left(t+\frac{\cot(2a+c)}{2b^2}\right)\right\}.$$

Ex. 3. If in addition to gravity, each element of the chain be acted on by a small normal force whose magnitude is Fg, prove that the equation of motion of the chain is

$$\frac{\rho}{g\cos a}\frac{d^2\phi}{dt^2}-\frac{d^2\phi}{da^2}-4\phi-2C=\frac{1}{\cos a}\frac{dF}{da}+2\int\frac{F}{\cos a}\,da.$$

If the chain is nearly horizontal, so that a is very small, and if $F=f\sin(at-ca)$, prove that the denominator of the corresponding term in the expression for ϕ is

$$g(c^2-4)-\rho a^2.$$

Ex. 4. A heavy chain of length $2l$ is suspended from two points A, B in the same horizontal line whose distance apart is not very different from $2l$. Each particle of the chain is slightly disturbed from its position of rest in a direction perpendicular to the vertical plane through AB. Find the small oscillations of the chain.

Ex. 5. A heavy string is suspended from two fixed points A and B and rests in equilibrium in the form of a catenary whose parameter is c. Let the string be initially displaced, the points of support A, B being also moved, so that

$$\phi=\sigma(1+\cos 2a)+\sigma'\sin 2a,$$

where σ and σ' are two small quantities and the other letters have the same meaning as in Art. 656. If the string be placed at rest in this new position, prove that it will always remain at rest.

Small Oscillations of a tight string.

663. *An elastic string whose weight may be neglected and whose unstretched length is* l *has its extremities fixed at two points whose distance apart is* l′. *The string being disturbed so that each particle is moved along the length of the string, find the equations of motion.*

Let A be one of the fixed points, and let AB be the string when unstretched and placed in a straight line. Let the extremity B be pulled until it reaches the other fixed point B'. Let PQ be any element of the unstretched string, $P'Q'$ the same element at the time t. Let $AP=x$ and let the abscissa AP' be x'. Let T and $T+dT$ be the tensions at P' and Q'. Let M be the mass of the whole string, m the mass of a unit of length of *unstretched* string. Then, as in Art. 637, the equation of motion is

$$m\frac{d^2x'}{dt^2}=\frac{dT}{dx} \quad\text{.......................................}\quad (1).$$

If E be the modulus of elasticity, we have by Hooke's law

$$\frac{dx'}{dx}=1+\frac{T}{E} \quad\text{.......................................}\quad (2).$$

Eliminating T, we have

$$\frac{d^2x'}{dt^2}=\frac{E}{m}\frac{d^2x'}{dx^2} \quad\text{.......................................}\quad (3).$$

If we put $E=ma^2$, the integral of this equation is

$$x'=f(at-x)+F(at+x),$$

where f and F are two arbitrary functions.

The discussion of this equation may be found in any treatise on Sound. The result is, that a function of the form $\phi(at-x)$ represents a wave which travels with a velocity equal to a. In the case therefore of the string, the motion will be represented by a series of waves travelling both ways along the string with the same velocity. This velocity is such that the time of traversing a length l of unstretched string or a length l' of stretched string is $l\sqrt{\frac{m}{E}}$. It should be noticed that this time is independent of the nature of the disturbance, and is the same whether the string be originally stretched or not.

It should also be noticed, that assuming as usual the truth of Hooke's law, the equation (3) and these results are not merely approximations, but are strictly accurate.

It is often more convenient to select some particular state of the string as a standard of reference and to express the actual position of any particle at the time t by its displacement from its position in this standard. Thus if the unstretched state AB of the string be chosen as the standard of reference, we put $x'=x+\xi$, so that ξ is the displacement of the particle whose abscissa in the unstretched state is x. The equation of motion now takes the form

$$\frac{d^2\xi}{dt^2}=\frac{E}{m}\frac{d^2\xi}{dx^2},$$

and the integral may be obtained as before.

664. ***An elastic string being stretched as in the last proposition is slightly disturbed in any manner, find the equations of motion.***

Following the same notation as before, let (x', y', z') be the co-ordinates of P'. Then, as in Art. 637, the equations of motion are

$$m\frac{d^2x'}{dt^2} = \frac{d}{dx}\left(T\frac{dx'}{ds'}\right) \text{.............................}(1),$$

$$m\frac{d^2y'}{dt^2} = \frac{d}{dx}\left(T\frac{dy'}{ds'}\right) \text{.............................}(2),$$

$$m\frac{d^2z'}{dt^2} = \frac{d}{dx}\left(T\frac{dz'}{ds'}\right) \text{.............................}(3),$$

where ds' is the length of the element $P'Q'$. If E be the modulus of elasticity we have by Hooke's law

$$\frac{ds'}{dx} = 1 + \frac{T}{E} \text{.....................................}(4).$$

Since the disturbance is very small $\frac{dy'}{ds'}$ and $\frac{dz'}{ds'}$ are very small and $\frac{dx'}{ds'}$ is very nearly equal to unity. Hence the first equation takes the form

$$m\frac{d^2x'}{dt^2} = \frac{dT}{dx},$$

and Hooke's equation takes the form

$$\frac{dx'}{dx} = 1 + \frac{T}{E},$$

which are the same equations as in the last proposition, so that when the disturbance is small the longitudinal motion is independent of the motion transverse to the string.

In the second equation we may regard T as constant, its small variations being multiplied by the small quantity $\frac{dy'}{ds'}$. Hence we may put $T = T_0$ where $T_0 = E\,\frac{l'-l}{l}$. This gives by equation (4) $\frac{ds'}{dx} = \frac{l'}{l}$. The equation of motion therefore becomes

$$\frac{d^2y'}{dt^2} = \frac{T_0}{m}\frac{l}{l'}\frac{d^2y'}{dx^2}.$$

The third equation may be treated in the same way.

The velocity of a transverse vibration measured in units of length of unstretched string is therefore $\sqrt{\frac{T_0 l}{ml'}}$. The time of traversing a length l of unstretched string or l' of stretched string is $\sqrt{\frac{mll'}{T_0}}$. This velocity is independent of the nature of the disturbance but depends on the tightness or tension of the string.

If the string be very slightly elastic we may, in this last formula, put $l' = l$. In this case we obtain the results given in all treatises on Sound.

665. There are two modes of applying the equations of motion to actual cases. We shall first illustrate these by solving a simple example by both methods, and we shall then make some remarks on the results.

An elastic string whose unstretched length is l *rests on a perfectly smooth table and has its extremities fixed at two points* A, B′ *whose distance apart is* l′, *where* l′ *is greater than* l. *The extremity* B′ *is suddenly released, find the motion.*

Following the same notation as in Art. 663, the motion is given by the equation

$$\xi = f(at - x) + F(at + x),$$

where ξ is the displacement of the particle whose abscissa in the unstretched string is x. The conditions to determine f and F are as follows.

1. When $x=0$, $\xi=0$ for all values of t.
2. When $x=l$, $T=0$ and $\therefore \dfrac{d\xi}{dx}=0$ for all values of t.
3. When $t=0$, $\xi=rx$ from $x=0$ to $x=l$, where $l'=(r+1)\,l$.
4. When $t=0$, $\dfrac{d\xi}{dt}=0$ from $x=0$ to $x=l$.

From the first condition it follows that the functions F and f are the same with opposite signs. From the second condition we have $f'(at+l)=-f'(at-l)$, so that the values of the function f' recur with opposite signs when the variable is increased by $2l$. If then we knew the values of $f'(z)$ for all values of z from $z=z_0$ to $z=z_0+2l$ where z_0 has any value, then the form of the function is altogether known. Now the third condition gives $f(-x)-f(x)=rx$ and the fourth gives $f'(-x)=f'(x)$ from $x=0$ to $x=l$. Hence $f'(x)=-\dfrac{r}{2}$ from $x=-l$ to $x=l$. It follows that $f'(z)=-\dfrac{r}{2}$ from $z=-l$ to l, $f'(z)=\dfrac{r}{2}$ from $z=l$ to $3l$ and so on changing sign every time the variable passes the values l, $3l$, $5l$, &c. Let us consider the motion of any point P of the string whose unstretched abscissa is x. Its velocity is given by the formula $\dfrac{v}{a}=f'(at-x)-f'(at+x)$. Since $x<l$ we have $\dfrac{v}{a}=-\dfrac{r}{2}+\dfrac{r}{2}=0$; hence the particle does not move until $at+x=l$. The second function then changes sign and we have $\dfrac{v}{a}=-\dfrac{r}{2}-\dfrac{r}{2}=-r$. The particle continues to move with this velocity until $at-x=l$, when the first function changes sign and so on. Let AB be the unstretched string, and let a point R starting from B move continually along the string and back again with velocity a. Then it is easy to see that when R is on the same side of P as the loose end of the string, P will be at rest, and when R is on the same side of P as the fixed end, P will be moving with a velocity alternately equal to $\pm ra$. The general character of the motion is; the equilibrium of the string being disturbed at B, a wave of length $4l$ travels along the string, so that P does not begin to move until the wave reaches it. This wave is reflected at A and returns.

666. The second method of conducting the solution is as follows. Taking as before the expression

$$\xi = f(at - x) + F(at + x),$$

let us expand each function in a series of sines and cosines, so that we have

$$\xi = \Sigma\,[A \sin\{n(at-x)+\alpha\} + B \sin\{n(at+x)+\beta\}],$$

where Σ implies summation for all values of n, and A, B, α and β are constants which are different in every term and may conveniently be regarded as functions of n.

Since the motion is oscillatory, we may suppose that all the values of n are real and it is clear that without loss of generality we may restrict n to be positive. We do not propose to discuss the circumstances under which these suppositions may be correctly made. For these we must refer the reader to Fourier's theorem. We may here regard the assumptions as justified by the result, because we can thus satisfy all the data of the question.

The four conditions of the problem enable us to determine the constants. From the first condition we have $\beta=\alpha+\kappa\pi$, $B=(-1)^{\kappa+1}A$ where κ is any integer. It easily follows, by expanding, that ξ may be written in the form

$$\xi=\Sigma\,(C\sin nat+D\cos nat)\sin nx,$$

where C and D are to be regarded as functions of n. From the second condition we have $\cos nl=0$, hence $nl=(2i+1)\dfrac{\pi}{2}$ where i is any positive integer. The possible harmonic periods (see Arts. 412 and 450) of the string, with proper initial disturbances, one end being fixed and the other loose, are therefore included in the form $\dfrac{4l}{(2i+1)a}$.

The initial disturbance is given by the third and fourth conditions. We have

$$\Sigma D\sin nx=rx, \qquad \Sigma Cn\sin nx=0.$$

To find the value of D in any term we multiply the first equation by the coefficient of D in that term and integrate throughout the length of the string, i.e. from $x=0$ to $x=l$. This gives

$$D\,\frac{l}{2}=r\int_0^l x\sin nx\,dx=r\,\frac{\sin nl}{n^2}.$$

The other terms all vanish since $\int_0^l \sin nx\sin n'x\,dx=0$, when n and n' are numerically unequal.

Treating the second equation in the same way, we find $C=0$. Hence the motion is given by

$$\xi=\Sigma\,\frac{2r}{l}\,\frac{\sin nl}{n^2}\cos nat\sin nx.$$

Writing for i its values 1, 2, 3, &c. successively, this equation becomes when written at length

$$\xi=\frac{8rl}{\pi^2}\left\{\cos\frac{\pi at}{2l}\sin\frac{\pi x}{2l}-\frac{1}{3^2}\cos\frac{3\pi at}{2l}\sin\frac{3\pi x}{2l}+\frac{1}{5^2}\cos\frac{5\pi at}{2l}\sin\frac{5\pi at}{2l}-\&c.\right\}.$$

This is a convergent series for ξ and it may be a sufficient approximation to the motion to take only the first few terms. For example, suppose we reject all beyond the first two terms, and in order to compare the result with that obtained in the first solution let us put $at=\dfrac{1}{2}l$. If we trace the curve whose ordinate is $-\dfrac{d\xi}{dt}$ and abscissa x, we find that it resembles $\xi=0$ for small values of x, then rises with a point of contrary flexure and becomes nearly horizontal as x approaches l. This agrees very well with the former result.

667. If we examine these solutions, we shall see that we have two kinds of conditions to determine the arbitrary functions; (1) There are the conditions at the two extremities of the string. The peculiarity of these is, that they hold for all values of t. (2) There are the initial conditions of motion. The peculiarity of

these is, that they do *not* hold for all values of x, but only for all values within a certain range limited by the length of the string. The first set of conditions is used to determine the mode in which the values of the functions recur, so that when their values are known through a certain limited range, they will become known for all those values of the variable which occur in the problem. The second set of conditions is used to determine their values during this limited range.

The functions were found to be discontinuous. It may be objected that no notice was taken of any possible discontinuity in forming the equations of motion; and that therefore these equations cannot be applied, without further examination, to any cases which require the arbitrary functions introduced into the solution to be discontinuous. This question has been much discussed, but we have not space here to enter into it. We must refer the reader to De Morgan's *Differential Calculus*, Chap. XXI. Art. 92, where both a short history of the dispute between Lagrange and D'Alembert and a discussion of the difficulty may be found. See also the *Mécanique Analytique, Seconde Partie*, Sect. VI. § iv.

In the second form of the solution we replace the arbitrary functions by a convergent series of harmonic vibrations. Taking a finite number of terms as an approximation, we have a perfectly continuous solution whose initial conditions differ but slightly from those of the proposed problem. This difference is less and less, the more terms of the series are included in the solution.

In comparing the two results, we see that each form has its advantages. The first determines the motion by a simple formula. The second is more convenient when the harmonic periods are required.

668. Ex. A heavy elastic string AB whose unstretched length is l is suspended from a point A under the action of gravity. If ξ be the vertical displacement of any point whose distance from A is x when the string is unstretched, and if a be the velocity of a wave measured in units of unstretched length, prove that

$$\xi = -\frac{gx^2}{2a^2} + \frac{glx}{a^2} + f(at-x) - f(at+x),$$

where $f(z)$ recurs with an opposite sign when z is increased by $2l$. If the string is initially unstretched and at rest, prove that

$$f(z) = \pm\frac{gz^2}{4a^2} + \frac{glz}{2a^2},$$

the upper sign being taken when z lies between $-l$ and 0, and the lower when z lies between 0 and l. Thence show that the whole length oscillates between l and $l+\dfrac{gl^2}{a^2}$.

Taking the other form of solution, show that the harmonic periods are $p=\dfrac{4l}{(2i+1)a}$ where i is any integer.

Show also that

$$\xi = -\frac{gx^2}{2a^2} + \frac{glx}{a^2} - \frac{16gl^2}{\pi^3a^2}\Sigma\frac{\sin\left(\frac{2i+1}{2}\frac{\pi x}{l}\right)\cos\left(\frac{2i+1}{2}\frac{\pi at}{l}\right)}{(2i+1)^3},$$

the summation extending from $i=0$ to $i=\infty$.

669. *Three elastic strings* AB, BC, CD *of different materials are attached to each other at* B *and* C *and stretched in a straight line between two fixed points* A, D.

If the particles of the string receive any longitudinal displacements and start from rest, find the subsequent motion.

Let A be the origin, AD the direction in which x is measured. Let the unstretched lengths of AB, BC, CD be l_1, l_2, l_3. Let E_1, E_2, E_3 be their respective coefficients of elasticity, m_1, m_2, m_3 the masses of a unit of length of each string. For the sake of brevity let $E_1 = m_1 a_1^2$, $E_2 = m_2 a_2^2$, $E_3 = m_3 a_3^2$. Let the rest of the notation be the same as before.

When the string is stretched *in equilibrium* between the two fixed points A and D, let T_0 be the tension of the string. In this position the displacements of the elements of each string from their positions when unstretched may be written

$$\xi_1 = \frac{l_1' - l_1}{l_1} x = \frac{T_0}{E_1} x,$$

$$\xi_2 = l_1' - l_1 + \frac{l_2' - l_2}{l_2}(x - l_1) = \frac{T_0}{E_1} l_1 + \frac{T_0}{E_2}(x - l_1),$$

$$\xi_3 = \&c. = \frac{T_0}{E_1} l_1 + \frac{T_0}{E_2} l_2 + \frac{T_0}{E_3}(x - l_1 - l_2).$$

At the time t after the equilibrium has been disturbed, let these displacements be respectively $\xi_1 + \xi_1'$, $\xi_2 + \xi_2'$, $\xi_3 + \xi_3'$. We then have

$$\xi_1' = \Sigma L_1 \sin(n_1 x + M_1)\cos n_1 a_1 t,$$
$$\xi_2' = \Sigma L_2 \sin\{n_2(x - l_1) + M_2\}\cos n_2 a_2 t,$$
$$\xi_3' = \Sigma L_3 \sin\{n_3(x - l_1 - l_2) + M_3\}\cos n_3 a_3 t,$$

where Σ implies summation for all the harmonics. In order to compare the coefficients of the same harmonic we must suppose $n_1 a_1 = n_2 a_2 = n_3 a_3 = \dfrac{2\pi}{p}$, where p is the period of the harmonic.

To find the constants we have the conditions

when $x = 0$,	$x = l_1$,	$x = l_1 + l_2$,	$x = l_1 + l_2 + l_3$,
$\xi_1' = 0$,	$\xi_1' = \xi_2'$,	$\xi_2' = \xi_3'$,	$\xi_3' = 0$,
	$E_1 \dfrac{d\xi_1'}{dx} = E_2 \dfrac{d\xi_2'}{dx}$,	$E_2 \dfrac{d\xi_2'}{dx} = E_3 \dfrac{d\xi_3'}{dx}$.	

These give

$$M_1 = 0$$

$$\left.\begin{aligned} L_2 \sin M_2 &= L_1 \sin(n_1 l_1 + M_1) \\ E_2 n_2 L_2 \cos M_2 &= E_1 n_1 L_1 \cos(n_1 l_1 + M_1) \end{aligned}\right\},$$

$$\left.\begin{aligned} L_3 \sin M_3 &= L_2 \sin(n_2 l_2 + M_2) \\ E_3 n_3 L_3 \cos M_3 &= E_2 n_2 L_2 \cos(n_2 l_2 + M_2) \end{aligned}\right\},$$

$$0 = L_3 \sin(n_3 l_3 + M_3).$$

These give the following equations to find the M's;

$$0 = M_1, \quad \frac{\tan M_2}{E_2 n_2} = \frac{\tan(n_1 l_1 + M_1)}{E_1 n_1}, \quad \frac{\tan M_3}{E_3 n_3} = \frac{\tan(n_2 l_2 + M_2)}{E_2 n_2}, \quad 0 = \frac{\tan(n_3 l_3 + M_3)}{E_3 n_3}.$$

Solving these we find

$$\frac{\tan n_1 l_1}{E_1 n_1} + \frac{\tan n_2 l_2}{E_2 n_2} + \frac{\tan n_3 l_3}{E_3 n_3} = (E_2 n_2)^2 \frac{\tan n_1 l_1}{E_1 n_1} \cdot \frac{\tan n_2 l_2}{E_2 n_2} \cdot \frac{\tan n_3 l_3}{E_3 n_3}.$$

Substituting for n_1, n_2, n_3 in terms of p we have an equation to find the harmonics.

The values of p being known, it is clear that the preceding equations determine all the constants except L_1. We have therefore one constant undetermined for each harmonic. To find these we must have recourse to the initial conditions. The equations may be written in the forms

$$\xi_1' = \Sigma P_n \cos nat, \qquad \xi_2' = \Sigma Q_n \cos nat, \qquad \xi_3' = \Sigma R_n \cos nat,$$

where P_n, Q_n and R_n satisfy the equation $\dfrac{d^2P}{dx^2} = -n^2P$. We have therefore, after integration by parts,

$$m^2 \int P_m P_n dx = -\int P_n \frac{d^2P_m}{dx^2} dx = -P_n \frac{dP_m}{dx} + \frac{dP_n}{dx} P_m + n^2 \int P_m P_n dx.$$

Similar theorems apply to Q_n and R_n.

We also have the conditions

$$\begin{array}{lllll} \text{when} & x=0, & x=l_1, & x=l_1+l_2, & x=l_1+l_2+l_3, \\ & P=0, & P=Q, & Q=R, & R=0, \end{array}$$

$$E_1 \frac{dP}{dx} = E_2 \frac{dQ}{dx}, \qquad E_2 \frac{dQ}{dx} = E_3 \frac{dR}{dx},$$

whatever the suffixes may be, provided they are the same in each equation. If then we put

$$\phi(m, n) = \int_0^{l_1} E_1 P_m P_n dx + \int_{l_1}^{l_1+l_2} E_2 Q_m Q_n dx + \int_{l_1+l_2}^{l_1+l_2+l_3} E_3 R_m R_n dx,$$

we have $m^2\phi(m, n) = n^2\phi(m, n)$, and therefore each is necessarily zero when m and n are different. A precisely similar theorem would apply if one or both ends of the string were loose, or if the string were vibrating transversely instead of longitudinally.

Suppose now that we have initially $\xi_1' = f_1(x)$, $\xi_2' = f_2(x)$, $\xi_3' = f_3(x)$. We easily find

$$\int_0^{l_1} E_1 f_1(x) \sin(n_1 x + M_1)\, dx + \int_{l_1}^{l_1+l_2} E_2 f_2(x) \sin\{n_2(x - l_1) + M_2\}\, dx$$
$$+ \int_{l_1+l_2}^{l_1+l_2+l_3} E_3 f_3(x) \sin\{n_3(x - l_1 - l_2) + M_3\}\, dx$$
$$= E_1 L_1 \int_0^{l_1} \sin^2(n_1 x + M_1)\, dx + E_2 L_2 \int_{l_1}^{l_1+l_2} \sin^2\{n_2(x - l_1) + M_2\}\, dx$$
$$+ E_3 L_3 \int_{l_1+l_2}^{l_1+l_2+l_3} \sin^2\{n_3(x - l_1 - l_2) + M_3\}\, dx,$$

these integrations may be easily effected and give an additional equation to find the L, which corresponds to any value of p.

If the strings did not start from rest, we should merely have to *add* to the expressions for ξ_1', ξ_2', ξ_3' similar functions of x but with $\sin nat$ written for $\cos nat$.

670. Ex. 1. If the three strings vibrate transversely, and a_1, a_2, a_3 be the velocities of a wave along them measured in units of length of unstretched string, prove that the periods of the notes are given by the equation

$$\frac{\tan n_1 l_1}{n_1} + \frac{\tan n_2 l_2}{n_2} + \frac{\tan n_3 l_3}{n_3} = n_2^2 \frac{\tan n_1 l_1}{n_1} \cdot \frac{\tan n_2 l_2}{n_2} \cdot \frac{\tan n_3 l_3}{n_3},$$

where $n_1a_1 = n_2a_2 = n_3a_3 = \frac{2\pi}{p}$. If the initial disturbance is given show how to find the subsequent motion.

Ex. 2. Two heavy strings AB, BC of different materials are attached together at B and suspended under gravity from a fixed point A. Prove that the periods of the vertical oscillations are given by the equation

$$\tan\frac{2\pi l_1}{a_1 p} \cdot \tan\frac{2\pi l_2}{a_2 p} = \frac{E_1 a_2}{E_2 a_1},$$

the notation being the same as before. If the two strings be initially unstretched, find their lengths at any time.

671. *An elastic string is stretched between two fixed points* A *and* B′ *and is set in vibration, it is required to find the energy.*

Let the notation be the same as that used in Arts. 663 and 664.

First let the vibrations be longitudinal. The equation of motion is

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}.$$

Hence we have

$$\xi = \frac{l'-l}{l}x + \Sigma[A\sin\{n(at-x)+\alpha\} + B\sin\{n(at+x)+\beta\}].$$

Since ξ must vanish when $x=0$ and be equal to $l'-l$ when $x=l$ we find, as in Art. 666,

$$\xi = \frac{l'-l}{l}x + \Sigma C\sin nx\sin(nat+\gamma),$$

where $nl = i\pi$ and Σ implies summation for all positive integer values of i. The letters C and γ are constants which may be different in every term and which depend on the initial disturbance.

The kinetic energy of the whole string is

$$= \int_0^l \frac{1}{2} m dx\left(\frac{d\xi}{dt}\right)^2 = \int_0^l \frac{1}{2} m dx\{\Sigma Cna\sin nx\cos(nat+\gamma)\}^2.$$

Now $\int_0^l \sin nx\sin n'x dx = 0$ when n and n' are numerically unequal since nl and $n'l$ are both integer multiples of π. Hence, when the square of the series is expanded, the integral of the product of any two terms is zero.

Also $\int_0^l \sin^2 nx dx = \frac{1}{2}l$, hence the kinetic energy becomes

$$= \frac{1}{4} mla^2\,\Sigma C^2 n^2\cos^2(nat+\gamma).$$

To find the potential energy; we notice that the work done in stretching an element from its unstretched length dx to its length $dx + d\xi$ is, by Art. 327, equal to $\frac{1}{2}E\left(\frac{d\xi}{dx}\right)^2 dx$. Hence the whole work done in stretching the string is

$$= \int_0^l \frac{1}{2} E dx\left(\frac{d\xi}{dx}\right)^2 = \int_0^l \frac{1}{2} E dx\left\{\frac{l'-l}{l} + \Sigma Cn\cos nx\sin(nat+\gamma)\right\}^2.$$

Now $\int_0^l \cos nx\cos n'x dx = 0$ or $\frac{1}{2}l$ according as n and n' are numerically unequal

or equal to each other; also $\int_0^l \cos nx dx = 0$. Hence as before, the integral becomes

$$=\frac{1}{2}E\frac{(l'-l)^2}{l}+\frac{1}{4}El\Sigma C^2n^2\sin^2(nat+\gamma).$$

The first term is the work done in stretching the string from the unstretched length l to the stretched length l'. If we refer the potential energy to the position of the string when stretched in equilibrium between the extreme points A and B' as the standard position, we retain the latter term only.

The energy is the sum of the kinetic and potential energies. Since $E=ma^2$, this becomes

$$\text{energy}=\frac{1}{4}mla^2\Sigma C^2n^2.$$

This result might have been deduced more simply from Art. 458, where it is shown that the energy of a compound vibration is the sum of the energies of the simple vibrations into which it may be resolved. See also Art. 451. The kinetic energy of any *single* harmonic is easily seen by integration to be

$$\frac{1}{4}mla^2C^2n^2\cos^2(nat+\gamma).$$

Hence the whole energy is $\qquad \frac{1}{4}mla^2\Sigma C^2n^2.$

We may also notice that, as in Art. 457, the mean kinetic energy is equal to the mean potential energy, the means being taken for any very long period.

672. *Next, let the vibrations be transversal.*

Following the notation of Art. 664, the motion is given, as before, by

$$y'=\Sigma C\sin nx\sin(nat+\gamma),$$

where $nl=i\pi$ and Σ implies summation for all positive integer values of i.

The kinetic energy by the same reasoning as in Art. 671 is equal to

$$\frac{1}{4}mla^2\Sigma C^2n^2\cos^2(nat+\gamma).$$

To find the potential energy, we notice that the work done in stretching an element from its unstretched length dx to its stretched length ds' is by Art. 327 equal to $\frac{1}{2}E\left(\frac{ds'}{dx}-1\right)^2dx$. Now

$$(ds')^2=(dx')^2+(dy')^2=\left(\frac{l'}{l}dx\right)^2+dy'^2,$$

$$\therefore\ \frac{ds'}{dx}=\frac{l'}{l}\left\{1+\frac{1}{2}\frac{l^2}{l'^2}\left(\frac{dy'}{dx}\right)^2\right\}\text{ nearly.}$$

Remembering that, by Art. 664, $ma^2=E\frac{l'-l}{l'}$; we find that the whole work done in stretching the string is

$$\int_0^l\frac{1}{2}dx\left\{E\left(\frac{l'-l}{l}\right)^2+ma^2\left(\frac{dy'}{dx}\right)^2\right\}.$$

Substituting for y' and integrating we find that the work is equal to

$$\frac{1}{2}E\frac{(l'-l)^2}{l}+\frac{1}{4}mla^2\Sigma C^2n^2\sin^2(nat+\gamma).$$

If we take the position of equilibrium of the string when stretched between the extreme points A and B' as the position of reference, we find that the

$$\text{energy} = \frac{1}{4} mla^2 \Sigma C^2 n^2.$$

This we may call the energy of the disturbance.

Prof. Donkin in his treatise on Acoustics, page 128, has found the energy of a string vibrating transversely, by an ingenious application of the method of subtractions.

Ex. 1. An elastic rod AB has the end A fixed and B free. Being placed on a perfectly smooth table, it vibrates longitudinally. Show that the energy of a disturbance represented by $\xi = \Sigma C \sin nx \sin (nat + \gamma)$ where $nl = (2i+1)\frac{\pi}{2}$ is $\frac{1}{4} mla^2 \Sigma C^2 n^2$.

NOTES.

On D'Alembert's Principle, by Sir G. B. Airy.

I HAVE seen some statements of or remarks on this principle which appear to me to be erroneous. The principle itself is not a new physical principle, nor any addition to existing physical principles; but is a convenient principle of combination of mechanical considerations, which results in a comprehensive process of great elegance.

The tacit idea, which dominates through the investigation, is this:—That every mass of matter in any complex mechanical combination may be conceived as containing in itself two distinct properties:—one that of connexion in itself, of susceptibility to pressure-force, and of connexion with other such masses, but not of inertia nor of impressions of momentum:—the other that of discrete molecules of matter, held in their places by the connexion-frame, susceptible to externally impressed momentum, and possessing inertia. The union produces an imponderable skeleton, carrying ponderable particles of matter.

Now the action of external momentum-forces on any one particle tends to produce a certain momentum-acceleration in that particle, which (generally) is not allowed to produce its full effect. And what prevents it from producing its full effect? It is the pressure of the skeleton-frame, which pressure will be measured by the difference between the impressed momentum-acceleration and the actual momentum-acceleration for the same. Thus every part of the skeleton sustains a pressure-force depending on that difference of momenta. And the whole mechanical system, however complicated, may now be conceived as a system of skeletons, each sustaining pressure-forces, and (by virtue of their combination) each impressing forces on the others.

And what will be the laws of movement resulting from this connexion? The forces are pressure-forces, acting on imponderable skeletons, and they must balance according to the laws of statical equilibrium. For if they did not, there would be instantaneous change from the understood motion, which change would be accompanied with instantaneous change of momentum-acceleration of the molecules, that would produce different pressures corresponding to equilibrium. (It is to be remarked that momentum cannot be changed instantaneously, but momentum-acceleration can be changed instantaneously.)

We come thus to the conclusion, that, taking for every molecule the difference between the impressed momentum-acceleration and the actual momentum-acceleration, those differences through the entire machine will statically balance. And—combining in one group all the impressed momentum-accelerations, and in another group all the actual momentum-accelerations—it is the same thing as saying that the impressed momentum-accelerations through the entire machine will balance the actual momentum-accelerations through the entire machine. This is the usual expression of D'Alembert's principle.

Euler's Geometrical Equations.

Art. 235. It is sometimes necessary to express the angular velocities of the body about the *fixed* axes OX, OY, OZ in terms of θ, ϕ, ψ. This may be effected in the following manner. Let ω_x, ω_y, ω_z be the angular velocities about the fixed axes, Ω the resultant any velocity. If we impress on space and also on the body in addition to its existing motion, an angular velocity equal to $-\Omega$ about the resultant axis of rotation, the axes OA, OB, OC will become fixed, and the axes OX, OY, OZ will move with angular velocities $-\omega_x$, $-\omega_y$, $-\omega_z$. Hence, in the formulæ of the text, if we change

$$\left.\begin{array}{lll}\phi & \text{into} & -\psi \\ \theta & \ldots & -\theta \\ \psi & \ldots & -\phi\end{array}\right\}, \qquad \left.\begin{array}{llll}\omega_1 & \text{will become} & & -\omega_x \\ \omega_2 & \ldots & \ldots & -\omega_y \\ \omega_3 & \ldots & \ldots & -\omega_z\end{array}\right\}.$$

Thus, we have

$$\omega_x = -\frac{d\theta}{dt}\sin\psi + \frac{d\phi}{dt}\sin\theta\cos\psi,$$

$$\omega_y = \frac{d\theta}{dt}\cos\psi + \frac{d\phi}{dt}\sin\theta\sin\psi,$$

$$\omega_z = \frac{d\phi}{dt}\cos\theta + \frac{d\psi}{dt}.$$

Sometimes it will be more convenient to measure the angular coordinates θ, ϕ, ψ in a different manner. Suppose, for example, we wish to refer the axes fixed in space to the axes fixed in the body as co-ordinate axes. To obtain the standard figure corresponding to this case, we must in the figure of Art. 235 interchange the letters X, Y, Z with A, B, C each with each. The angles θ, ϕ, ψ being measured as indicated in the figure after this change, the relations connecting them with the angular velocities about the axes fixed in space, are obtained from those in the text by simply changing ω_1, ω_2, ω_3 into $-\omega_x$, $-\omega_y$, $-\omega_z$. If we choose to measure θ in the opposite direction to that indicated in the figure, the expressions for ω_x, ω_y, become identical with those for ω_1, ω_2, in the text.

On the Impact of Bodies.

Arts. 156 and 305. The problem of the impact of two *smooth inelastic* bodies is considered by Poisson in his *Traité de Mécanique*, Seconde Edition, 1833. The motion of each body just before impact being supposed given, he forms six equations of motion for each body to determine the motion just after impact. These contain thirteen unknown quantities, viz. the resolved velocities of the centre of gravity of each body along three rectangular axes, the three resolved angular velocities of each body about the same axes, and lastly the mutual reaction of the two bodies. Thus the equations are insufficient to determine the motion. A thirteenth equation is then obtained from the principle that the impact terminates at the moment of greatest compression, i.e. at the moment when the normal velocities of the points of contact of the two bodies which impinge, are equal.

When the bodies are *elastic*, Poisson divides the impact into two periods. The first begins at the first contact of the bodies and terminates at the moment of greatest compression. The second begins at the moment of greatest compression and terminates when the bodies separate. The motion at the end of the first period is found exactly as if the bodies were inelastic. The motion at the end of the second period is found from the principle that the whole momentum communicated by one body to the other during the second period, bears a constant ratio to that communicated during the first period of the impact. This ratio depends on the elasticity of the two bodies and can be found only by experiments made on some bodies of the same material in some simple cases of impact.

When the bodies are *rough* and *slide on each other* during the impact, Poisson remarks that there will also be a frictional impulse. This is to be found from the principle that the magnitude of the friction at each instant must bear a constant ratio to the normal pressure and the direction must be opposite to that of the relative motion of the points in contact. He applies this to the case of a sphere, either inelastic or perfectly elastic, impinging on a rough plane, the sphere turning before the impact about a horizontal axis perpendicular to the direction of motion of the centre of gravity. He points out that there are several cases to be considered; (1) when the sliding is the same in direction during the whole of the impact and does not vanish, (2) when the sliding vanishes during the impact and remains zero, (3) when the sliding vanishes and changes sign. This third case, however, contains an unknown quantity and his formulæ therefore fail to determine the motion. Poisson points out that the problem would be very complicated if the sphere had an initial rotation about an axis not perpendicular to the vertical plane in which the centre of gravity moves. This case he does not attempt to solve, but passes on to discuss at greater length the impact of *smooth* bodies.

M. Coriolis in his *Jeu de Billard* (1835) considers the impact of two *rough* spheres *sliding* on each other during the whole of the impact. He obtains the result given in Art. 312, Ex. 3.

M. Ed. Phillips in the fourteenth volume of *Liouville's Journal*, 1849, considers the problem of the impact of two *rough inelastic* bodies of any form when the direction of the friction is not necessarily the same throughout the impact, provided the *sliding does not vanish* during the impact. He divides the period of impact into elementary portions and applies Poisson's rule for the magnitude and direction of the friction to each elementary period. He points out how the solution of the equations may be effected, and in particular he discusses the case in which the two bodies have their principal axes at the point of contact parallel each to each and also each body has its centre of gravity on the common normal at the point of contact. He deduces from this the two results, given in Art. 312, Ex. 4 and 5.

M. Phillips does not examine in detail the impact of *elastic* bodies, though he remarks that the period of impact must be divided into two portions which must be considered separately. These however, he considers, do not present any further peculiarities.

The case in which the sliding vanishes and the friction becomes discontinuous, does not appear to have been examined by him.

Sir W. R. Hamilton's Equations.

Art. 378. The demonstration as given by Sir W. R. Hamilton requires that T should be a homogeneous quadratic function of the accented letters and this is generally the case in dynamics. The extension to the case in which the geometrical equations do not contain the time explicitly is due to Prof. Donkin. Prof. Donkin has made a further extension of this theorem which is sometimes useful. If T_1 be a function of any other letter, say ξ, as well as θ, ϕ, &c., then we shall have $\dfrac{dT_1}{d\xi} = -\dfrac{dT_2}{d\xi}$, the differentiation with respect to ξ being in each case performed only so far as ξ appears explicitly. The theorem may be demonstrated as in the note to page 374.

On the Principle of Least Action.

The argument in Art. 394 shows that $\delta\int Tdt = 0$ under certain conditions. According to the usual phraseology it is asserted that $\int Tdt$ is either a maximum or a minimum. But this is not strictly correct. It seems clear that since the *Vis Viva* cannot be negative, there must be some mode of motion from one given position to another, for which the action is the least possible. When, therefore, the equations supplied by the Calculus of Variations lead to but one possible motion, that motion must make $\int Tdt$ a minimum. But when there are several

possible modes of motion, though none can be a maximum for the reason given in the text, some of these may be neither maxima nor minima.

To determine whether the integral is a maximum or a minimum or neither, we must examine the terms of the second order in the variation of the integral to ascertain if their sum keeps one sign or not for all variations of the independent variables. This is a very troublesome process, and we do not propose to discuss it. It will be sufficient to call the reader's attention to some remarks of Jacobi, given in the seventeenth volume of *Crelle's Journal*, 1837, and translated in Mr Todhunter's *History of the Calculus of Variations*, page 250.

Suppose a dynamical system to start from any given position which we shall call A, and to arrive at some position B. If the time be given, the motion is found by making $\delta \int L dt = 0$; if the energy be given, by making $\delta \int T dt = 0$. The constants which occur in integrating the differential equations supplied by the Calculus of Variations are to be determined by means of the given limiting values; but as this involves the solution of equations there will in general be several systems of values for the arbitrary constants, so that several possible modes of motion from A to B may be found which satisfy the same differential equation and the same limiting conditions. Now let one of these modes of motion be chosen, and let the position B approach A, so as to be always on this chosen mode of motion. Suppose that when B reaches the position C another possible mode of motion from A to B is indefinitely near to the chosen motion. Then C determines the boundary up to which or beyond which the integration must not extend if the integral is to be a minimum.

The reason seems to be as follows. If U be equal to the integral under consideration, we have along each of the motions from A to B $\delta U = 0$. Hence when B coincides with C, we have both $\delta U = 0$ and $\delta (U + \delta U) = 0$. It easily follows that the terms of the second order can be made to vanish by a proper variation. When the limits of integration are more extended than AC, it is not difficult to show that the terms of the second order can be made not merely to vanish, but to change sign.

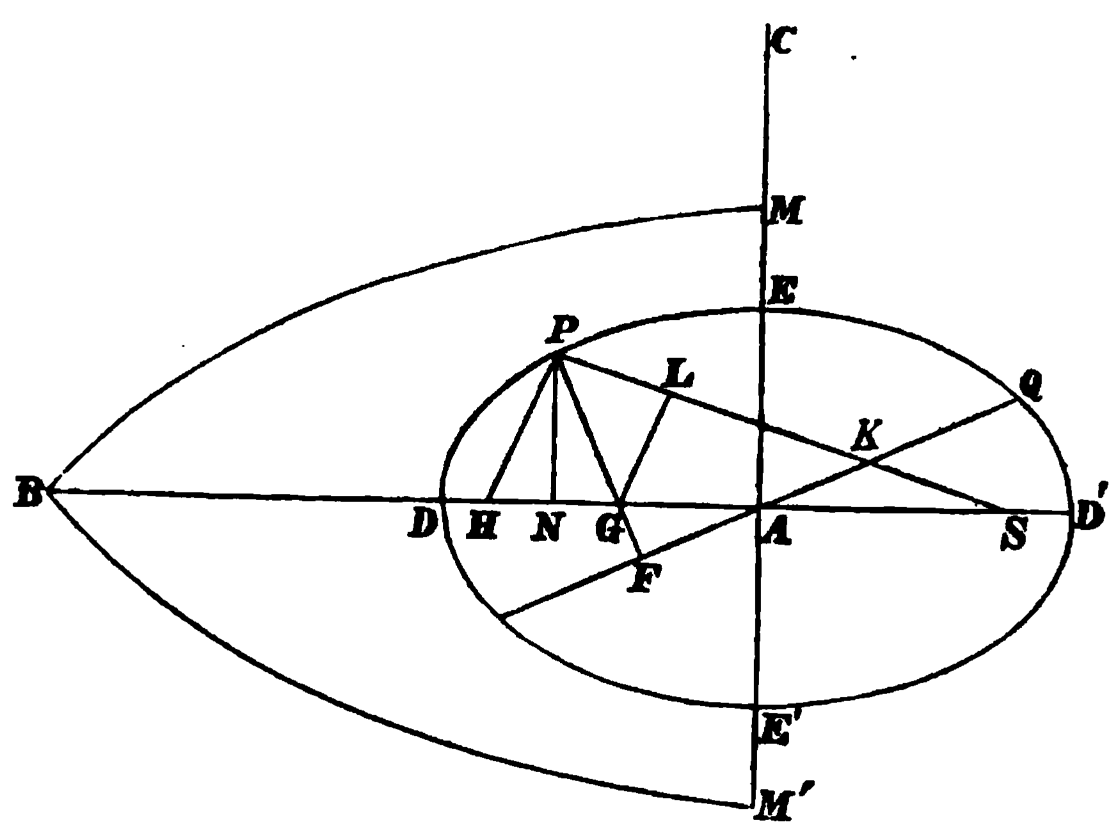
Jacobi illustrates his rule by considering the principle of least action in the elliptic motion of a planet. Let S be the sun, and let the particle start from A towards aphelion to arrive at a point B. The path is known to be an ellipse with S for focus. Since we use the principle of least action, the energy of the motion is given: hence the major axis of the ellipse is known, let this be $2a$. The other focus H of the ellipse is the intersection of two circles described with centres A and B and radii $2a - SA$, $2a - SB$ respectively. The two intersections give two solutions which only coincide when the circles touch, that is when the line AB passes through the focus H. Thus if we draw a chord AC through H to cut the ellipse described by the particle in C, then the terminal position B must fall between A and C if the integral which occurs in the principle of least action is really to be a minimum for this ellipse. If B

coincide with C, then the second variation cannot become negative, but it can become zero, so that the variation of the integral is then of the third order, and may therefore be either positive or negative. If B be beyond C the second variation itself can become negative.

If the particle start from A towards perihelion, then the extreme point C is determined by drawing a chord AC through the focus S to cut the ellipse in C. For if A and C are the limits we can obtain an infinite number of solutions by the revolution of the ellipse round AC. If then in the last case the second limit B fall beyond C there will be a curve of double curvature between the two given points for which $\int Tdt$ is less than it is for the ellipse.

On Sphero-Conics.

The following properties of a sphero-conic will be found useful in connexion with the theorems of Art. 527. They appear to be new. The curve is represented by the line $DED'E'$. As in the text, the eye is supposed to be situated in the radius through A, viewing the sphere from a considerable distance. The three principal planes of the cone intersect the sphere in the three quadrants AB, BC, CA, and any one of the three points A, B, C might be called the centre. The arcs AD and AE are represented by a and b.



1. Equation to the conic. Draw the arc PN perpendicular to AD and let $PN=y$, $AN=x$. Let NP produced cut the small circle described on DD' as diameter in P', let NP' be called the eccentric ordinate and be represented by y'. We then have

$$\left.\begin{aligned}\frac{\tan y}{\tan y'} = \text{constant} = \frac{\tan b}{\tan a}\\ \cos a = \cos y' \cos x\end{aligned}\right\}.$$

2. The projection of the normal PG on the focal radius vector SP, i.e. PL, is constant and equal to half the latus rectum.

If $2l$ be the latus rectum, then $\tan l = \dfrac{\tan^2 b}{\tan a}$.

Also $$\frac{\tan GL}{\sin PN} = \text{constant}.$$

3. If QAF be an arc cutting PG at right angles, QA may be called the semi-conjugate of AP. Then

$$\tan PG \,.\, \tan PF = \tan^2 b.$$

4. The length PK cut off the focal radius vector by the conjugate diameter is constant and equal to a. This follows from (2) and (3).

5. If $1 - e^2 = \dfrac{\sin^2 b}{\sin^2 a}$, e may be called the eccentricity of the spheroconic. Then

$$\tan AG = e^2 \tan AN.$$

6. Also S being a focus

$$\tan (SP - a) = e \tan AN.$$

7. Polar equation to the conic

$$\frac{\tan l}{\tan SP} = 1 - \frac{e}{\cos^2 b} \cos P\hat{S}A.$$

8. If ρ be the radius of curvature at P, then

$$\tan \rho = \frac{\tan^3 n}{\tan^2 l}.$$

9. Regarding AP, AQ as conjugate semi-diameters,

$$\left.\begin{array}{l} \sin^2 AP + \sin^2 AQ = \sin^2 a + \sin^2 b \\ \sin AQ \,.\, \sin PF = \sin a \,.\, \sin b \end{array}\right\}.$$

10. If p be the perpendicular from the centre A on the tangent at P,

$$\frac{\tan^2 a \tan^2 b}{\tan^2 p} = \tan^2 a + \tan^2 b - \tan^2 AP.$$

11. Also $$\tan^2 PG - \tan^2 l = \frac{e^2}{\cos^4 b} \sin^2 PN.$$

12. $$\left.\begin{array}{l} \sin^2 a - \sin^2 AP \\ = \sin^2 AQ - \sin^2 b \end{array}\right\} = \frac{e^2}{1 - e^2} \sin^2 PN.$$

Cor. $$\tan^2 PG = \frac{\tan^2 b}{\cos^2 b \sin^2 a} (\cos^2 AP - \cos^2 a \cos^2 b).$$

If $\sin AM = \sin AM' = \dfrac{\sin b}{\sin a}$, the planes of the arcs BM and BM' are parallel to the circular sections of the cone. Some of the properties of these arcs resemble those of asymptotes when B is regarded as the centre of the conic. The properties which connect the sphero-conic with the arcs BM and BM' will be found in Dr Salmon's *Solid Geometry*.

Many other properties of sphero-conics will also be found in Mr Frost's *Solid Geometry*.

Miscellaneous Notes.

Art. 3. The term *moment of inertia with regard to a plane* seems to have been first used by M. Binet in the *Journal Polytechnique*, 1813.

Arts. 19 *and* 182. So much has been written on the ellipsoids of inertia and on the kinematics of a solid body that it is difficult to determine what is due to each of the various authors. The reader will find much information on this point in Prof. Cayley's report to the British Association on the Special Problems of Dynamics, 1862.

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